PHYSICS 200B : CLASSICAL MECHANICS
SOLUTION SET #1

[1] Consider one-dimensional motion in the potential $V(x) = -V_0 \,\text{sech}^2(x/a)$ with $V_0 > 0$.

(a) Sketch the potential $V(x)$. Over what range of energies may action-angle variables be used?

(b) Find the action J and the Hamiltonian $H(J)$.

(c) Find the angle variable ϕ in terms of x and the energy E.

(d) Find the Solution for $x(t)$ by first solving for the motion of the action-angle variables.

Helpful mathematical identities :

$$\int\limits_0^{\bar{u}(E)} du\,\sqrt{E + V_0\,\text{sech}^2 u} = \frac{\pi}{2}\left(\sqrt{V_0} - \sqrt{-E}\right) \quad \text{if} \quad -V_0 < E < 0$$

$$\int du\,\left(E + V_0\,\text{sech}^2 u\right)^{-1/2} = \begin{cases} (-E)^{-1/2}\,\sin^{-1}\left(\sqrt{\frac{-E}{V_0+E}}\,\sinh u\right) & \text{if} \quad -V_0 < E < 0 \\ E^{-1/2}\,\sinh^{-1}\left(\sqrt{\frac{E}{V_0+E}}\,\sinh u\right) & \text{if} \quad E > 0 \end{cases}$$

where $\bar{u}(E) = \cosh^{-1}\sqrt{V_0/(-E)}$ in the first integral.

Solution :

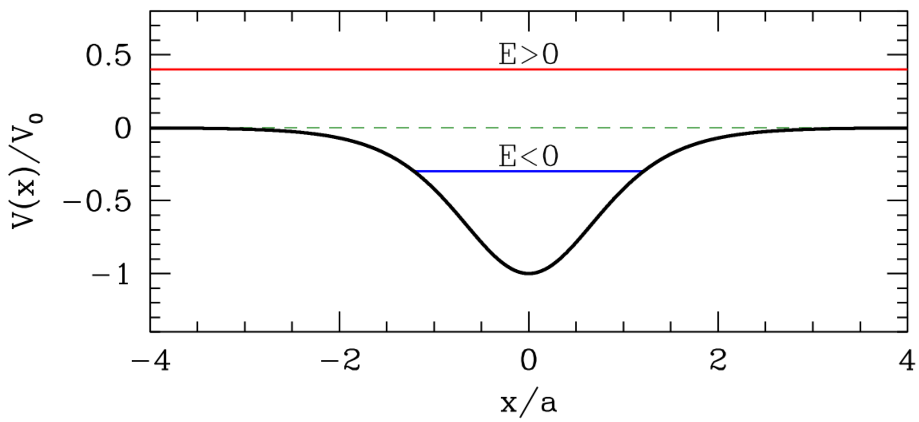
(a) The figure is shown below. For $E < -V_0$ there are no Solutions. For $E > 0$ the motion is unbound, neither librating nor rotating. Action-angle variables may be applied in the region $-V_0 < E < 0$.

(b) Using conservation of energy $E = \frac{p^2}{2m} + V(x)$, the momentum is

$$p = \sqrt{2m\big(E - V(x)\big)} \quad .$$

The action is

$$\begin{aligned} J = \frac{1}{2\pi}\oint p\,dq &= \frac{2}{\pi}\sqrt{2m}\int\limits_0^{\bar{x}(E)} dx\,\sqrt{E - V(x)} \\ &= \frac{2a}{\pi}\sqrt{2m}\int\limits_0^{\bar{u}(E)} du\,\sqrt{E + V_0\,\text{sech}^2 u} \\ &= \sqrt{2ma^2}\left(V_0^{1/2} - |E|^{1/2}\right) \quad . \end{aligned}$$



Thus, from $H(J)=E$ we have

$$H(J)=-\bigg(\frac{J}{\sqrt{2ma^2}}-\sqrt{V_0}\bigg)^2 \quad ,$$

where $\bar{x}(E)=a\,\bar{u}(E)=a\cosh^{-1}\sqrt{V_0/|E|}$.

(c) We have

$$W(x,J)=\int\!dq\,p=\sqrt{2m}\int\limits^{x}\!dx'\,\sqrt{E-V(x')} \quad .$$

Then

$$\begin{aligned}\phi=\frac{\partial W}{\partial J}&=\tfrac{1}{2}\sqrt{2m}\,\frac{\partial E}{\partial J}\int\limits^{x}\!dx'\,\Big(E+V_0\,\text{sech}^2(x'/a)\Big)^{-1/2}\\ &=\phi_0-\sin^{-1}\Bigg(\sqrt{\frac{|E|}{V_0-|E|}}\,\sinh(x/a)\Bigg) \quad ,\end{aligned}$$

where ϕ_0 is an arbitrary constant.

(d) Since $\dot{\phi}=\frac{\partial H}{\partial J}\equiv\nu(J)$, we have $\phi(t)=\nu(J)\,t$ and

$$x(t)=a\sinh^{-1}\Bigg(\sqrt{\frac{V_0-|E|}{|E|}}\,\sin(\omega t+\phi_0)\Bigg) \quad ,$$

where

$$\omega=-\nu(J)=-\frac{\partial E}{\partial J}=\sqrt{\frac{-2E}{ma^2}} \quad .$$

Note that $x(t)$ oscillates between $\pm\bar{x}(E)$, where $\sinh(\bar{x}/a)=\sqrt{\big(V_0-|E|\big)/|E|}$, which is equivalent to $\cosh(\bar{x}/a)=\sqrt{V_0/|E|}$, as we found in part (b).

[2] A particle of mass m moves in the potential $U(q)=A\,|q|$. The Hamiltonian is thus

$$H_0(q,p)=\frac{p^2}{2m}+A\,|q| \quad ,$$

where A is a constant.

(a) List all independent conserved quantities.

(b) Show that the action variable J is related to the energy E according to $J = \beta\, E^{3/2}/A$, where β is a constant, involving m. Find β.

(c) Find $q = q(\phi, J)$ in terms of the action-angle variables.

(d) Find $H_0(J)$ and the oscillation frequency $\nu_0(J)$.

(e) The system is now perturbed by a quadratic potential, so that

$$H(q,p) = \frac{p^2}{2m} + A\,|q| + \epsilon\, B\, q^2 \quad ,$$

where ϵ is a small dimensionless parameter. Compute the shift $\Delta\nu$ to lowest nontrivial order in ϵ, in terms of ν_0 and constants.

Solution :

(a) The only conserved quantity is the Hamiltonian itself:

$$\frac{dH_0}{dt} = \frac{\partial H_0}{\partial t} = 0 \quad .$$

We write $H_0(q,p) = E$, the total energy. Clearly $E \geq 0$, and $E = 0$ is particularly boring.

(b) Since the energy is conserved, we have

$$p(q) = \pm\sqrt{2m(E - A|q|)} \quad .$$

There are two turning points, at $q_\pm(E) = E/A$. We can integrate to get the action:

$$J = \frac{1}{2\pi}\oint\limits_{\mathcal{C}} p\, dq = \frac{2}{\pi}\int\limits_0^{E/A} dq\,\sqrt{2m(E - Aq)} = \frac{4\sqrt{2m}}{3\pi A}\, E^{3/2} \equiv \frac{\beta}{A}\, E^{3/2} \quad ,$$

with $\beta = 4\sqrt{2m}/3\pi$. Note that the integral over a complete cycle is written above as four times the integral over a quarter cycle, *i.e.* from $q = 0$ to $q = q_+(E) = E/A$.

(c) We first obtain the characteristic function $W(q, E(J))$. We have

$$p = \frac{dW}{dq} = \pm\sqrt{2m(E - A|q|)} \quad \Rightarrow \quad W(q) = \mp\frac{\pi\beta}{2A}\,\big(E - A|q|\big)^{3/2}\,\mathrm{sgn}\,(q) \quad ,$$

where we've used $\frac{2}{3}\sqrt{2m} = \frac{\pi}{2}\beta$. The angle variable is

$$\phi = \frac{\partial W}{\partial J} = \frac{\partial W}{\partial E}\,\frac{\partial E}{\partial J} = \mp\frac{\pi\beta^{1/3}}{2A^{1/3}}\,J^{-1/3}\,\big(E - A|q|\big)^{1/2}\,\mathrm{sgn}\,(q) \quad .$$

Squaring, we find

$$\left(\frac{2\phi}{\pi}\right)^2 = \left(\frac{\beta}{AJ}\right)^{2/3} \big(E - A|q|\big)$$
$$= 1 - \left(\frac{\beta^2 A}{J^2}\right)|q| \quad .$$

Thus,

$$q(\phi, J) = \frac{J^{2/3}}{(\beta^2 A)^{1/3}} \left\{1 - \tfrac{4}{\pi^2}\,\phi^2\right\} \qquad \phi \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \quad .$$

This is valid on the interval $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, where q is positive. In fact, this is all we need to solve the problem, but it is worthwhile writing down the continuation of this relation for the other half of the cycle, *i.e.* for $\phi \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. This can be done by inspection, taking advantage of the symmetry of the orbit $\mathcal{C}$:

$$q(\phi, J) = \frac{J^{2/3}}{(\beta^2 A)^{1/3}} \left\{\tfrac{4}{\pi^2}\,(\phi - \pi)^2 - 1\right\} \qquad \phi \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \quad .$$

(d)We have

$$H_0(J) = E = \beta^{-2/3}\, A^{2/3}\, J^{2/3} \quad ,$$

so

$$\nu_0(J) = \frac{\partial H_0(J)}{\partial J} = \tfrac{2}{3}\,\beta^{-2/3}\, A^{2/3}\, J^{-1/3} \quad .$$

(e) Expressed in terms of the action-angle variables (ϕ, J), the perturbing Hamiltonian is $\epsilon\, H_1(\phi, J)$, with

$$H_1(\phi, J) = B\, q^2 = B \cdot \left(\frac{J^2}{\beta^2 A}\right)^{2/3} \left(1 - \frac{4\phi^2}{\pi^2}\right)^2 \quad .$$

This holds for all ϕ provided we periodically extend the function ϕ^2 from the interval $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ to the entire real line. Due to the parity $(q \to -q)$ symmetry, we can average over a quarter cycle, and we obtain

$$\big\langle H_1(\phi, J)\big\rangle = B \cdot \left(\frac{J^2}{\beta^2 A}\right)^{2/3} \int\limits_0^1 ds\, (1 - s^2)^2 = \frac{8B}{15(\beta^2 A)^{2/3}}\, J^{4/3} \quad ,$$

where we've substituted $s = \frac{2}{\pi}\,\phi$. The energy shift is $\Delta E = \epsilon \big\langle H_1 \big\rangle$. Thus,

$$\nu(J) = \nu_0(J) + \tfrac{32}{45}\,\epsilon\, \frac{B J^{1/3}}{(\beta^2 A)^{3/2}} = \nu_0(J) + \epsilon \cdot \frac{2\pi^2 B}{15\, m} \cdot \frac{1}{\nu_0(J)} \quad .$$

[3] Consider the nonlinear oscillator described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 + \tfrac{1}{4}\epsilon a q^4 + \tfrac{1}{4}\epsilon b p^4 \quad ,$$

where ε is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ, where A is the amplitude of the q motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ.

Solution :

With $k \equiv m\nu_0^2$, recall the AA variables

$$\phi_0 = \tan^{-1}\!\left(\frac{m\nu_0 q}{p}\right) \qquad , \qquad J_0 = \frac{p^2}{2m\nu_0} + \tfrac{1}{2}m\nu_0 q^2 \quad .$$

Thus, $q = (2J_0/m\nu_0)^{1/2} \sin\phi_0$ and $p = (2m\nu_0 J_0)^{1/2}\cos\phi_0$, so the Hamiltonian is

$$\widetilde{H}(\phi_0, J_0) = \nu_0 J_0 + \epsilon \widetilde{H}_1(\phi_0, J_0) \quad ,$$

where

$$\widetilde{H}_1(\phi_0, J_0) = \frac{aJ_0^2}{m^2\nu_0^2}\sin^4\!\phi_0 + b\, m^2\nu_0^2 J_0^2 \cos^4\!\phi_0 \quad .$$

(a) Averaging over ϕ_0, we have $\langle \sin^4\!\phi_0 \rangle = \langle \cos^4\!\phi_0 \rangle = \frac{3}{8}$, so

$$E_1(J) = \big\langle \widetilde{H}_1(\phi_0, J) \big\rangle = \left(\frac{a}{mk} + bmk\right) \times \tfrac{3}{8} J^2 \quad .$$

The perturbed frequencies are $\nu(J) = \nu_0 + \epsilon\nu_1$ where $\nu_1 = \frac{\partial E_1}{\partial J}$. Thus,

$$\nu(J) = \sqrt{\frac{k}{m}} + \left(\frac{a}{mk} + bmk\right) \times \tfrac{3}{4}\epsilon J \quad .$$

(b) We only need J to zeroth order in ϵ. Setting $p = 0$ and $q = A$ gives $J = \frac{1}{2}m\nu_0 A^2 + \mathcal{O}(\epsilon)$, in which case

$$\nu(A) = \sqrt{\frac{k}{m}} + \left(\frac{a}{mk} + bmk\right) \times \tfrac{3}{8}\epsilon m\nu_0 A^2 \quad .$$

(c) Recall the desired type-II CT is generated by $S(\phi_0, J) = \phi_0 J + \epsilon S_1(\phi_0, J) + \ldots$, with

$$\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \widetilde{H}_1 \rangle - \widetilde{H}_1}{\nu_0(J)} \quad .$$

Thus,

$$\frac{\partial S_1}{\partial \phi_0} = \frac{aJ^2}{m^2\nu_0^3}\big(\tfrac{3}{8} - \sin^4\phi_0\big) + bm^2\nu_0 J\big(\tfrac{3}{8} - \cos^4\phi_0\big) \quad .$$

Integrating, we have

$$S_1(\phi_0, J) = \frac{aJ^2}{m^2\nu_0^3}\Big(\tfrac{1}{4}\sin(2\phi_0) - \tfrac{1}{32}\sin(4\phi_0)\Big) - bm^2\nu_0 J^2\Big(\tfrac{1}{4}\sin(2\phi_0) + \tfrac{1}{32}\sin(4\phi_0)\Big) \quad .$$

The constant may be set to zero as it leads to a constant shift of the angle variable ϕ. Thus, we have

$$\begin{aligned} J_0 &= J + \epsilon\,\frac{\partial S_1}{\partial \phi_0} + \mathcal{O}(\epsilon^2) \\ &= J + \left(\frac{a - bm^4\nu_0^4}{2m^2\nu_0^3}\right)\epsilon J^2\cos(2\phi_0) - \left(\frac{a + bm^2\nu_0^4}{8m^2\nu_0^3}\right)\epsilon J^2\cos(4\phi_0) + \mathcal{O}(\epsilon^2) \quad . \end{aligned}$$

Thus,

$$J = J_0 - \left(\frac{a - bm^4\nu_0^4}{2m^2\nu_0^3}\right)\epsilon J_0^2\cos(2\phi_0) + \left(\frac{a + bm^2\nu_0^4}{8m^2\nu_0^3}\right)\epsilon J_0^2\cos(4\phi_0) + \mathcal{O}(\epsilon^2) \quad .$$

We then have

$$\begin{aligned} \phi &= \phi_0 + \epsilon\,\frac{\partial S_1}{\partial J} + \mathcal{O}(\epsilon^2) \\ &= \phi_0 + \left(\frac{a - bm^4\nu_0^4}{2m^2\nu_0^3}\right)\epsilon J_0\sin(2\phi_0) - \left(\frac{a + bm^2\nu_0^4}{16m^2\nu_0^3}\right)\epsilon J_0\sin(4\phi_0) + \mathcal{O}(\epsilon^2) \quad . \end{aligned}$$