30. $$p=\frac{mv}{\sqrt{1-v^2/c^2}}=\frac{1}{c}\frac{(mc^2)(v/c)}{\sqrt{1-v^2/c^2}}=\frac{1}{c}\frac{(938.3\text{ MeV})(0.835)}{\sqrt{1-(0.835)^2}}=1424\text{ MeV}/c$$

$$K = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 = \frac{938.3\ \text{MeV}}{\sqrt{1-(0.835)^2}} - 938.3\ \text{MeV} = 767\ \text{MeV}$$

$$E = K + mc^2 = 767\ \text{MeV} + 938.3\ \text{MeV} = 1705\ \text{MeV}$$

31. $E = K + mc^2 = 0.923\ \text{MeV} + 0.511\ \text{MeV} = 1.434\ \text{MeV}$

Solving Equation 2.36 for v, we obtain

$$v = c\sqrt{1-\left(\frac{mc^2}{E}\right)^2} = c\sqrt{1-\left(\frac{0.511\ \text{MeV}}{1.434\ \text{MeV}}\right)^2} = 0.934c$$

33. For what range of velocities is $K - \frac{1}{2}mv^2 \le 0.01K$? At the upper limit of this range, where $K - \frac{1}{2}mv^2 = 0.01K$, we have

$$0.99K = 0.99\left(\frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2\right) = \tfrac{1}{2}mv^2$$

With $x = v^2/c^2$, $0.99\left(\frac{1}{\sqrt{1-x}} - 1\right) = \frac{1}{2}x$ which gives $\frac{1}{1-x} = \left(1 + \frac{0.5}{0.99}x\right)^2$

$$1 = (1-x)(1 + 1.0101x + 0.2551x^2) \qquad \text{or} \qquad 0.2551x^2 + 0.7550x - 0.0101 = 0$$

Solving using the quadratic formula, we find x = 0.0133 or -2.97. Only the positive solution is physically meaningful, so

$$v = \sqrt{0.0133}\,c = 0.115c$$

That is, for speeds smaller than 0.115c, the classical kinetic energy is accurate to within 1%. For a different approach to that same type of calculation, see Problem 36.

34. As in Problem 33, let us now find the *lower* limit on the momentum such that

$$\sqrt{(pc)^2+(mc^2)^2}-pc \le 0.01\sqrt{(pc)^2+(mc^2)^2}$$

From the lower limit, we obtain $0.99\sqrt{(pc)^2+(mc^2)^2} = pc$, which can be written as

$$(pc)^2 = \frac{m^2c^4}{1/(0.99)^2-1} \qquad \text{or} \qquad pc = 7.02mc^2$$

With $mvc/\sqrt{1-v^2/c^2} = 7.02mc^2$, we obtain

$$\frac{v^2}{c^2} = 49.25\left(1-\frac{v^2}{c^2}\right) \qquad \text{or} \qquad v/c = 0.990$$

Whenever $v/c \ge 0.990$, the expression $E = pc$ will be accurate to within 1%.

38.

$$K_{\mathrm{f}} - K_{\mathrm{i}} = (E_{\mathrm{f}} - mc^2) - (E_{\mathrm{i}} - mc^2) = E_{\mathrm{f}} - E_{\mathrm{i}}$$

$$= \frac{mc^2}{\sqrt{1 - v_{\mathrm{f}}^2 / c^2}} - \frac{mc^2}{\sqrt{1 - v_{\mathrm{i}}^2 / c^2}} = \frac{0.511\,\mathrm{MeV}}{\sqrt{1 - (0.91)^2}} - \frac{0.511\,\mathrm{MeV}}{\sqrt{1 - (0.85)^2}} = 0.262\ \mathrm{MeV}$$

41. Because the electrons and the protons have charges of the same magnitude e, after acceleration through a potential difference of magnitude ΔV = 12.0 million volts (a positive difference for the electron, a negative difference for the proton), each loses potential energy of $\Delta U = -e\Delta V$ = -12.0 MeV and thus each acquires a kinetic energy of K = +12.0 MeV. For the electron, $E = K + mc^2$ = 12.0 MeV + 0.511 MeV = 12.5 MeV. The momentum is then

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(12.5\,\text{MeV})^2 - (0.511\,\text{MeV})^2} = 12.5\,\text{MeV}/c$$

The classical formula $K = p^2/2m$ gives

$$p = \sqrt{2mK} = \sqrt{2(0.511\,\text{MeV}/c^2)(12.0\,\text{MeV})} = 3.50\,\text{MeV}/c$$

which is far from the correct result (a discrepancy we would expect for such highly relativistic electrons). For the protons, $E = K + mc^2$ = 12.0 MeV + 938.3 MeV= 950.3 MeV, and the momentum is

$$p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(950.3\,\text{MeV})^2 - (938.3\,\text{MeV})^2} = 150.5\,\text{MeV}/c$$

The classical formula gives

$$p = \sqrt{2mK} = \sqrt{2(938.3\,\text{MeV}/c^2)(12.0\,\text{MeV})} = 150.1\,\text{MeV}/c$$

The difference between the classical and relativistic formulas appears only in the fourth significant figure.

42. The mass of a uranium atom is about (235 u)(1.66×10^{-27} kg/u) = 3.90×10^{-25} kg, so 1.50 kg contains 1.50 kg/3.90×10^{-25} kg = 3.84×10^{24} atoms. The total energy released is

$$\Delta E = (210\text{ MeV/atom})(3.84\times10^{24}\text{ atoms}) = 8.06\times10^{26}\text{ MeV}$$

and the change in mass is

$$\Delta m = \frac{\Delta E}{c^2} = \frac{(8.06\times10^{26}\text{ MeV})(1.602\times10^{-13}\text{ J/MeV})}{(2.998\times10^{8}\text{ MeV})^2} = 1.44\times10^{-3}\text{ kg}$$

About one gram of matter vanishes for each kilogram that fissions!

44. (a)

$$E = E_\pi + E_p = \frac{m_\pi c^2}{\sqrt{1 - v^2/c^2}} + m_p c^2 = \frac{139.6\text{ MeV}}{\sqrt{1-(0.921)^2}} + 938.3\text{ MeV} = 1296.7\text{ MeV}$$

(b)

$$p = p_\pi + p_p = \frac{m_\pi v}{\sqrt{1-v^2/c^2}} = \frac{1}{c}\frac{m_\pi c^2 (v/c)}{\sqrt{1-v^2/c^2}} = \frac{1}{c}\frac{(139.6\text{ MeV})(0.921)}{\sqrt{1-(0.921)^2}} = 330.0\text{ MeV}/c$$

(c)

$$mc^2 = \sqrt{E^2 - (pc)^2} = \sqrt{(1296.7\text{ MeV})^2 - (330.0\text{ MeV})^2} = 1254\text{ MeV}$$

45. Before the collision, the total relativistic energy of each electron is

$$E_e = \frac{m_e c^2}{\sqrt{1-v^2/c^2}} = \frac{0.511\text{ MeV}}{\sqrt{1-(0.99999)^2}} = 114.3\text{ MeV}$$

The total energy in the collision is therefore 2×114.3 MeV = 228.6 MeV. The total momentum is zero before the collision, because the two particles moves with equal and opposite velocities and have equal masses. After the collision, the total momentum is still zero, so we know that the two muons must move with equal speeds and thus have equal energies. The total energy of each muon is then 114.3 MeV and its kinetic energy is

$$K_\mu = E_\mu - m_\mu c^2 = 114.3\text{ MeV} - 105.7\text{ MeV} = 8.6\text{ MeV}$$

47. For particle 1, moving in the positive x direction,

$$E_1 = K_1 + mc^2 = 282\text{ MeV} + 140\text{ MeV} = 422\text{ MeV}$$
$$cp_1 = \sqrt{E_1^2 - (mc^2)^2} = \sqrt{(422\text{ MeV})^2 - (140\text{ MeV})^2} = +398\text{ MeV}$$

For particle 2, moving in the negative x direction,

$$E_2 = K_2 + mc^2 = 25\text{ MeV} + 140\text{ MeV} = 165\text{ MeV}$$
$$cp_2 = -\sqrt{E_2^2 - (mc^2)^2} = -\sqrt{(165\text{ MeV})^2 - (140\text{ MeV})^2} = -87\text{ MeV}$$

The net final momentum is $p_\mathrm{f} = p_1 + p_2 = 398\text{ MeV}/c - 87\text{ MeV}/c = 311\text{ MeV}/c$, and the net final energy is $E_\mathrm{f} = E_1 + E_2 = 422\text{ MeV} + 165\text{ MeV} = 587\text{ MeV}$. Because of the conservation laws, these must be equal to the momentum and the energy of the initial particle, so that its rest energy is then

$$m_\mathrm{i}c^2 = \sqrt{E_\mathrm{i}^2 - (cp_\mathrm{i})^2} = \sqrt{(587\text{ MeV})^2 - (311\text{ MeV})^2} = 498\text{ MeV}$$

Solving Equation 2.36 for v, we obtain

$$v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = c\sqrt{1 - \left(\frac{498\text{ MeV}}{587\text{ MeV}}\right)^2} = 0.529c$$

59. (a) Before the first acceleration, $E = E_0 = mc^2$. After the acceleration, the energy is

$$E_1 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-(0.99)^2}} = 3.6\text{ MeV}$$

The change in energy is $\Delta E = E_1 - E_0 = 3.6\text{ MeV} - 0.5\text{ MeV} = 3.1\,\text{MeV}$, so the first stage adds 3.1 MeV to the energy of the electron.

(b) $$E_2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-(0.999)^2}} = 11.4\text{ MeV}$$

The change in energy is $\Delta E = E_2 - E_1 = 11.4\text{ MeV} - 3.6\text{ MeV} = 7.8\text{ MeV}$, so the second stage adds about 2.5 times as much energy as the first stage, even though the second stage increases the velocity by only 0.9%.

62. The initial energy is

$$E_\pi = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{135\ \text{MeV}}{\sqrt{1-(0.98)^2}} = 678\ \text{MeV}$$

and the momentum is

$$p_\pi = \frac{1}{c}\sqrt{E_\pi^2 - (mc^2)^2} = \frac{1}{c}\sqrt{(678\text{ MeV})^2 - (135\text{ MeV})^2} = 664\text{ MeV}/c$$

Because the two gamma ray photons have equal energies, each has an energy of $\frac{1}{2}(678\text{ MeV})$, so $E_\gamma = 339$ MeV. Each gamma ray photon has a momentum of $p_\gamma = E_\gamma / c = 339\text{ MeV}/c$, which has a component $p_\gamma \cos\theta$ along the direction of the initial π meson. Conservation of momentum then gives $p_\pi = 2p_\gamma \cos\theta$, so the angle is

$$\theta = \cos^{-1}\frac{p_\pi}{2p_\gamma} = \cos^{-1}\frac{664\text{ MeV}/c}{2(339\text{ MeV}/c)} = 11.7°$$

64. (a) We use subscripts – and + to represent respectively the electron (e^-) and positron (e^+). Then before the collision the momenta are

$$p_- = \frac{mv_-}{\sqrt{1-v_-^2/c^2}} = \frac{1}{c}\frac{mc^2(v_-/c)}{\sqrt{1-v_-^2/c^2}} = \frac{1}{c}\frac{(0.511\,\text{MeV})(0.834)}{\sqrt{1-(0.834)^2}} = 0.772\ \text{MeV}/c$$

$$p+ = \frac{mv_+}{\sqrt{1-v_+^2/c^2}} = \frac{1}{c}\frac{mc^2(v_+/c)}{\sqrt{1-v_+^2/c^2}} = \frac{1}{c}\frac{(0.511\,\text{MeV})(-0.428)}{\sqrt{1-(0.428)^2}} = -0.242\ \text{MeV}/c$$

The total momentum before the collision is

$$p = p_- + p_+ = 0.772\ \text{MeV}/c - 0.242\ \text{MeV}/c = 0.530\ \text{MeV}/c\,.$$

By conservation of momentum, this must also be the momentum of the new particle after the collision. The total energies before the collision are

$$E_- = \frac{mc^2}{\sqrt{1-v_-^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-(0.834)^2}} = 0.926\ \text{MeV}$$

$$E_+ = \frac{mc^2}{\sqrt{1-v_+^2/c^2}} = \frac{0.511\,\text{MeV}}{\sqrt{1-(0.428)^2}} = 0.565\ \text{MeV}$$

The total energy before the collision is

$$E = E_- + E_+ = 0.926\ \text{MeV} + 0.565\ \text{MeV} = 1.492\ \text{MeV}\,.$$

By conservation of energy, this must be the same as the total energy of the new particle after the collision.
(b) We can find the mass of the new particle from its momentum and total energy using Eq. 2.39:

$$M = c^{-2}\sqrt{E^2-(pc)^2} = c^{-2}\sqrt{(1.492\ \text{MeV})^2-(0.530\ \text{MeV})^2} = 1.394\ \text{MeV}/c^2$$

(c) The initial and final kinetic energies are

$$K_{\mathrm{i}} = (E_{-} - mc^2) + (E_{+} - mc^2) = 1.492\ \mathrm{MeV} - 2(0.511\ \mathrm{MeV}) = 0.470\ \mathrm{MeV}$$
$$K_{\mathrm{f}} = E - Mc^2 = 1.492\ \mathrm{MeV} - 1.394\ \mathrm{MeV} = 0.097\ \mathrm{MeV}$$

The change in kinetic energy is

$$\Delta K = K_{\mathrm{i}} - K_{\mathrm{f}} = 0.470\ \mathrm{MeV} - 0.097\ \mathrm{MeV} = 0.372\ \mathrm{MeV}$$

The change in mass is

$$\Delta m = M - 2m = 1.394\ \mathrm{MeV}/c^2 - 2(0.511\ \mathrm{MeV}/c^2) = 0.372\ \mathrm{MeV}/c^2$$

The additional mass of the new particle comes from the loss in kinetic energy in the collision.

(d) The momentum and energy of the original particles and the new particle would have different values in the new frame, but the values for M, Δm, and ΔK would be the same. Mass is an invariant in special relativity (all observers measure the same value) and since the mass value comes from the change in kinetic energy, all observers must also find the same ΔK.