PHYSICS 221A : NONLINEAR DYNAMICS
HW ASSIGNMENT #3

(1) Riccati equations are nonlinear nonautonomous ODEs of the form

$$\frac{dx}{dt} = a(t)\, x^2 + b(t)\, x + c(t) \ .$$

(a) Make a change of dependent variable from $x(t)$ to $y(t)$, where

$$x(t) = -\frac{1}{a(t)}\, \frac{\dot{y}(t)}{y(t)} \ ,$$

which is known as the *Riccati transformation*. Show that $y(t)$ obeys a *linear* nonautonomous second order ODE. Write the formal solution to this ODE by writing it in the form $\dot{\boldsymbol{\varphi}} = M(t)\, \boldsymbol{\varphi}$ and expressing the solution in terms of a time ordered exponential.

(b) Solve the Riccati equation

$$\dot{x} = e^t\, x^2 - x + e^{-t} \ .$$

(c) Suppose we have a solution $X(t)$ to the Riccati equation. Show that by writing $x(t) = X(t) + u(t)$ we obtain the solvable Bernoulli equation

$$\dot{u} = a(t)\, u^2 + \Big(b(t) + 2\, a(t) X(t)\Big)\, u \ ,$$

which can then be solved using the method from problem (5) of homework set #1.

(d) Consider the Riccati equation

$$\dot{x} = x^2 - tx + 1 \ .$$

By inspection, we have that $x(t) = t$ is a solution. Using the method of part (c) above, find a general solution for arbitrary $x(0) \equiv x_0$.

(2) Consider the equation

$$\ddot{x} + x = \epsilon\, x^5$$

with $\epsilon \ll 1$.

(a) Develop a two term straightforward expansion for the solution and discuss its uniformity.

(b) Using the Poincaré-Lindstedt method, find a uniformly valid expansion to first order.

(c) Using the multiple time scale method, find a uniformly valid expansion to first order.

(3) Consider the equation

$$\ddot{x} + \epsilon\,\dot{x}^3 + x = 0$$

with $\epsilon \ll 1$. Using the multiple time scale method, find a uniformly valid expansion to first order.

(4) Analyze the forced oscillator

$$\ddot{x} + x = \epsilon\left(\dot{x} - \tfrac{1}{3}\dot{x}^3\right) + \epsilon\, f_0\, \cos(t + \epsilon\nu t)$$

using the discussion in §4.3.1 and §4.3.2 of the notes as a template.