PHYSICS 140A : STATISTICAL PHYSICS
FINAL EXAMINATION

(1) Provide clear, accurate, and brief answers for each of the following:0

(a) What is the Gibbs-Duhem for a single-component system and how is it derived from homogeneity of the energy function $E(S,V,N)$? `[5 points]`

(b) What are the separate conditions guaranteeing thermal, mechanical, and chemical (or particle) equilibrium between two single-component systems, and what equalities do they entail? `[5 points]`

(c) What is a virial equation of state? What are the dimensions of the $k^{\rm th}$ virial coefficient B_k, and on what intensive quantity or quantities does B_k depend? `[5 points]`

(d) Consider a noninteracting classical system consisting of distinguishable particles situated on each of $N_{\sf s}$ sites. The available energy states for each particle are $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, $\varepsilon_3 = \Delta$, and $\varepsilon_4 = \Omega$. What is the free energy $F(T, N_{\sf s})$? What is the entropy at $T = 0$?
`[5 points]`

(e) For a noninteracting Bose gas with density of states $g(\varepsilon) = A\,\varepsilon^5$, find the condensation temperature $T_{\rm c}(n)$. `[5 points]` You may find the following helpful:

$$\int\limits_0^\infty \! dt\, \frac{t^{r-1}}{\exp(t)-1} = \Gamma(r)\,\zeta(r) \quad .$$

(2) A surface consists of a collection of $N_{\sf s}$ sites, each of which hosts an electric dipole $\boldsymbol{p}_j = \mu_0\,\hat{\boldsymbol{n}}_j$. In an electric field $\boldsymbol{E}$, the energy of the $j^{\rm th}$ dipole is $-\boldsymbol{E}\cdot\boldsymbol{p}_j$. Each $\hat{\boldsymbol{n}}_j$ is a unit vector in $d = 3$ dimensions and can be expressed in terms of Cartesian components as $\hat{\boldsymbol{n}}_j = (\sin\theta_j\cos\phi_j, \sin\theta_j\sin\phi_j, \cos\theta_j)$ with polar angle $\theta_j \in [0,\pi]$ and azimuthal angle $\phi_j \in [0, 2\pi)$. The dipoles are situated at unique locations in the crystal, and are thus distinguishable.

(a) What is the partition function $Z(T, N_{\sf s}, E)$? You may assume $\boldsymbol{E}$ is parallel to $\hat{\boldsymbol{z}}$. *Hint: First find the single site partition function* $\zeta(T, E)$. `[5 points]`

(b) Find the average dipole moment $\langle \boldsymbol{p} \rangle$. `[5 points]`

(c) Now suppose that each site is either empty, with energy zero, or contains a dipole with energy $-\boldsymbol{E}\cdot\boldsymbol{p}$. The chemical potential for dipoles is μ. Find the grand potential $\Omega(T, N_{\sf s}, \mu, \boldsymbol{E})$. `[10 points]`

(d) Next, let the surface be in equilibrium with a nonrelativistic monatomic ideal gas of number density n. Gas atoms can be adsorbed on the surface, in which case they acquire a dipole moment and are bound to a surface adsorption site with energy $-\Delta < 0$. They can also desorb and join the gas as mass m atoms with $\boldsymbol{p} = 0$. Find the surface site occupation fraction $f = N_{\rm surface}/N_{\rm s}$ in terms of T, E, Δ, the gas number density n, and other constants. [5 points]

(3) Consider an ultrarelativistic gas of N identical and indistinguishable particles in three space dimensions. The Hamiltonian of each particle is $\hat{h} = c|\boldsymbol{p}|$.

(a) What is the single particle partition function ζ. Assume the system is confined to a box of volume V. [5 points]

(b) What is the Helmholtz free energy $F(T,V,N)$? [5 points]

(c) What is the entropy $S(T,V,N)$? [5 points]

(d) What is the chemical potential $\mu(T,V,N)$? [5 points]

(e) What is the heat capacity at constant volume $C_{V,N}(T,V,N)$? [5 points]

(4) Consider a three-dimensional Bose gas of particles which have two internal polarization states, labeled by $\sigma = \pm 1$. The single particle energies are given by (with $\Delta > 0$)

$$\varepsilon(\boldsymbol{p},\sigma) = \frac{\boldsymbol{p}^2}{2m} + \sigma\Delta \quad ,$$

(a) Find the density of states per unit volume $g(\varepsilon)$. Recall that the DOS for nonrelativistic particles in three dimensions is given by

$$g_0(\varepsilon) = \frac{\sqrt{2m^3}}{2\pi^2\hbar^3}\,\varepsilon^{1/2}\,\Theta(\varepsilon) \quad .$$

Hint: The answer can be expressed as a sum of the DOS for $\sigma = \pm 1$ polarization states. [10 points]

(b) Into what single particle state does the gas condense? What is the value of the chemical potential μ in the condensed phase? [5 points]

(c) Find an implicit expression for the condensation temperature $T_{\rm c}(n,\Delta)$, and from it obtain an analytical expression for $T_{\rm c}(n,\Delta=\infty)$. You may find the following useful:

$$\int\limits_{-\infty}^{\infty} d\varepsilon\, \frac{g_0(\varepsilon)}{e^{(\varepsilon-\mu)/k_{\rm B}T} - 1} = \lambda_T^{-3}\,\mathrm{Li}_{3/2}\big(e^{\mu/k_{\rm B}T}\big)$$

as well as $\mathrm{Li}_s(1) = \zeta(s)$. *Hint: This obviates the need for you to do any actual integrals!* [10 points]

(d) When $\Delta = \infty$, the condensation temperature should agree with the familiar result for three-dimensional Bose condensation. Assuming $\Delta \gg k_{\rm B}T_{\rm c}(n,\Delta=\infty)$, find analytically the leading order difference $T_{\rm c}(n,\Delta) - T_{\rm c}(n,\Delta=\infty)$. [100 quatloos extra credit]