PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #4 SOLUTIONS

(1) Consider the one-dimensional Ising model with next-nearest neighbor interactions,

$$\hat{H} = -J\sum_n \sigma_n \sigma_{n+1} - K\sum_n \sigma_n \sigma_{n+2} \, ,$$

on a ring with N sites, where N is even. By considering consecutive pairs of sites, show that the partition function may be written in the form $Z = \mathsf{Tr}(R^{N/2})$, where R is a 4×4 transfer matrix. Find R. *Hint:* It may be useful to think of the system as a railroad trestle, depicted in fig. 1, with Hamiltonian

$$\hat{H} = -\sum_j \Big[J\sigma_j \mu_j + J\mu_j \sigma_{j+1} + K\sigma_j \sigma_{j+1} + K\mu_j \mu_{j+1} \Big] \, .$$

Then $R = R_{(\sigma_j \mu_j),(\sigma_{j+1}\mu_{j+1})}$, with $(\sigma\mu)$ a composite index which takes one of four possible values $(++)$, $(+-)$, $(-+)$, or $(--)$.

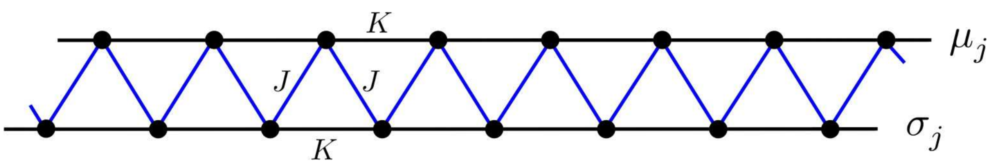

Figure 1: Railroad trestle representation of next-nearest neighbor chain.

The transfer matrix can be read off from the Hamiltonian:

$$R_{(\sigma\mu),(\sigma'\mu')} = e^{\beta J \mu(\sigma+\sigma')}\, e^{\beta K(\sigma\sigma' + \mu\mu')} \, .$$

Expressed as a matrix of rank four, with rows and columns corresponding to $\{++, +-, -+, --\}$, we have

$$R = \begin{pmatrix} e^{2\beta(J+K)} & e^{2\beta J} & 1 & e^{-2\beta K} \\ e^{-2\beta J} & e^{-2\beta(J-K)} & e^{-2\beta K} & 1 \\ 1 & e^{-2\beta K} & e^{-2\beta(J-K)} & e^{-2\beta J} \\ e^{-2\beta K} & 1 & e^{2\beta J} & e^{2\beta(J+K)} \end{pmatrix} .$$

Querying WolframAlpha for the eigenvalues, we find

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}\Big[uv - (1+u^{-1})\sqrt{u^2v^2 - 2uv^2 + 4u + v^2} + 2v^{-1} + u^{-1}v\Big] \\ \lambda_2 &= \tfrac{1}{2}\Big[uv + (1+u^{-1})\sqrt{u^2v^2 - 2uv^2 + 4u + v^2} + 2v^{-1} + u^{-1}v\Big] \\ \lambda_3 &= \tfrac{1}{2}\Big[uv - (1-u^{-1})\sqrt{u^2v^2 + 2uv^2 - 4u + v^2} - 2v^{-1} + u^{-1}v\Big] \\ \lambda_4 &= \tfrac{1}{2}\Big[uv + (1-u^{-1})\sqrt{u^2v^2 + 2uv^2 - 4u + v^2} - 2v^{-1} + u^{-1}v\Big] \, , \end{aligned}$$

where $u = e^{2\beta J}$ and $v = e^{2\beta K}$. The partition function on a ring of N sites, with N even, is

$$Z = \mathsf{Tr}\left(R^{N/2}\right) = \lambda_1^{N/2} + \lambda_2^{N/2} + \lambda_3^{N/2} + \lambda_4^{N/2} \, .$$

(2) Compute the partition function for the one-dimensional Tonks gas of hard rods of length a on a ring of circumference L. This is slightly tricky, so here are some hints. Once again, assume a particular ordering so that $x_1 < x_2 < \cdots < x_N$. Due to translational invariance, we can define the positions of particles $\{2, \ldots, N\}$ relative to that of particle 1, which we initially place at $x_1 = 0$. Then periodicity means that $x_N \le L - a$, and in general one then has

$$x_{j-1} + a \le x_j \le L - (N - j + 1)a \, .$$

Now integrate over $\{x_2, \ldots, x_N\}$ subject to these constraints. Finally, one does the x_1 integral, which is over the entire ring, but which must be corrected to eliminate overcounting from cyclic permutations. How many cyclic permutations are there?

There are N cyclic permutations, hence the last x_1 integral yields L/N, and

$$Z(T, L, N) = \lambda_T^{-N} \, \frac{L}{N} \int\limits_a^{Y_2} \! dx_2 \int\limits_{x_2+a}^{Y_3} \! dx_3 \cdots \int\limits_{x_{N-1}+a}^{Y_N} \! dx_N = \frac{L(L - Na)^{N-1} \lambda_T^{-N}}{N!} \, .$$

(3) For each of the cluster diagrams in Fig. 2, find the symmetry factor s_γ and write an expression for the cluster integral b_γ.

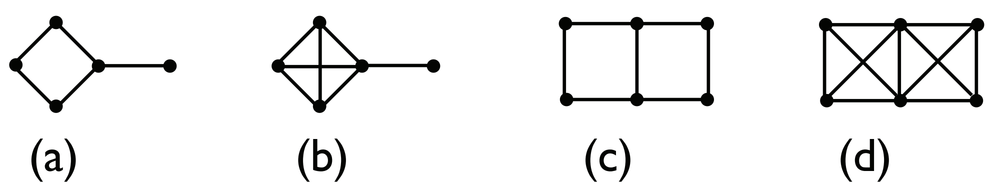

Figure 2: Cluster diagrams for problem 3.

Choose labels as in Fig. 3, and set $\boldsymbol{x}_{n_\gamma} \equiv 0$ to cancel out the volume factor in the definition of b_γ.

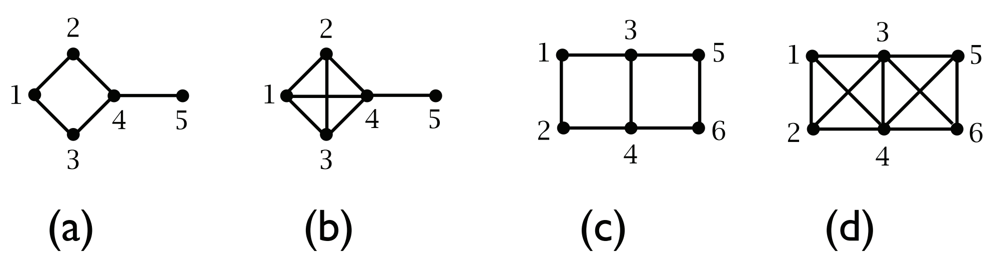

Figure 3: Labeled cluster diagrams.

(a) The symmetry factor is $s_\gamma = 2$, so

$$b_\gamma = \tfrac{1}{2} \int \! d^d\!x_1 \! \int \! d^d\!x_2 \! \int \! d^d\!x_3 \! \int \! d^d\!x_4 \; f(r_{12}) \, f(r_{13}) \, f(r_{24}) \, f(r_{34}) \, f(r_4) \, .$$

(b) Sites 1, 2, and 3 may be permuted in any way, so the symmetry factor is $s_\gamma = 6$. We then have

$$b_\gamma = \tfrac{1}{6}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \; f(r_{12})\, f(r_{13})\, f(r_{24})\, f(r_{34})\, f(r_{14})\, f(r_{23})\, f(r_4)\ .$$

(c) The diagram is symmetric under reflections in two axes, hence $s_\gamma = 4$. We then have

$$b_\gamma = \tfrac{1}{4}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \!\int\! d^d\!x_5 \; f(r_{12})\, f(r_{13})\, f(r_{24})\, f(r_{34})\, f(r_{35})\, f(r_4)\, f(r_5)\ .$$

(d) The diagram is symmetric with respect to the permutations (12), (34), (56), and $(15)(26)$. Thus, $s_\gamma = 2^4 = 16$. We then have

$$b_\gamma = \tfrac{1}{16}\int\! d^d\!x_1 \!\int\! d^d\!x_2 \!\int\! d^d\!x_3 \!\int\! d^d\!x_4 \!\int\! d^d\!x_5 \; f(r_{12})\, f(r_{13})\, f(r_{14})\, f(r_{23})\, f(r_{24})\, f(r_{34})\, f(r_{35})\, f(r_{45})\, f(r_3)\, f(r_4)\, f(r_5)\ .$$