PHYSICS 200B : CLASSICAL MECHANICS
SOLUTION SET #2

[1] Consider the standard map on the unit torus,

$$\begin{aligned} x_{n+1} &= x_n + y_n \ \mathsf{mod}\ 1 \\ y_{n+1} &= y_n + \kappa\, \sin(2\pi x_{n+1}) \ \mathsf{mod}\ 1 \quad . \end{aligned}$$

Find all the fixed points and identify their stability as a function of the control parameter κ.

Solution :

The Jacobian of the map is

$$M_n \equiv \frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} = \begin{pmatrix} 1 & 1 \\ 2\pi\kappa\cos(2\pi x_{n+1}) & 1 + 2\pi\kappa\cos(2\pi x_{n+1}) \end{pmatrix} \quad .$$

Note that $\mathsf{det}\, M_n = 1$. A fixed point (x^*, y^*) must satisfy

$$y^* \cong 0 \qquad , \qquad \kappa\sin(2\pi x^*) \cong 0 \quad ,$$

where $A \cong B$ means $A = B$ mod 1. Thus, fixed points on the unit torus are located at $x^* = \sin^{-1}(n/\kappa)/2\pi$ and $y^* = 0$, where $n \in \mathbb{Z}$ and $\kappa \geq |n|$. Thus $\kappa\cos(2\pi x^*) = \pm\sqrt{\kappa^2 - n^2}$. For a 2×2 matrix M, the characteristic polynomial is $P(\lambda) = \lambda^2 - T\lambda + D$, where $T = \mathsf{Tr}\, M$ and $D = \mathsf{det}\, M$. Since $D = 1$, we have $\lambda_\pm = \frac{1}{2}T \pm \frac{1}{2}\sqrt{T^2 - 4}$, with $T = 2 \pm 2\pi\sqrt{\kappa^2 - n^2}$. Stability requires $|T| < 2$ so that $\lambda_\pm = e^{\pm i\theta}$, with $\cos\theta = \frac{1}{2}T$. Thus, the solution with $\cos(2\pi x^*) > 0$ is always unstable. For $\cos(2\pi x^*) < 0$, we must have

$$T_- = 2 - 2\pi\sqrt{\kappa^2 - n^2} > -2 \qquad \Rightarrow \qquad n^2 < \kappa^2 < n^2 + \frac{4}{\pi^2} \quad .$$

[2] Write a computer program to iterate the map from problem [1]. For each value of κ you consider, iterate starting from N^2 initial conditions $(x_0, y_0) = (j/N, k/N)$, where j and k each run from 0 to $N - 1$. You can take $N = 10$.

(a) By experimenting, see if you can find the value of κ where there are no unbroken KAM tori which span the x-direction $x \in [0, 1]$.

(b) Next, consider the standard map on the cylinder,

$$\begin{aligned} x_{n+1} &= x_n + y_n \ \mathsf{mod}\ 1 \\ y_{n+1} &= y_n + \kappa\, \sin(2\pi x_{n+1}) \quad , \end{aligned}$$

where the y variable now may take values on the entire real line. For each given κ, plot $\langle y_n^2 \rangle$ *versus* n, where the average is over the N^2 initial conditions. Assuming the evolution is diffusive in the chaotic regime, compute the diffusion constant $D(\kappa)$ from the formula $\langle y_n^2 \rangle = 2Dn$. Plot $D(\kappa)$ *versus* κ over the range $\kappa \in [1, 10]$. Compare to the value from the quasilinear approximation, $D_{\text{ql}} = \frac{1}{4}\kappa^2$.

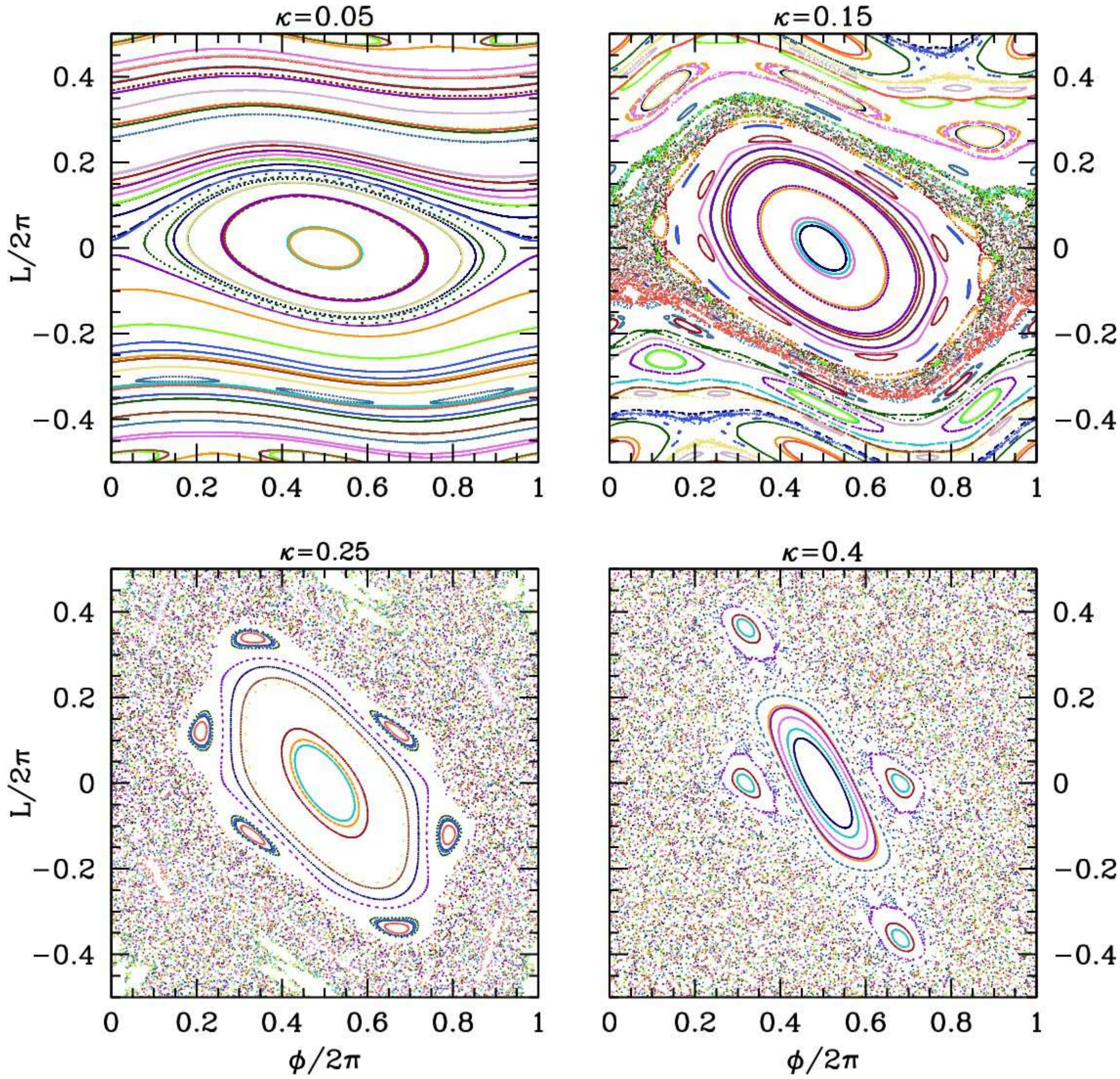

Figure 1: The standard map on the torus for different values of κ. The critical value where no unbroken KAM tori span the x-direction is found to be $\kappa \approx 0.16$.

Solution :

(a) The critical value of κ is found to be $\kappa_c \approx 0.16$. See fig. 1.

(b) See fig. 2 for the results on the cylinder, and figs. 3 and 4 for the diffusion constant results.

[**3**] For the logistic map $x_{n+1} = f(x_n)$ with $f(x) = rx(1-x)$, plot the functions $f^{(n)}(x)$ for $n = 1$, 2, and 4 and plot the intersections of $y = f^{(n)}(x)$ with $y = x$. Show how varying the control parameter r results in bifurcations corresponding to the appearance of 2-cycles and 4-cycles.

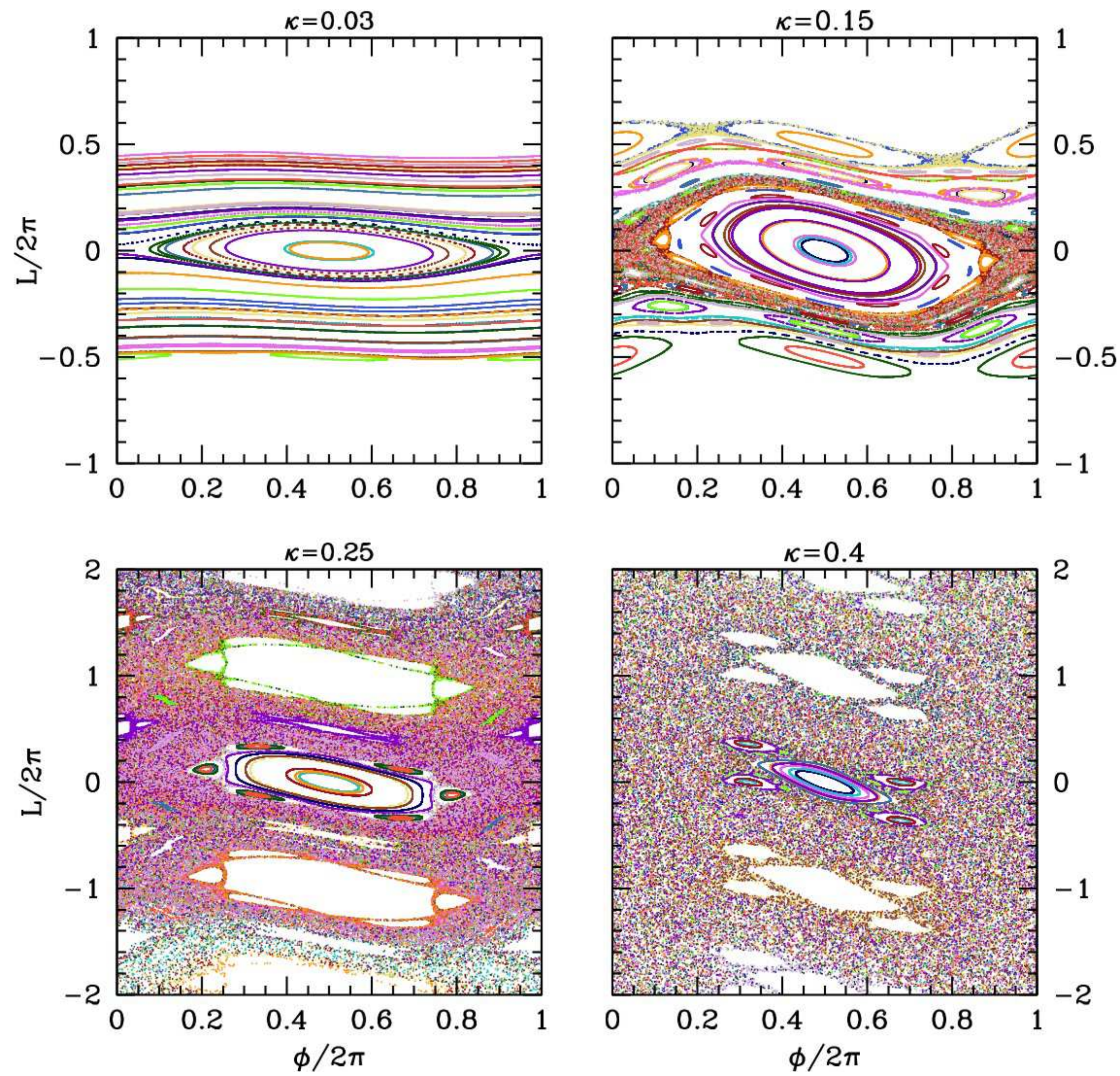

Figure 2: The standard map on the cylinder for different values of κ.

Solution :

See figs. 5, 6, 7, 8. The data are consistent with the Feigenbaum results cited in §2.5.1 of the lecture notes:

$$r_1 = 3 \quad , \; r_2 = 1 + \sqrt{6} = 3.4494897 \quad , \; r_3 = 3.544096 \quad , \; r_4 = 3.564407 \quad ,$$
$$r_5 = 3.568759 \quad , \; r_6 = 3.569692 \quad , \; r_7 = 3.569891 \quad , \; r_8 = 3.569934 \quad , \; \ldots$$

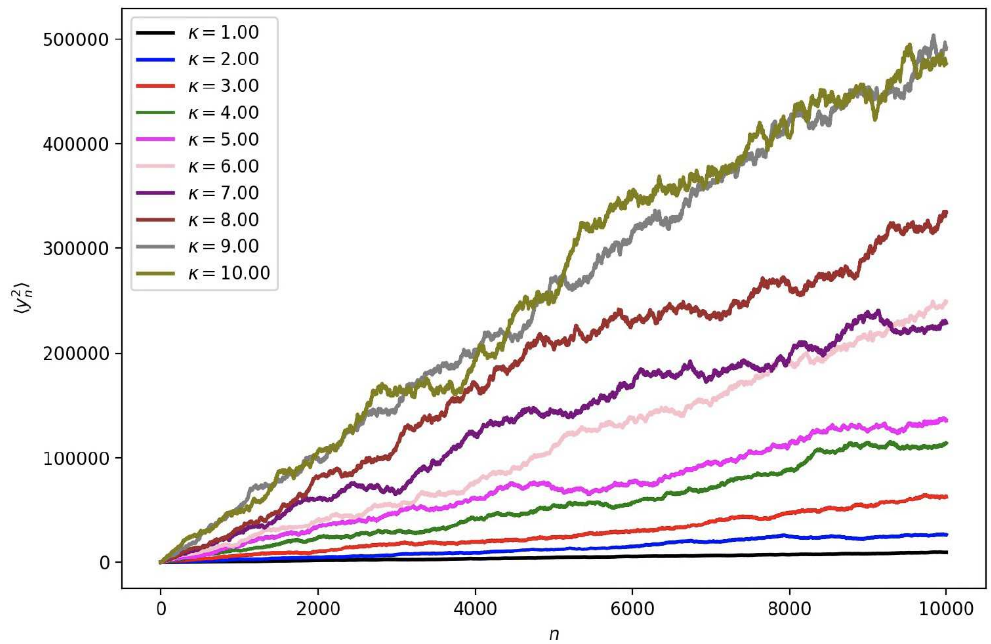

Figure 3: $\langle y_n^2 \rangle$ *versus* n for different values of κ. Credit: J. Gidugu.

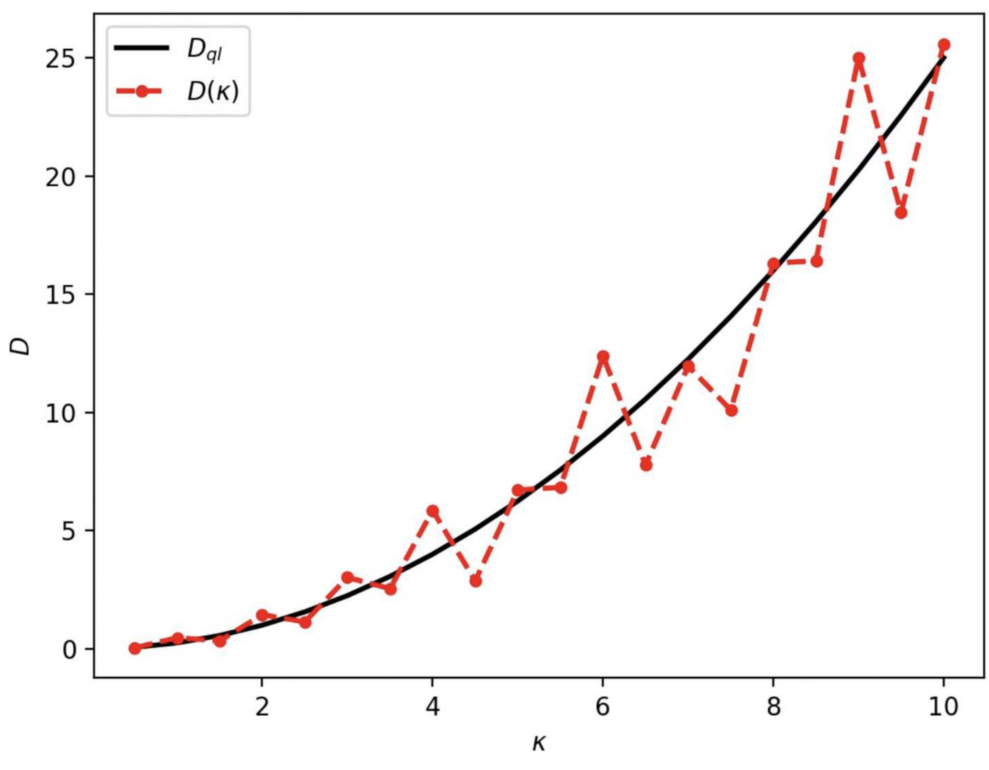

Figure 4: The standard map on the cylinder for different values of κ. Credit: J. Gidugu.

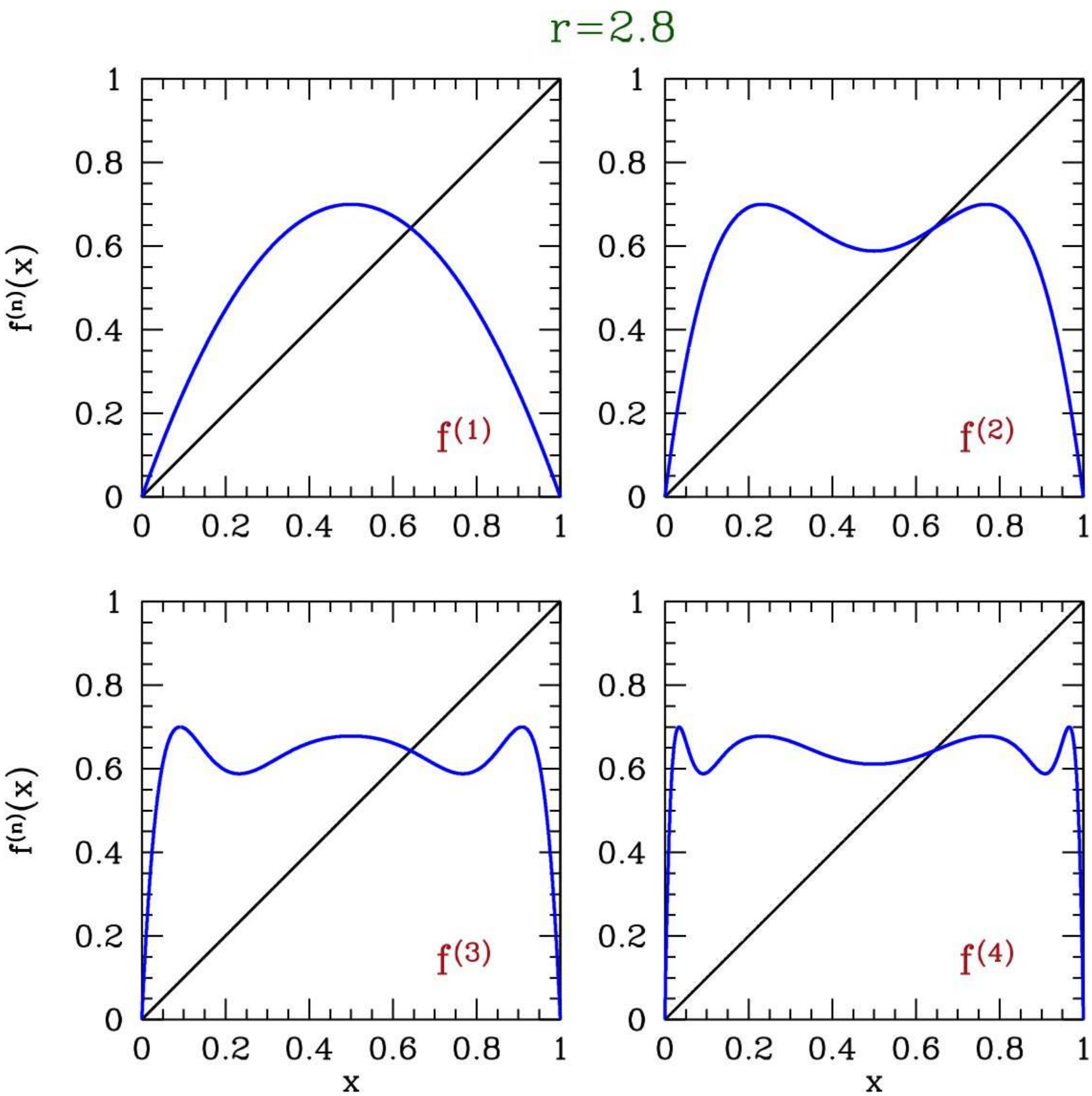

Figure 5: The first four iterates of the logistic map for $r = 2.8$.

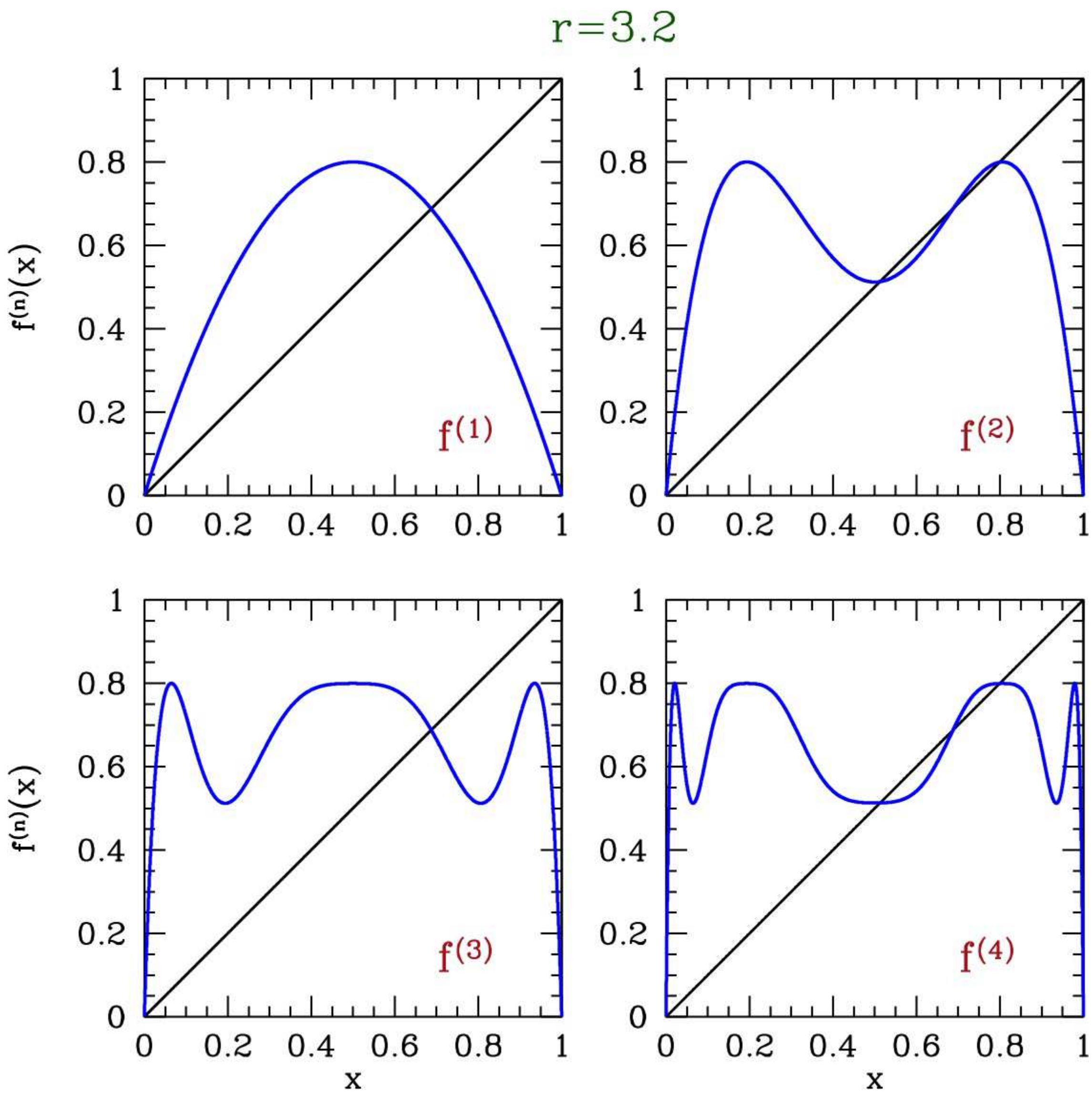

Figure 6: The first four iterates of the logistic map for $r = 3.2$.

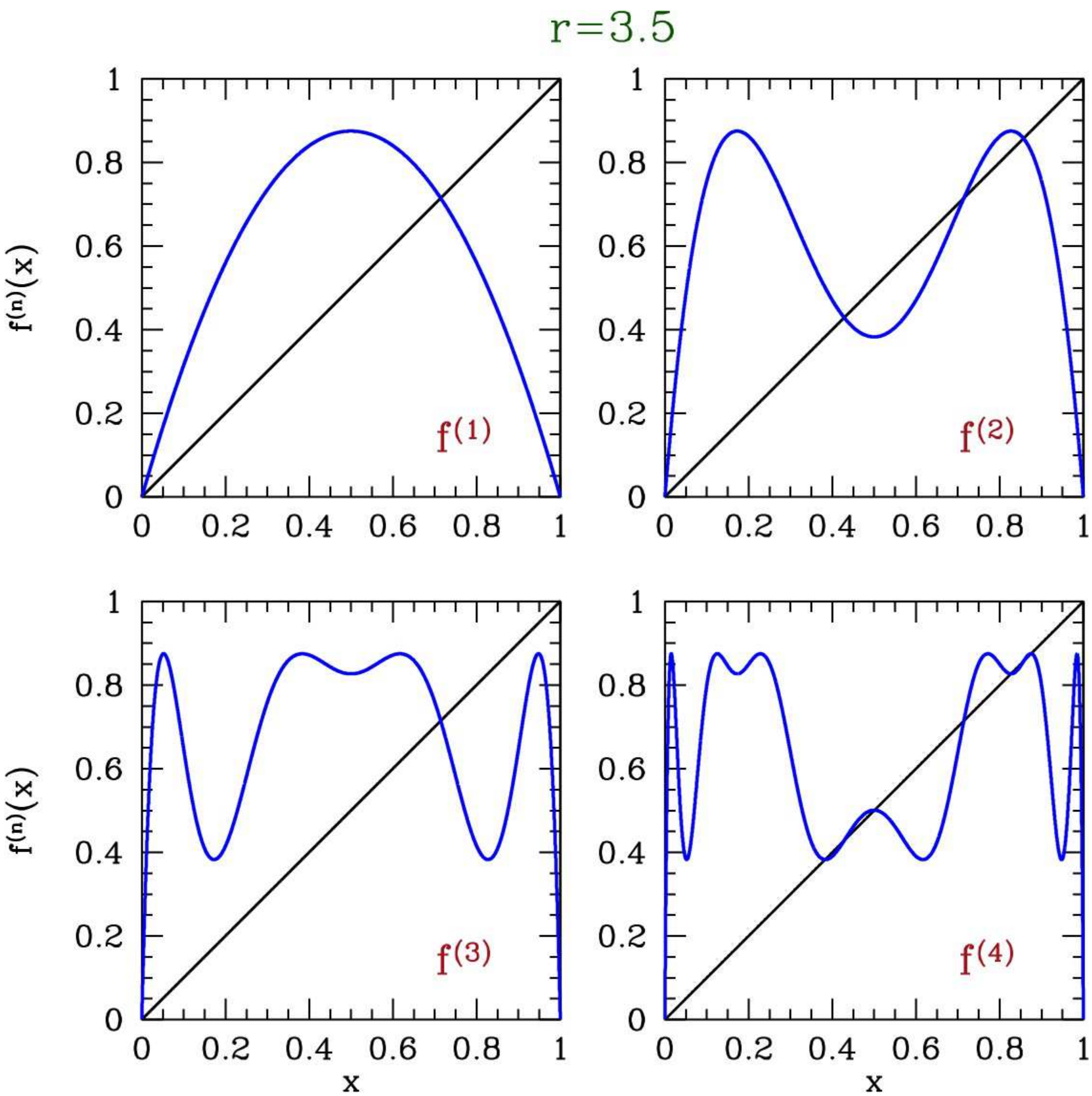

Figure 7: The first four iterates of the logistic map for $r = 3.5$.

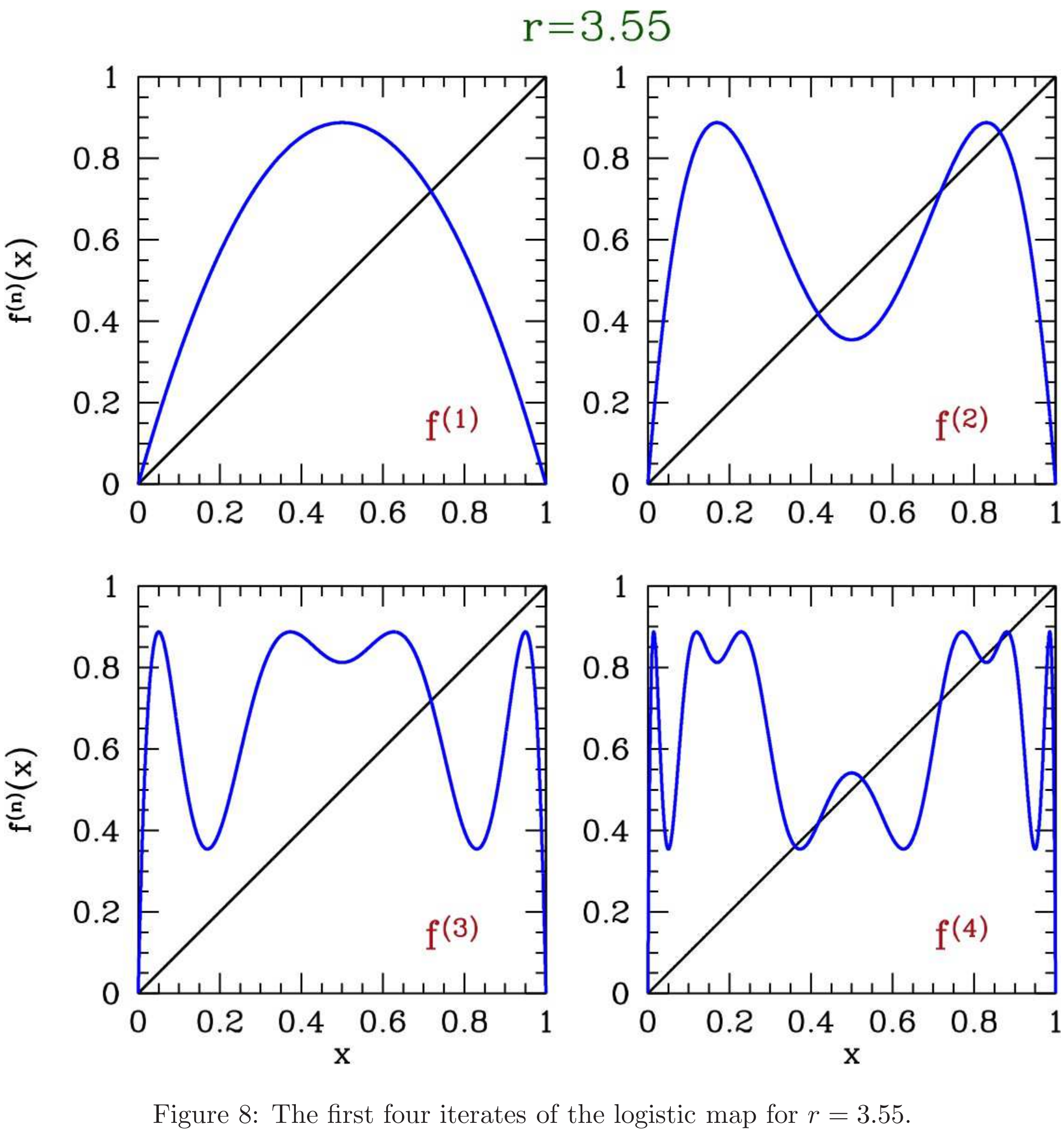

Figure 8: The first four iterates of the logistic map for $r = 3.55$.