PHYSICS 140A : STATISTICAL PHYSICS
HW SOLUTIONS #8

(1) Thanksgiving turkey typically cooks at a temperature of $350^\circ\,\mathrm{F}$. Calculate the total electromagnetic energy inside an over of volume $V = 1.0\,\mathrm{m}^3$ at this temperature. Compare it to the thermal energy of the air in the oven at the same temperature.

The total electromagnetic energy is

$$E = 3pV = \frac{\pi^2}{15}\,\frac{V\,(k_\mathrm{B}T)^4}{(\hbar c)^3} = 3.78 \times 10^{-5}\,\mathrm{J}\ . \qquad (1)$$

For air, which is a diatomic ideal gas, we have $E = \frac{5}{2}pV$. What do we take for p? If we assume that oven door is closed at an initial temperature of $63^\circ\,\mathrm{F}$ which is $300\,\mathrm{K}$, then with a final temperature of $350^\circ\,\mathrm{F} = 450\,\mathrm{K}$, we have an increase in the absolute temperature by 50%, hence a corresponding pressure increase of 50%. So we set $p = \frac{3}{2}$ atm and we have

$$E = \tfrac{5}{2} \cdot \tfrac{3}{2}(1.013 \times 10^5\,\mathrm{Pa})(1.0\,\mathrm{m}^3) = 3.80 \times 10^5\,\mathrm{J}\ , \qquad (2)$$

which is about ten orders of magnitude larger.

(2) In §5.4.4 of the lecture notes we derived the spectral energy density $\rho_\varepsilon(\nu, T)$ for a three-dimensional blackbody. We found that it was peaked at a frequency $\nu^* = s^* k_\mathrm{B}T/h$ where $s^* = 2.83144$ extremizes the function $s^3/(e^s - 1)$. Consider instead the function $\tilde\rho_\varepsilon(\lambda, T)$ as a function of wavelength λ and temperature T, where $\lambda = c/\nu$. To relate $\rho_\varepsilon(\nu, T)$ and $\tilde\rho_\varepsilon(\lambda, T)$, set the fraction of energy of EM radiation between frequencies ν and $\nu + d\nu$ equal to the fraction of energy between wavelengths λ and $\lambda + d\lambda$. Show that this is maximized at a wavelength $\lambda^* = t^* hc/k_\mathrm{B}T$, where t^* is a constant. Find t^* numerically. Is $t^* = 1/s^*$? Why or why not?

Solution:

We must have

$$\begin{aligned}\tilde\rho_\varepsilon(\lambda, T) &= \rho_\varepsilon(\nu, T)\left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2}\,\rho_\varepsilon(\nu, T) \\ &= \frac{15}{\pi^4}\,\frac{k_\mathrm{B}T}{hc}\,\frac{(hc/\lambda k_\mathrm{B}T)^5}{e^{hc/\lambda k_\mathrm{B}T} - 1} \equiv \frac{15}{\pi^4}\,\frac{k_\mathrm{B}T}{hc}\,\frac{(\lambda_T/\lambda)^5}{e^{\lambda_T/\lambda} - 1}\quad ,\end{aligned}$$

where $\lambda_T \equiv hc/k_\mathrm{B}T$ is not to be confused with the thermal de Broglie wavelength for a massive particle. The maximum value occurs for $\lambda^*(T) = u\,k_\mathrm{B}T$ where

$$\frac{d}{du}\left(\frac{u^5}{e^u - 1}\right) = 0 \qquad \Rightarrow \qquad u = \frac{u}{1 - e^{-u}} = 5 \qquad \Rightarrow \qquad u = 4.9651 \quad .$$

Thus $\lambda^* = t^* ch/k_\mathrm{B}T$ where $t^* = 1/u^* = 0.2014$. Note that $\lambda^*(T) \neq c/\nu^* = 0.3544\,hc/k_\mathrm{B}T$. This is because the spectral density $\tilde\rho_\varepsilon(\lambda, T)$ is given by $\tilde\rho_\varepsilon(\lambda, T) = (c/\lambda^2)\,\rho_\varepsilon(\nu = c/\lambda, T)$ and so the stationary point for λ is obtained by extremizing a different function.

(3) A three-dimensional gas of particles obeys the dispersion relation $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{7/4}$. The internal degeneracy is $\mathrm{g} = 1$.

(a) Compute the single particle density of states $g(\varepsilon)$.

(b) For photon statistics, compute the pressure $p(n)$.

(c) For photon statistics, compute the entropy density $s(n) = S/V$.

(d) For Bose-Einstein statistics, compute the condensation temperature $T_{\text{BEC}}(n)$.

Solution:

(a) For a general power law dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{\sigma}$ in $d = 3$ dimensions, the density of states $g(\varepsilon)$ is given by[1]

$$g(\varepsilon) = \frac{1}{2\pi^2\sigma A^{3/\sigma}}\,\frac{k^{d-1}}{d\varepsilon/dk}$$

Thus for $\sigma = \frac{7}{4}$ we have

$$g(\varepsilon) = \frac{2}{7\pi^2 A^{12/7}}\,\varepsilon^{5/7}\,\Theta(\varepsilon) \quad .$$

(b) From the results in §5.4.1 of the lecture notes, we have

$$n(T) = \frac{1}{7\pi^2 A^{12/7}}\,\zeta\big(\tfrac{12}{7}\big)\,\Gamma\big(\tfrac{12}{7}\big)\,(k_{\rm B}T)^{12/7} \qquad , \qquad p(T) = \frac{1}{7\pi^2 A^{12/7}}\,\zeta\big(\tfrac{19}{7}\big)\,\Gamma\big(\tfrac{12}{7}\big)\,(k_{\rm B}T)^{19/7} \quad .$$

Dividing, we have

$$p(n) = \frac{\zeta(19/7)}{\zeta(12/7)}\,n k_{\rm B}T = 0.6267\,n k_{\rm B}T \quad .$$

(c) Since $\mu = 0$, we have $d\mu = -s\,dT + v\,dp = 0$ with $v = 1/n$. Thus

$$s = \frac{1}{n}\,\frac{dp}{dT} = \frac{19\,\zeta(19/7)}{7\,\zeta(12/7)}\,k_{\rm B} = 1.7011\,k_{\rm B} \quad .$$

(d) The condition for Bose-Einstein condensation is

$$n = n(T_{\rm c}, \mu = 0) = \int\limits_0^\infty \frac{g(\varepsilon)}{e^{\varepsilon/k_{\rm B}T_{\rm c}} - 1} = \frac{2\,\zeta(12/7)\,\Gamma(12/7)}{7\pi^2 A^{12/7}}\,(k_{\rm B}T_{\rm c})^{12/7} \quad ,$$

hence

$$k_{\rm B}T_{\rm c} = \left(\frac{7\pi^2 A^{12/7} n}{2\,\zeta(12/7)\,\Gamma(12/7)}\right)^{7/12} = 5.5198\,A\,n^{7/12} \quad .$$

[1] See eqn. 5.57 of the lecture notes.

(4) A branch of excitations for a three-dimensional system has a dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{2/3}$. The excitations are bosonic and are not conserved; they therefore obey photon statistics.

(a) Find the single excitation density of states per unit volume, $g(\varepsilon)$. You may assume that there is no internal degeneracy for this excitation branch.

(b) Find the heat capacity $C_V(T, V)$.

(c) Find the ratio E/pV.

(d) If the particles are bosons with number conservation, find the critical temperature T_{c} for Bose-Einstein condensation.

Solution:

(a) We have, for three-dimensional systems,

$$g(\varepsilon) = \frac{1}{2\pi^2}\,\frac{k^2}{d\varepsilon/dk} = \frac{3}{4\pi^2 A}\,k^{7/3} \,.$$

Inverting the dispersion to give $k(\varepsilon) = (\varepsilon/A)^{3/2}$, we obtain

$$g(\varepsilon) = \frac{3}{4\pi^2}\,\frac{\varepsilon^{7/2}}{A^{9/2}}$$

(b) The energy is then

$$\begin{aligned} E &= V\int\limits_0^\infty d\varepsilon\; g(\varepsilon)\,\frac{\varepsilon}{e^{\varepsilon/k_{\mathrm{B}}T}-1} \\ &= \frac{3V}{4\pi^2}\,\Gamma\big(\tfrac{11}{2}\big)\,\zeta\big(\tfrac{11}{2}\big)\,\frac{(k_{\mathrm{B}}T)^{11/2}}{A^{9/2}} \,. \end{aligned}$$

Thus,

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{3V}{4\pi^2}\,\Gamma\big(\tfrac{13}{2}\big)\,\zeta\big(\tfrac{11}{2}\big)\,k_{\mathrm{B}}\left(\frac{k_{\mathrm{B}}T}{A}\right)^{9/2}$$

(c) The pressure is

$$
\begin{aligned}
p = -\frac{\Omega}{V} &= -k_{\mathrm{B}}T \int_0^\infty d\varepsilon\; g(\varepsilon)\; \ln\left(1 - e^{-\varepsilon/k_{\mathrm{B}}T}\right) \\
&= -k_{\mathrm{B}}T \int_0^\infty d\varepsilon\; \frac{3}{4\pi^2}\, \frac{\varepsilon^{7/2}}{A^{9/2}}\; \ln\left(1 - e^{-\varepsilon/k_{\mathrm{B}}T}\right) \\
&= -\frac{3}{4\pi^2}\, \frac{(k_{\mathrm{B}}T)^{11/2}}{A^{9/2}} \int_0^\infty ds\; s^{7/2}\; \ln\left(1 - e^{-s}\right) \\
&= \frac{3V}{4\pi^2}\, \Gamma\left(\tfrac{9}{2}\right) \zeta\left(\tfrac{11}{2}\right) \frac{(k_{\mathrm{B}}T)^{11/2}}{A^{9/2}} \,.
\end{aligned}
$$

Thus,

$$
\frac{E}{pV} = \frac{\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \tfrac{9}{2}
$$

(d) To find T_{c} for BEC, we set $z = 1$ (*i.e.* $\mu = 0$) and $n_0 = 0$, and obtain

$$
n = \int_0^\infty d\varepsilon\; g(\varepsilon)\; \frac{\varepsilon}{e^{\varepsilon/k_{\mathrm{B}}T_{\mathrm{c}}} - 1}
$$

Substituting in our form for $g(\varepsilon)$, we obtain

$$
n = \frac{3}{4\pi^2}\, \Gamma\left(\tfrac{9}{2}\right) \zeta\left(\tfrac{9}{2}\right) \left(\frac{k_{\mathrm{B}}T}{A}\right)^{9/2} ,
$$

and therefore

$$
T_{\mathrm{c}} = \frac{A}{k_{\mathrm{B}}} \left(\frac{4\pi^2 n}{3\,\Gamma\left(\frac{9}{2}\right) \zeta\left(\frac{9}{2}\right)}\right)^{2/9}
$$