PHYSICS 200B : CLASSICAL MECHANICS
SOLUTION SET #4

[**1**] Blasius' theorem says that the force per unit length of a body of constant cross-sectional profile Σ is given by

$$\bar{\mathcal{F}} = \mathcal{F}_x - i\mathcal{F}_y = \tfrac{i}{2}\,\rho \oint\limits_{\mathcal{C}} dz \left(\frac{dW}{dz}\right)^2 \quad ,$$

where $\mathcal{C} = \partial\Sigma$ is a closed curve which traces the boundary of Σ, and $W(z)$ is the complex potential.

Consider a 2D flow with stream function $\psi(x,y) = A(x-c)y$, where A and c are real constants. A circular cylinder of radius a is introduced into this flow, with its center at the origin. Find $W(z)$ for the resulting flow. Use Blasius' theorem to calculate the force per unit length exerted on the cylinder.

We first find the conjugate harmonic function $\phi(x,y)$ satisfying

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = A(x-c) \quad , \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = -Ay \quad .$$

We conclude

$$\phi(x,y) = \tfrac{1}{2}Ax^2 - Acx - \tfrac{1}{2}Ay^2 \quad ,$$

and thus

$$w(z) = \phi(x,y) + i\psi(x,y) = \tfrac{1}{2}Az^2 - Acz \quad .$$

Now we introduce a cylinder of radius a. The boundary of the circle must be a streamline, but as $|z| \to \infty$ we have $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ where $\phi(x,y)$ is given above. To make this so, we invert $w(z)$ in the circle $|z| = a$ and write

$$\begin{aligned} W(z) &= w(z) + \overline{w(a^2/\bar{z})} \\ &= \tfrac{1}{2}Az^2 - Acz + \frac{Aa^2}{2z^2} - \frac{Aca^2}{z} \quad . \end{aligned}$$

Using Cauchy's theorem, we then find

$$\begin{aligned} \bar{\mathcal{F}} &= \tfrac{i}{2}\,\rho \oint\limits_{\mathcal{C}} dz \left(\frac{dW}{dz}\right)^2 \\ &= \tfrac{i}{2}\,\rho \oint\limits_{|z|=a} dz \left(Az - Ac + \frac{Aca^2}{z^2} - \frac{Aa^2}{2z^3}\right)^2 \\ &= \tfrac{i}{2}\rho \cdot 2\pi i \cdot 2A^2ca^2 = -2\pi\rho A^2ca^2 \quad . \end{aligned}$$

Thus, $\mathcal{F}_x = -2\pi\rho A^2ca^2$ and $\mathcal{F}_y = 0$.

[**2**] Show that the Joukowski transformation $Z = z + a^2/z$ can be written in the form

$$\frac{Z-2a}{Z+2a} = \left(\frac{z-a}{z+a}\right)^2 \quad ,$$

so that

$$\text{arg}(Z-2a) - \text{arg}(Z+2a) = 2\Big\{\text{arg}(z-a) - \text{arg}(z+a)\Big\} \quad . \tag{1}$$

Consider the circle in the (x, y) plane which passes through $z = -a$ and a with its center at $z_0 = ia\,\text{ctn}\,\beta$. Show that the above transformation takes this circle into a circular arc between $Z = -2a$ and $Z = +2a$, with subtended angle 2β (see figure). Obtain an expression for the complex potential in the Z plane when the flow is uniform at speed V and parallel to the real axis. Show that the velocity will be finite at both the leading and tailing edges if $\Gamma - -4\pi Va\,\text{ctn}\,\beta$.

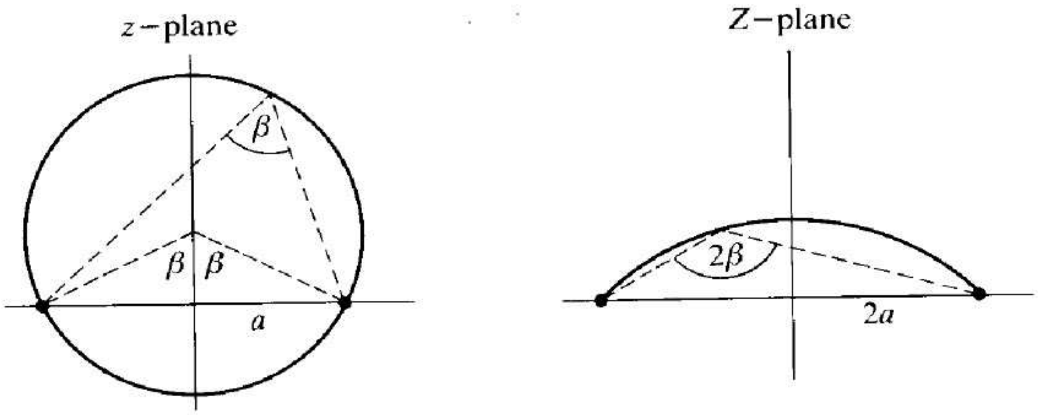

Figure 1: Geometry of the circle and its image in problem 2.

We have

$$\frac{Z-2a}{Z+2a} = \frac{z+a^2z^{-1}-2a}{z+a^2z^{-1}+2a} = \frac{(z-a)^2/z}{(z+a)^2/z} = \frac{(z-a)^2}{(z+a)^2} \quad .$$

Taking the argument and using $\text{arg}(z_1/z_2) = \text{arg}(z_1) - \text{arg}(z_2)$ the desired result follows immediately.

Next let $z_0 = ia\,\text{ctn}\,\beta$. The radius of the circle in the z-plane is b, where

$$b^2 = a^2 + (a\,\text{ctn}\,\beta)^2 = a^2\csc^2\beta \quad ,$$

so $b = a/\sin\beta$. The locus of points on this circle may be written as $z(\theta) = z_0 - ib\,e^{i\theta}$, where $\theta \in [0, 2\pi)$. Thus,

$$z \pm a = \left(e^{\mp i\beta} - e^{i\theta}\right) \cdot ia\csc\beta \quad ,$$

and we have

$$\frac{Z+2a}{Z-2a} = \left(\frac{e^{-i\beta} - e^{i\theta}}{e^{i\beta} - e^{i\theta}}\right)^2 \quad .$$

Now

$$\frac{e^{-i\beta}-e^{i\theta}}{e^{i\beta}-e^{i\theta}} = -e^{-i\beta}\cdot\frac{e^{i(\beta+\theta)}-1}{e^{i\beta}-e^{i\theta}}$$
$$= -e^{-i\beta}\cdot\frac{e^{i(\beta+\theta)/2}-e^{-i(\beta+\theta)/2}}{e^{i(\beta-\theta)/2}-e^{-i(\beta-\theta)/2}}\cdot\frac{e^{i(\beta+\theta)/2}}{e^{i(\beta+\theta)/2}}$$
$$= -e^{-i\beta}\cdot\frac{\sin\left[\frac{1}{2}(\beta+\theta)\right]}{\sin\left[\frac{1}{2}(\beta-\theta)\right]} \quad .$$

Thus,

$$\mathsf{arg}(Z+2a)-\mathsf{arg}(Z-2a)=2\pi-2\beta \quad ,$$

which says that the circle in the z-plane maps to an arc in the Z-plane as shown in fig. 1.

Now consider the complex potential

$$W(z)=V(z-z_0)+\frac{Vb^2}{z-z_0}+\frac{\Gamma}{2\pi i}\log(z-z_0) \quad ,$$

corresponding to uniform flow at infinity with a streamline along $|z|=b$. Then the complex potential in the Z-plane is $\mathcal{W}(Z)=W(F(Z))$ where

$$F(Z)=z=\tfrac{1}{2}\Big(Z\pm\sqrt{Z^2-4a^2}\Big) \quad .$$

Thus the complex velocity

$$\overline{\mathcal{V}}(Z)=\mathcal{W}'(Z)=W'(z)\,F'(Z)$$

Consider the case $Z=2a$, corresponding to $z=a$. Since

$$F'(Z)=\frac{1}{2}\pm\frac{Z}{2\sqrt{Z^2-4a^2}}=\pm\frac{z}{\sqrt{Z^2-4a^2}} \quad ,$$

we have that $F(Z)$ diverges with an inverse square root singularity as Z approaches $\pm 2a$. We now show that $W'(z)$ vanishes when $Z=\pm 2a$, cancelling the singularity, provided $\Gamma=-4\pi Va\,\mathrm{ctn}\,\beta$. In this case,

$$W'(z=a)=V\left\{1-\frac{b^2/a^2}{(1-i\,\mathrm{ctn}\,\beta)^2}+\frac{\Gamma/aV}{2\pi i}\,\frac{1}{1-i\,\mathrm{ctn}\,\beta}\right\}$$
$$=V\left\{1-\frac{1}{(\sin\beta-i\cos\beta)^2}+\frac{2i\cos\beta}{\sin\beta-i\cos\beta}\right\}$$
$$=V\left\{1+e^{-2i\beta}-2\cos\beta e^{-i\beta}\right\}=0 \quad ,$$

which vanishes! To find out the value of the velocity at the leading and trailing edges, set $z=a+\delta z$. An intelligent parameterization here is to take $\delta z=-i\epsilon a\csc\beta\,e^{i\beta}$ and see what happens for complex ϵ. We then have

$$z-z_0=z-ia\,\mathrm{ctn}\,\beta=a-ia\,\mathrm{ctn}\,\beta-i\epsilon a\csc\beta\,e^{i\beta}$$
$$=-\frac{ia}{\sin\beta}\,(1+\epsilon)\,e^{i\beta} \quad .$$

Then

$$W'(z) = V\left\{1 + \frac{e^{-2i\beta}}{(1+\epsilon)^2} - \frac{2\cos\beta\, e^{-i\beta}}{1+\epsilon}\right\}$$
$$= V\left\{1 - \frac{e^{-2i\beta}}{1+\epsilon}\right\} \cdot \frac{\epsilon}{1+\epsilon} = 2iV\sin\beta\, e^{-i\beta}\,\epsilon + \mathcal{O}(\epsilon^2) \quad .$$

Next, we have $F'(Z) = z/\sqrt{Z^2 - 4a^2}$. We write $Z^2 - 4a^2 = (Z+2a)(Z-2a)$. For $Z \approx 2a$ we may write $Z + 2a = 4a + \mathcal{O}(\epsilon)$, and

$$Z - 2a = z + \frac{a^2}{z} - 2a = \frac{(z-a)^2}{z} = a\left(-i\epsilon\csc\beta\, e^{i\beta}\right)^2 \quad .$$

Thus,

$$F'(Z) = \frac{z}{\sqrt{Z+2a}} \cdot \frac{1}{\sqrt{Z-2a}} = \frac{a}{2\sqrt{a}} \cdot \frac{1}{-i\epsilon\sqrt{a}\,\csc\beta\, e^{i\beta}} = \frac{i\, e^{-i\beta}}{2\,\epsilon\csc\beta} \quad .$$

Thus we see that $W'(z)$ vanishes as ϵ^1 and $F'(Z)$ diverges as ϵ^{-1}. Multiplying and taking the limit $\epsilon \to 0$, we obtain the complex velocity at the edge $Z = 2a$ to be

$$\overline{\mathcal{V}} = -V\sin^2\!\beta\, e^{-2i\beta} \quad .$$

[**3**] Show that an array of N identical point vortices of circulation Γ, placed equally about a circle of radius a, will rotate at a constant angular frequency Ω. Find the value of Ω.

Let $\omega = e^{2\pi i/N}$. The locations of the vortices are taken to be $z_n = a\omega^n$ where $n \in \{1, N\}$; note that $z_{n+N} = z_n$. The complex potential for a vortex located at the origin is $W(z) = (\Gamma/2\pi i)\log z$, and the corresponding complex velocity field is $\bar{v}(z) = \Gamma/2\pi i z$. The complex velocity of the j^{th} vortex is a sum of contributions for all the others and is given by

$$\bar{v}_j = \frac{\Gamma\bar{\omega}^j}{2\pi i a}\sum_{n=1}^{N-1}\frac{1}{1-\omega^n} \quad .$$

Suppose N is odd. Then we pair the terms in the above sum: n with $N - n$. Note that

$$\frac{1}{1-\omega^n} + \frac{1}{1-\omega^{N-n}} = \frac{1}{1-\omega^n} + \frac{\omega^n}{\omega^n - 1} = 1 \quad ,$$

since $\omega^N = 1$. There are $(N-1)/2$ such pairs, so we conclude that

$$\bar{v}_j = \frac{N-1}{4\pi i a}\,\Gamma\bar{\omega}^j \quad .$$

When N is even, we again pair n with $N - n$. The value $n = N/2$ is its own mate, and there are $(N-2)/2$ bona fide pairs. Thus,

$$\sum_{n=1}^{N-1}\frac{1}{1-\omega^n} = \frac{N-2}{2} + \frac{1}{1-\omega^{N/2}} = \frac{N-1}{2} \quad ,$$

since $\omega^{N/2} = -1$. Thus once again we have $\bar{v}_j = (N-1)\Gamma\omega^j/4\pi i a$. Note that uniform rotation in the (x, y) plane about the origin with angular frequency Ω means

$$\boldsymbol{v}(\boldsymbol{r}) = \Omega\,\hat{\boldsymbol{z}} \times \boldsymbol{r} = \Omega\,(x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}) \quad ,$$

and thus the complex velocity is $\bar{v} = \Omega\,(-y - ix) = -i\Omega\bar{z}$. For the j^{th} vortex, $z_j = a\bar{\omega}^j$. Thus, we conclude $\Omega = (N-1)\Gamma/4\pi a^2$.

[4] Consider a large circular disk of radius R executing a prescribed angular motion $\theta(t)$. The disk is immersed in a fluid under conditions of constant pressure. Let the plane of the disk lie at $z = 0$. Assume that the fluid velocity takes the form

$$v_\phi(r, \phi, z, t) = r\,\Omega(z, t) \ , \tag{2}$$

with $v_r = v_z = 0$.

(a) Write down the Navier-Stokes equations for the fluid. Assume you can neglect the $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v}$ term. (Under what conditions is this true?) Show that you obtain the diffusion equation. What are the boundary conditions on the fluid motion?

The Navier-Stokes equations are

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v} = -\boldsymbol{\nabla} p + \nu\,\nabla^2 \boldsymbol{v} \ . \tag{3}$$

If we neglect the nonlinear term, we have the diffusion equation,

$$\frac{\partial \Omega}{\partial t} = \nu\,\frac{\partial^2 \Omega}{\partial z^2} \ . \tag{4}$$

In deriving this, it is useful to write

$$\boldsymbol{v} = r\,\Omega(z, t)\,\hat{\boldsymbol{\phi}} = (x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}})\,\Omega(z, t) \ . \tag{5}$$

The nonlinear term is

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v} = -\frac{\hat{\boldsymbol{r}}}{r}\,v_\phi^2 = -\Omega^2\,\boldsymbol{r} \ . \tag{6}$$

This may be neglected if

$$\big|\Omega\big| \ll \frac{\nu}{R^2} \ , \tag{7}$$

which is equivalent to $\text{Re} \ll 1$, where the Reynolds number is $\text{Re} = R\,v_\phi/\nu$.

(b) Our goal is next to find a complete solution to $\Omega(z, t)$ in terms of the function $\theta(t)$. To this end, we perform the following analysis. Define the spatial Laplace transform,

$$\check{\Omega}_{\mathsf{L}}(\kappa, t) \equiv \int_0^\infty \! dz\,e^{-\kappa z}\,\Omega(z, t) \ . \tag{8}$$

You may assume in this problem that the fluid motion is symmetric about $z = 0$, *i.e.* $\Omega(z,t) = \Omega(-z,t)$, so we only have to consider the region $z \geq 0$. The inverse Laplace transform is

$$\Omega(z,t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{d\kappa}{2\pi i}\, e^{+\kappa z}\, \check{\Omega}_{\mathsf{L}}(\kappa,t) \tag{9}$$

where the contour lies to the left of any branch cut or singularity on the line $\mathrm{Im}\,(\kappa) = 0$. Later on we will see that we can take $c = 0$, so the contour lies along the axis $\mathrm{Re}\,(\kappa) = 0$. Show directly that

$$\left(\partial_t - \nu\kappa^2\right) \check{\Omega}_{\mathsf{L}}(\kappa,t) = F_\kappa(t) \ , \tag{10}$$

where the function $F_\kappa(t)$ on the RHS depends on $\Omega(0,t)$ and $\Omega'(0,t)$ (prime denotes differentiation with respect to z). Find $F_\kappa(t)$.

We have that

$$\begin{aligned} 0 &= \int\limits_0^\infty \!dz\, e^{-\kappa z} \left\{ \frac{\partial\Omega}{\partial t} - \nu\, \frac{\partial^2\Omega}{\partial z^2} \right\} \\ &= \left(\partial_t - \nu\,\kappa^2\right) \check{\Omega}_{\mathsf{L}}(\kappa,t) + \nu\big[\Omega'(0,t) + \kappa\,\Omega(0,t)\big] \ . \end{aligned} \tag{11}$$

Thus,

$$\left(\partial_t - \nu\,\kappa^2\right) \check{\Omega}_{\mathsf{L}}(\kappa,t) = -\nu\big[\Omega'(0,t) + \kappa\,\Omega(0,t)\big] \ . \tag{12}$$

(c) Integrate the above first order equation from some arbitrary initial time $t = t_0$ to final time t and obtain $\Omega(z,t)$ in terms of the functions $\Omega(z,t_0)$, $\Omega(0,t)$, and $\Omega'(0,t)$. Show that the term involving $\Omega(z,t_0)$ is a transient which decays to zero in the limit $t_0 \to -\infty$. Dropping the transient, performing the inverse Laplace transform, and rotating the κ contour so that $\kappa = ik$, where k runs along the real axis, show that

$$\Omega(z,t) = -\nu \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikz} \int\limits_{-\infty}^{t} \!dt'\, e^{-\nu k^2(t-t')} \Big[\Omega'(0,t') + ik\Omega(0,t')\Big] \ . \tag{13}$$

Integrating, we obtain

$$\check{\Omega}_{\mathsf{L}}(\kappa,t) = e^{\nu\kappa^2(t-t_0)}\, \check{\Omega}_{\mathsf{L}}(\kappa,t_0) - \nu \int\limits_{t_0}^{t} \!dt'\, e^{\nu\kappa^2(t-t')} \Big[\Omega'(0,t') + ik\Omega(0,t')\Big] \ . \tag{14}$$

The first term is a transient which is negligible in the limit $t_0 \to -\infty$. Remember that κ is purely imaginary along its integration contour, so we can set $\kappa \equiv ik$ with k real. Applying the inverse Laplace transform, we recover the desired result.

(d) Find the total torque on the disk $N(t)$. You will need to integrate $\boldsymbol{r} \times \boldsymbol{f}$ over the surface of the disk, using the viscous stress tensor of the fluid. Show that

$$N_{\text{fluid}}(t) = \pi\eta\, R^4\, \Omega'(0,t) \ , \tag{15}$$

where $\eta = \rho\,\nu$ is the shear viscosity.

The viscous force per unit surface area is $f_i = \tilde{\sigma}_{ij}\,n_j$, where n_j is the surface normal and

$$\tilde{\sigma}_{ij} = \eta\bigg(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \tfrac{2}{3}\,\delta_{ij}\,\boldsymbol{\nabla}\cdot\boldsymbol{v}\bigg) + \zeta\,\delta_{ij}\,\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{16}$$

is the viscous stress tensor. For the flow $\boldsymbol{v} = r\,\Omega(z,t)\,\hat{\boldsymbol{\phi}}$, the divergence vanishes. The differential viscous torque $d\boldsymbol{N} = dN\,\hat{\boldsymbol{z}}$ on the disk is then

$$\begin{aligned} dN &= \big(x f_y - y f_x\big)\,dA \\ &= \eta\bigg(x\,\frac{\partial v_y}{\partial z} - y\,\frac{\partial v_x}{\partial z}\bigg)\,dA = \eta\,r\,\frac{\partial\Omega}{\partial z}\,dA\ . \end{aligned} \tag{17}$$

Integrating, we find the total viscous torque:

$$N = 2\int\limits_0^R\!dr\,2\pi r\,r\,\eta\,\frac{\partial v_\phi}{\partial z} = \pi\eta R^4\,\Omega'(0,t)\ . \tag{18}$$

Note the factor of two, which arises from integration over both sides of the disk.

(e) By going to Fourier space in frequency, the k integral can be done. Show that

$$\hat{\Omega}(z,\omega) = -\frac{i\,e^{ik_+ z}}{k_+ - k_-}\,\Big\{\hat{\Omega}'(0,\omega) + ik_+\hat{\Omega}(0,\omega)\Big\}\ , \tag{19}$$

where $k_\pm = \pm\,e^{i\pi/4}\,\sqrt{\omega/\nu}$. Thus, setting $z \to 0^+$, we obtain

$$\hat{\Omega}'(0,\omega) = -ik_-\hat{\Omega}(0,\omega)\ . \tag{20}$$

Taking the Fourier transform, we have

$$\begin{aligned} \hat{\boldsymbol{\Omega}}(z,\omega) &= -\nu\int\limits_{-\infty}^{\infty}\!dt\,e^{i\omega t}\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,e^{ikz}\int\limits_{-\infty}^{t}\!dt'\,e^{-\nu k^2(t-t')}\,\Big[\Omega'(0,t') + ik\Omega(0,t')\Big] \\ &= -\nu\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,e^{ikz}\int\limits_0^{\infty}\!ds\,e^{-\nu k^2 s}\,e^{i\omega s}\int\limits_{-\infty}^{\infty}\!dt\,e^{i\omega(t-s)}\,\Big[\Omega'(0,t-s) + ik\Omega(0,t-s)\Big] \\ &= -\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,\frac{e^{ikz}}{k^2 - \frac{i\omega}{\nu}}\,\big[\hat{\boldsymbol{\Omega}}'(0,\omega) + ik\hat{\boldsymbol{\Omega}}(0,\omega)\big] = \frac{-i\,e^{ik_+ z}}{k_+ - k_-}\,\Big[\hat{\boldsymbol{\Omega}}'(0,\omega) + ik_+\,\hat{\boldsymbol{\Omega}}(0,\omega)\Big]\ , \end{aligned} \tag{21}$$

where we assume $z > 0$ in the last line. There is a subtlety here which is worth mentioning. In the above derivation, we have assumed ω is real and positive. For general ω, the roots are $k = \pm\sqrt{i\omega/\nu}$ and we define k_+ to be the root with the positive imaginary part.

(f) Suppose the disk is suspended from a torsional fiber. Let the disk's moment of inertia be I and the restoring torque due to the fiber be $N_{\text{fiber}} = -K\theta$. Show that the equation for the oscillation frequency of the disk is

$$\omega^2 + e^{i\pi/4}\,\omega_\nu^{1/2}\,\omega^{3/2} - \omega_0^2 = 0 \;, \tag{22}$$

where $\omega_0 = (K/I)^{1/2}$, and

$$\omega_\nu = \frac{\pi^2 \rho^2 R^8\,\nu}{I^2} \;. \tag{23}$$

Analyze this equation in the limits $\omega_0 \ll \omega_\nu$ and $\omega_0 \gg \omega_\nu$, and find the frequency of damped oscillations. *Hint:* The former case is easy – simply neglect the ω^2 term. For the latter case, perturb about the $\omega_\nu = 0$ solutions $\omega = \pm\omega_0$. Find the real and imaginary parts of the oscillation frequency ω in each case.

We have $\Omega(0,t) = \dot\theta(t)$, hence $\hat{\boldsymbol{\Omega}}(0,\omega) = -i\,\omega\,\hat\theta(\omega)$. Then

$$\begin{aligned}\hat{\boldsymbol{\Omega}}'(0,\omega) &= i\,e^{i\pi/4}\sqrt{\frac{\omega}{\nu}}\;\hat{\boldsymbol{\Omega}}(0,\omega)\\ &= e^{i\pi/4}\,\frac{\omega^{3/2}}{\nu^{1/2}}\,\hat\theta(\omega) \;.\end{aligned} \tag{24}$$

The Fourier transform of the torque is then

$$\hat N(\omega) = \pi\rho R^4 \cdot e^{i\pi/4}\,\nu^{1/2}\,\omega^{3/2}\,\hat\theta(\omega) \;. \tag{25}$$

Newton's second law for the disk is then

$$-I\omega^2\,\hat\theta(\omega) = -K\,\hat\theta(\omega) + \hat N(\omega) \;, \tag{26}$$

from which we obtain the desired result of eqn. **??**. To be perfectly correct, we should write this as

$$\omega^2 + e^{i\pi/4}\,\omega_\nu^{1/2}\,\omega^{3/2}\,\text{sgn}(\text{Re}\,\omega) - \omega_0^2 = 0 \;, \tag{27}$$

Suppose $\omega_0 = 0$. Then we have two solutions, $\omega = 0$ and $\omega = -i\omega_\nu$. For small ω_0, the latter will continue to be highly overdamped. The former solution becomes finite, and neglecting the $\mathcal{O}(\omega^2)$ term (since ω is small), we find

$$\omega = e^{-i\pi/6}\,\omega_0^{4/3}\,\omega_\nu^{-1/3} \;. \tag{28}$$

The damping rate is then $\gamma = -\text{Im}\,\omega = \frac{1}{2}\,\omega_0^{4/3}\,\omega_\nu^{-1/3}$.

In the opposite limit, where $\omega_\nu \ll \omega_0$, write $\omega = \omega_0 + \delta\omega$ and solve to first order in $\delta\omega$, obtaining

$$\delta\omega = -\tfrac{1}{2}\,e^{i\pi/4}\,\sqrt{\omega_0\,\omega_\nu} \;. \tag{29}$$

The viscous damping leads to a frequency shift and damping rate $-\Delta\omega = \gamma = \sqrt{\omega_0\,\omega_\nu/8}$. Note that $\Delta\omega < 0$, as is the case with a simple damped harmonic oscillator.

Note: There is an easier way to solve this problem, if we use some intuition. The diffusion equation $\Omega_t = \nu\Omega_{zz}$ and the boundary conditions are linear, which suggests we write our solution as

$$\Omega(z,t) = A(\omega)\, e^{-Q|z|}\, e^{-i\omega t} \ . \tag{30}$$

This is a solution to the diffusion equation if $\nu Q^2 = -i\omega$. Of the two roots for $Q(\omega)$, we need the one with the positive real part, so $Q = e^{-i\pi/4}\sqrt{\omega/\nu}$. Setting $z = 0$ and using $\dot{\Omega} = \theta$, we find $A(\omega) = -i\omega\,\hat{\theta}(\omega)$. The Fourier component of the viscous torque on the disk is then

$$\hat{N}_{\text{fluid}}(\omega) = \pi\rho\nu R^4 \cdot (-Q)(-i\omega)\,\hat{\theta}(\omega) \tag{31}$$

$$= e^{i\pi/4}\,\pi\rho R^4 \nu^{1/2}\omega^{3/2}\,\hat{\theta}(\omega) \ , \tag{32}$$

which when plugged into the equation of motion for the disk yields the above equation for the oscillation frequency.