PHYSICS 200B : CLASSICAL MECHANICS
FINAL EXAM : due 12:00 pm Sunday, March 20 by email

Instructions: You may consult only the course lecture notes but no other written or human sources. You are expected to use a computer and whatever software you find convenient for the numerical work.

[1] Consider the time-dependent 'kicked' Hamiltonian $H(t) = T(p) + V(q)\,K(t)$, where $K(t) = \tau \sum_n \delta(t - n\tau)$ is a Dirac comb. Let $q_n = q(n\tau^-)$ and $p_n = p(n\tau^-)$, *i.e.* just before each kick.

(a) Find the matrix

$$M = \frac{\partial(q_{n+1}, p_{n+1})}{\partial(q_n, p_n)} \quad ,$$

and show that it is symplectic.

(b) Find the condition that a fixed point (q^*, p^*) is unstable.

(c) Define the function

$$g(x) = x - \mathsf{nint}(x) \quad ,$$

where $\mathsf{nint}(x)$ is the nearest integer to x. Thus $g(\pm 0.4) = \pm 0.4$ since $\mathsf{nint}(\pm 0.4) = 0$, but $g(0.6) = -0.4$, $g(-3.7) = 0.3$, *etc.* Now consider the case

$$T(p) = \frac{P^2}{2m} \cdot \big[g(p/P)\big]^2 \qquad , \qquad V(q) = \tfrac{1}{2} k Q^2 \cdot \big[g(q/Q)\big]^2 \quad .$$

This effectively renders the phase space a torus of area PQ. Find the conditions for all fixed points of the map $(q_n, p_n) \to (q_{n+1}, p_{n+1})$. Which fixed points are unstable?

[2] Consider the 1D map $x_{n+1} = f(x_n)$, where

$$f(x) = rx(1-x)(1-2x)^2 \quad .$$

(a) Numerically explore the stability of the fixed 1-cycle by plotting cobweb diagrams for various values of r. Note that $f(x) = f(1-x)$, $f'(0) = r$, but $f(\frac{1}{2}) = 0$. Thus, as r changes, new solutions to the fixed point equation $f(x) = x$ may appear discontinuously. Can you numerically identify the ranges of stability?

(b) Another way to investigate is the following. Write a computer program which makes a plot like in fig. 2.10 of the lecture notes. Here is how I made that figure:

i. The outer loop is over the r values. For this problem, choose $r \in [1, 16]$. Loop over at least 500 values.

ii. For each r value, iterate the map $x' = f(x)$ one thousand times, but do not plot the results. Start with a random seed x_0. (You can even try using the same seed for each r value.)

iii. After iterating so many times, your program should have settled in on a stable cycle or else it is in a regime of chaos. Plot the next 400 iterates of the map.

iv. Advance r to its next value $r + \Delta r$ and go back to step (ii). Terminate after $r = 16$.

(c) Analytically obtain the region of stability in the control parameter r and the corresponding set of fixed points $x^*(r)$. *Hint: Simultaneously set* $f(x) = x$ *and* $f'(x) = \pm 1$.

(d) Show that for $r = 16$, if we define $x \equiv \sin^2\theta$, with $\theta \in [0, \pi]$, there is a simple relationship between θ_{n+1} and θ_n. Writing the binary expansion of $\theta_{n=0}$ as

$$\theta_0 = \pi \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad ,$$

and given that $\sin^2\theta$ is periodic under $\theta \to \theta + \pi$, find the corresponding binary expansion of θ_n.

[3] *The Burgers vortex* – Seek an exact, steady state solution to the Navier-Stokes equations (with $\zeta = 0$) of the form

$$\boldsymbol{v}(r, \phi, z) = -\tfrac{1}{2}\alpha r \hat{\boldsymbol{e}}_r + v_\phi(r)\, \hat{\boldsymbol{e}}_\phi + \alpha z\, \hat{\boldsymbol{e}}_z \quad .$$

(a) Verify that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ and that $\boldsymbol{\omega} = \omega(r)\, \hat{\boldsymbol{e}}_z$. Show that the equations of motion imply a first order ODE for the vorticity $\omega(r)$. Obtain that equation.

(b) Find $\omega(r)$ and $v_\phi(r)$.