23. (a) $E_0 = \frac{1}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$ so $x_0 = \sqrt{\hbar\omega_0/k}$

(b) $E_1 = \frac{3}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$ so $x_0 = \sqrt{3\hbar\omega_0/k}$

$E_2 = \frac{5}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$ so $x_0 = \sqrt{5\hbar\omega_0/k}$

24.
$$x_{\text{av}} = \int_{-\infty}^{\infty} |\psi(x)|^2\ x\,dx = A^2\int_{-\infty}^{\infty} e^{-2ax^2}\,x\,dx = 0$$

because the integrand is an odd function of x (the integral from $-\infty$ to 0 exactly cancels the integral from 0 to $+\infty$).

$$(x^2)_{\text{av}} = \int_{-\infty}^{\infty} |\psi(x)|^2\ x^2dx = A^2\int_{-\infty}^{\infty} e^{-2ax^2}x^2dx = 2A^2\int_0^{\infty} e^{-2ax^2}x^2dx = \frac{2A^2}{\sqrt{8a^3}}\int_0^{\infty} e^{-u^2}u^2du$$

with the substitution $u = x\sqrt{2a}$. The integral is a standard form found in tables and is equal to $\sqrt{\pi}/4$. Substituting $A = (\omega_0 m/\pi\hbar)^{1/4}$ and $a = \sqrt{km}/2\hbar = \omega_0 m/2\hbar$, we find

$$(x^2)_{\text{av}} = 2\left(\frac{\omega_0 m}{\pi\hbar}\right)^{1/2}\frac{1}{2\sqrt{2}}\left(\frac{2\hbar}{\omega_0 m}\right)^{3/2}\frac{\sqrt{\pi}}{4} = \frac{\hbar}{2\omega_0 m}$$

$$\Delta x = \sqrt{(x^2)_{\text{av}} - (x_{\text{av}})^2} = \sqrt{\hbar/2m\omega_0}$$

25. (a) Because the oscillating particle moves with equal probability in the positive and negative x directions, $p_{\text{av}} = 0$.

(b) $U_{\text{av}} = \frac{1}{2}k(x^2)_{\text{av}} = \frac{1}{2}k\frac{\hbar}{2\omega_0 m} = \frac{1}{2}\omega_0^2 m\frac{\hbar}{2\omega_0^2 m} = \frac{1}{4}\hbar\omega_0$

$K_{\text{av}} = E - U_{\text{av}} = \frac{1}{2}\hbar\omega_0 - \frac{1}{4}\hbar\omega_0 = \frac{1}{4}\hbar\omega_0$

$(p^2)_{\text{av}} = 2mK_{\text{av}} = 2m\left(\frac{1}{4}\hbar\omega_0\right) = \frac{\hbar\omega_0 m}{2}$

(c) $\Delta p = \sqrt{(p^2)_{\text{av}} - (p_{\text{av}})^2} = \sqrt{\hbar\omega_0 m/2}$

26. $E_0 = 1.24\text{ eV} = \frac{1}{2}\hbar\omega_0$ so $\hbar\omega_0 = 2.48\text{ eV}$

To $n = 2$ state: $\Delta E = E_2 - E_0 = \frac{5}{2}\hbar\omega_0 - \frac{1}{2}\hbar\omega_0 = 2\hbar\omega_0 = 2(2.48\text{ eV}) = 4.96\text{ eV}$

To $n = 4$ state: $\Delta E = E_4 - E_0 = \frac{9}{2}\hbar\omega_0 - \frac{1}{2}\hbar\omega_0 = 4\hbar\omega_0 = 4(2.48\text{ eV}) = 9.92\text{ eV}$

27. $P(x)\,dx = |\psi(x)|^2\,dx = A^2 e^{-2ax^2}dx$ so at $x = 0$ $P(0)dx = A^2dx$

At the classical turning points $x = \pm x_0$, $K = 0$ so $E = U$ or $\frac{1}{2}\hbar\omega_0 = \frac{1}{2}kx_0^2$

$$P(\pm x_0)dx = A^2 e^{-2(\sqrt{km}/2\hbar)(\hbar\omega_0/k)}dx = A^2e^{-1}dx = e^{-1}P(0)dx = 0.368P(0)dx$$

28. (a) If $E = 0$, then $p = 0$ and we would know the momentum exactly. Thus $\Delta p = 0$, which means $\Delta x = \infty$. But that would be inconsistent with a particle that is bound to a finite region of space.

(b)

$$E = \frac{1}{2}\hbar\omega_0 = \frac{1}{2}\hbar\sqrt{\frac{k}{m}} = \frac{1}{2}\hbar c\sqrt{\frac{k}{mc^2}} = 0.5(197\text{ eV}\cdot\text{nm})\sqrt{\frac{3.5\times10^3\text{ eV/nm}^2}{938\times10^3\text{ eV}}} = 0.19\text{ eV}$$

This is less than the binding energy, so this motion is not sufficient to dissociate the molecule.

(c) At the turning point of the motion, $E = \frac{1}{2}kx_0^2$, so

$$x_0 = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2(0.19\text{ eV})}{3.5\times10^3\text{ eV/nm}^2}} = 0.010\text{ nm}$$

This motion is not negligible at the atomic level.

32. $x < 0: \quad \psi_0 = A'e^{ik_0x} + B'e^{-ik_0x}$ with $k_0 = \sqrt{\frac{2mE}{\hbar^2}}$

$x > 0: \quad \psi_1(x) = C'e^{ik_1x} + D'e^{-ik_1x}$ with $k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$

If the particles are incident from them negative *x* direction, then D' (which is the coefficient of the term that represents a wave in the region of positive *x* traveling toward the origin) must be set to 0. We then apply the continuity conditions on ψ and $d\psi/dx$ at $x = 0$:

$$\psi_0(0) = \psi_1(0): \qquad A' + B' = C'$$

$$\left(\frac{d\psi_0}{dx}\right)_{x=0} = \left(\frac{d\psi_1}{dx}\right)_{x=0}: \qquad k_0(A' - B') = k_1C'$$

Solving these two equations, we obtain

$$C' = \frac{2A'}{1 + k_1/k_0}$$

$$B' = C' - A' = \frac{2A'}{1 + k_1/k_0} - A' = \frac{1 - k_1/k_0}{1 + k_1/k_0} A'$$

The squares of the amplitude ratios give the relative probabilities for the particles to be reflected at $x = 0$ or transmitted into the $x > 0$ region:

Reflection probability: $$\frac{|B'|^2}{|A'|^2} = \left(\frac{1 - k_1/k_0}{1 + k_1/k_0} \right)^2$$

Transmission probability: $$\frac{|C'|^2}{|A'|^2} = \frac{4}{(1 + k_1/k_0)^2}$$

42. (a) The x and y motions are independent, and each contributes an energy of $\hbar\omega_0(n+\frac{1}{2})$, but the integer n is not necessarily the same for the two independent motions. Thus the total energy is

$$E = \hbar\omega_0(n_x+\tfrac{1}{2}) + \hbar\omega_0(n_y+\tfrac{1}{2}) = \hbar\omega_0(n_x+n_y+1)$$

(b)

Energy	Degeneracy	(n_x, n_y)
$4\hbar\omega_0$	4	(0,3), (1,2), (2,1), (3,0)
$3\hbar\omega_0$	3	(0,2), (1,1), (2,0)
$2\hbar\omega_0$	2	(0,1), (1,0)
$\hbar\omega_0$	1	(0,0)

(c) The level with energy $N\hbar\omega_0$ has N different possible sets of quantum numbers n_x, n_y. Both n_x and n_y range from 0 to N–1 but with their sum fixed to N. The number of possible values of n_x is then N (the values are 0, 1, 2, …, N-2, N-1), and for each value of n_x the value of n_y is fixed. The total degeneracy of each level is thus $N = n_x + n_y + 1$.

43. (a) With $\Delta x = \sqrt{(x^2)_{\text{av}} - (x_{\text{av}})^2}$, clearly $x_{\text{av}} = 0$ for this wave function. Then

$$(x^2)_{\text{av}} = \int_{-\infty}^{+\infty} x^2\,|\psi(x)|^2\,dx = 2b^{-1}\int_0^{+\infty} x^2 e^{-2x/b}dx = 2b^{-1}\frac{2}{(2/b)^3} = \frac{b^2}{2}$$

So $\Delta x = b/\sqrt{2} = 0.71b$.

(b) The maximum probability density occurs at $x = 0$, where $P(x) = |\psi(x)|^2 = b^{-1}$. We now find the location where $P(x)$ drops to half that value, that is, where $e^{-2|x|/b} = 0.5$, or $-2|x|/b = \ln(0.5)$:

$$|x| = -(b/2)\ln(0.5) \qquad \text{or} \qquad x = \pm 0.347b$$

Our estimate for Δx is then the distance between the two points where the probability is half its maximum value, so $\Delta x = 0.69b$, which agrees very well with the result of the more rigorous calculation from part (a).