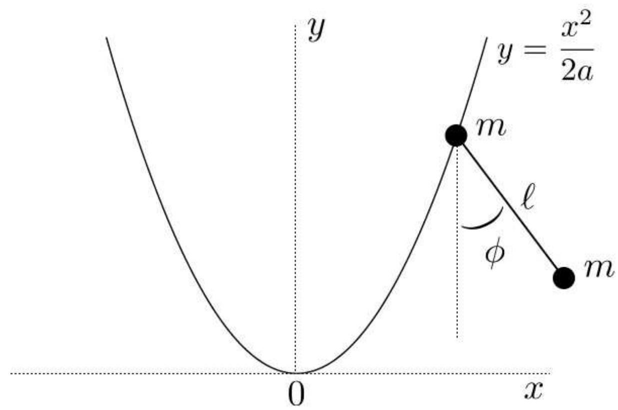
PHYSICS 110A : MECHANICS 1
PROBLEM SET #10 SOLUTIONS

[**1**] Recall problem #3 from HW #6, in which a mass m moves frictionlessly under the influence of gravity along the curve $y = x^2/2a$. Attached to the mass is a massless rigid rod of length ℓ, at the end of which is an identical mass m. The rod is constrained to swing in the (x, y) plane, as depicted in the figure below.



(a) Choose as generalized coordinates x and ϕ. Find the kinetic energy T and potential energy U.

(b) Choose as generalized displacements the coordinates $\eta_1 = x$ and $\eta_2 = \phi$ themselves. For small oscillations, find the T and V matrices. It may be convenient to define $\Omega_1 \equiv \sqrt{g/a}$ and $\Omega_2 \equiv \sqrt{g/\ell}$.

(c) Find the eigenfrequencies of the normal modes of oscillation.

(d) Suppose $\Omega_1 = \sqrt{3}\,\Omega_0$ and $\Omega_2 = 2\,\Omega_0$, where Ω_0 has dimensions of frequency. Find the modal matrix.

Solution :

(a) The coordinates of the mass on the curve are $(x_1\,,\,y_1) = (x\,,\,x^2/2a)$. Note $\dot{y} = (x/a)\,\dot{x}$. The coordinates for the hanging mass are $(x_2\,,\,y_2) = (x + \ell\sin\phi\,,\,x^2/2a - \ell\cos\phi)$. The kinetic energy is

$$
\begin{aligned}
T &= \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \\
&= m\bigg(1 + \frac{x^2}{a^2}\bigg)\dot{x}^2 + \tfrac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\bigg(\cos\phi + \frac{x}{a}\,\sin\phi\bigg)\dot{x}\dot{\phi} \quad .
\end{aligned}
$$

The potential energy is

$$
U = mg(y_1 + y_2) = \frac{mg}{a}\,x^2 - mg\ell\cos\phi \quad .
$$

(b) Equilibrium occurs for $x = \phi = 0$, hence

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial \dot{q}_\sigma \partial \dot{q}_{\sigma'}}\bigg|_{\bar{q}} = \begin{pmatrix} 2m & m\ell \\ m\ell & m\ell^2 \end{pmatrix}$$

and

$$\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma \partial q_{\sigma'}}\bigg|_{\bar{q}} = \begin{pmatrix} 2m\Omega_1^2 & 0 \\ 0 & m\ell^2\Omega_2^2 \end{pmatrix} \quad .$$

(c) We set $P(\omega^2) = \det(\omega^2\,\mathsf{T} - \mathsf{V}) = 0$, with

$$\omega^2\,\mathsf{T} - \mathsf{V} = \begin{pmatrix} 2m(\omega^2 - \Omega_1^2) & m\ell\omega^2 \\ m\ell\omega^2 & m\ell^2(\omega^2 - \Omega_2^2) \end{pmatrix}$$

Thus,

$$P(\omega^2) = m^2\ell^2 \Big\{ \omega^4 - 2\,(\Omega_1^2 + \Omega_2^2)\,\omega^2 + 2\,\Omega_1^2\Omega_2^2 \Big\} \quad .$$

Solving the quadratic equation, we have the two normal mode frequencies

$$\omega_\pm^2 = \Omega_1^2 + \Omega_2^2 \pm \sqrt{\Omega_1^4 + \Omega_2^4} \quad .$$

(d) With $\Omega_1 = \sqrt{3}\,\Omega_0$ and $\Omega_2 = 2\,\Omega_0$, we have $\omega_+^2 = 12\,\Omega_0^2$ and $\omega_-^2 = 2\,\Omega_0^2$. We then solve for the eigenvectors using $(\omega_i^2\,\mathsf{T} - \mathsf{V})_{\sigma\sigma'}\mathsf{A}_{\sigma' i} = 0$. From the form of $\omega^2\,\mathsf{T} - \mathsf{V}$, we see that

$$\mathsf{A}_{2,i} = \frac{\ell^{-1}\omega_i^2}{\Omega_2^2 - \omega_i^2}\,\mathsf{A}_{1,i} \quad ,$$

and imposing the normalization $\mathsf{A}^{\mathsf{t}}\mathsf{T}\mathsf{A} = \mathbb{I}$, we have

$$\mathsf{A} = \frac{1}{\sqrt{5m}} \begin{pmatrix} 2 & 1 \\ -3\ell^{-1} & \ell^{-1} \end{pmatrix} \quad .$$

[2] Two pendula each consisting of a point mass m hanging from a massless rigid rod of length ℓ are coupled by a massless spring of spring constant k (between the mass points). When the pendula hang vertically, the spring is unstretched. Compute the eigenfrequencies and the normal modes. Classify the normal modes according to whether they are even or odd with respect to the group $\mathbb{Z}_2$, generated by the elements $\mathbb{I}$ (identity) and P (reflection about a vertical line midway between the two pendula).

Solution :

The extension of the spring to lowest order in the angular displacements is $\ell(\theta_2 - \theta_1)$, hence the Lagrangian for small oscillations is

$$L = \tfrac{1}{2}m\ell^2\big(\dot{\theta}_1^2 + \dot{\theta}_2^2\big) - \tfrac{1}{2}k\ell^2\big(\theta_2 - \theta_1\big)^2 - \tfrac{1}{2}mg\ell\big(\theta_1^2 + \theta_2^2\big) \ .$$

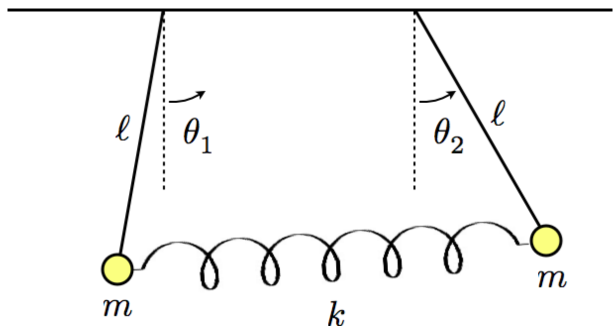


Figure 1: Coupled identical pendula.

Thus, the T and V matrices are

$$\mathsf{T} = m\ell^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad , \qquad \mathsf{V} = m\ell^2 \begin{pmatrix} \Omega^2 + \nu^2 & -\Omega^2 \\ -\Omega^2 & \Omega^2 + \nu^2 \end{pmatrix} \ ,$$

where

$$\Omega \equiv \sqrt{\frac{k}{m}} \qquad , \qquad \nu \equiv \sqrt{\frac{g}{\ell}} \ .$$

The modal matrix is obtained by inspection:

$$\mathsf{A} = \frac{1}{\ell\sqrt{2m}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ .$$

This satisfies

$$\mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad , \qquad \mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A} = \begin{pmatrix} \nu^2 & 0 \\ 0 & 2\Omega^2 + \nu^2 \end{pmatrix} \ .$$

The normal mode $\boldsymbol{\psi}^{(1)} \propto \binom{1}{1}$ with frequency $\omega_1 = \nu$ is symmetric (even parity), and the normal mode $\boldsymbol{\psi}^{(2)} \propto \binom{1}{-1}$ with frequency $\omega_2 = \sqrt{2\Omega^2 + \nu^2}$ is antisymmetric (odd parity).

[3] Two masses m_1 and m_2 are connected to a spring and a pendulum arm, as depicted in fig. 2. The unstretched length of the spring is a.

(a) Choosing generalized coordinates x and θ as shown, write the Lagrangian for this system.

(b) Expanding about equilibrium, write the Lagrangian for small oscillations as a quadratic form. *Suggestion: It may be convenient to define the generalized displacements* $\eta_1 \equiv x$ *and* $\eta_2 \equiv \ell\theta$.

(c) Let $\Omega \equiv \sqrt{k/m_1}$, $\nu \equiv \sqrt{g/\ell}$, and $r \equiv m_2/m_1$. Find the T and V matrices.

(d) Find the eigenfrequencies $\omega_{1,2}$.

(e) Find an expression for the ratios of the components of the normal mode eigenvectors $\psi_2^{(+)}/\psi_1^{(+)}$ and $\psi_2^{(-)}/\psi_1^{(-)}$.

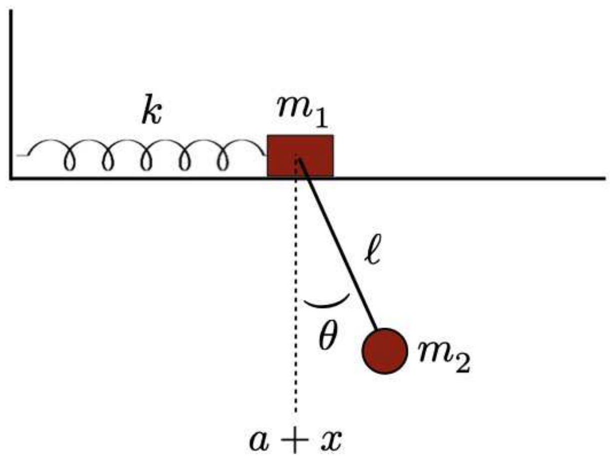


Figure 2: A spring, a pendulum, and two masses.

Solution :

(a) We have

$$x_1 = a + x \qquad , \qquad x_2 = a + x + \ell \sin\theta \qquad , \qquad y_2 = -\ell\cos\theta$$

and

$$\dot{x}_1 = \dot{x} \qquad , \qquad x_2 = \dot{x} + \ell\cos\theta\,\dot{\theta} \qquad , \qquad \dot{y}_2 = \ell\sin\theta\,\dot{\theta} \quad .$$

Thus

$$\begin{aligned} T &= \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + m_2\ell\cos\theta\,\dot{x}\,\dot{\theta} + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2 \\ U &= \tfrac{1}{2}kx^2 - m_2 g\ell\cos\theta \end{aligned}$$

(b) Equilibrium occurs for $x = 0$ and $\theta = 0$. For small x and θ, we have

$$\begin{aligned} T &= \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + m_2\ell\,\dot{x}\,\dot{\theta} + \tfrac{1}{2}m_2\ell^2\dot{\theta}^2 + \ldots \\ U &= \tfrac{1}{2}kx^2 + \tfrac{1}{2}m_2 g\ell\theta^2 - m_2 g\ell + \ldots \quad . \end{aligned}$$

Taking the hint and defining $\eta_1 \equiv x$ and $\eta_2 \equiv \ell\theta$, we have

$$L = \tfrac{1}{2}(m_1+m_2)\,\dot{\eta}_1^2 + m_2\,\dot{\eta}_1\dot{\eta}_2 + \tfrac{1}{2}m_2\,\dot{\eta}_2^2 - \tfrac{1}{2}k\,\eta_1^2 - \tfrac{1}{2}m_2\,\frac{g}{\ell}\,\eta_2^2 \quad ,$$

where we have dropped a constant term $m_2 g\ell$.

(c) Writing

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^{\mathrm{t}}\,\mathsf{T}\,\dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\boldsymbol{\eta}^{\mathrm{t}}\,\mathsf{V}\,\boldsymbol{\eta} \quad ,$$

we can read off

$$\mathsf{T} = m_1 \begin{pmatrix} 1+r & r \\ r & r \end{pmatrix} \qquad , \qquad \mathsf{V} = m_1 \begin{pmatrix} \varOmega^2 & 0 \\ 0 & r\nu^2 \end{pmatrix} \quad .$$

(d) We solve $\mathsf{det}(\omega^2 \mathsf{T} - \mathsf{V}) = 0$, with

$$\omega^2 \mathsf{T} - \mathsf{V} = m_1 \begin{pmatrix} (1+r)\,\omega^2 - \varOmega^2 & r\omega^2 \\ r\omega^2 & r(\omega^2 - \nu^2) \end{pmatrix} \quad .$$

Thus,

$$P(\omega^2) = \mathsf{det}(\omega^2 \mathsf{T} - \mathsf{V}) = m_1^2 \Big(\omega^4 - \big[\varOmega^2 + (1+r)\,\nu^2\big]\,\omega^2 + \varOmega^2 \nu^2 \Big) = 0 \quad .$$

Clearly the m_1^2 coefficient doesn't affect the calculation of the eigenfrequencies, which are given by

$$\omega_{1,2}^2 = \tfrac{1}{2}\big[\varOmega^2 + (1+r)\,\nu^2\big] \pm \tfrac{1}{2}\sqrt{\big[\varOmega^2 + (1+r)\,\nu^2\big]^2 - 4\varOmega^2\nu^2} \quad .$$

(e) We have

$$\begin{pmatrix} (1+r)\,\omega_\pm^2 - \varOmega^2 & r\omega_\pm^2 \\ r\omega_\pm^2 & r(\omega_\pm^2 - \nu^2) \end{pmatrix} \begin{pmatrix} \psi_1^{(\pm)} \\ \psi_2^{(\pm)} \end{pmatrix} = 0 \quad .$$

Thus, taking the equation arising from the bottom row, we have

$$r\omega_\pm^2\, \psi_1^{(\pm)} + r(\omega_\pm^2 - \nu^2)\, \psi_2^{(\pm)} = 0 \quad ,$$

and thus

$$\frac{\psi_2^{(\pm)}}{\psi_1^{(\pm)}} = \frac{\omega_\pm^2}{\nu^2 - \omega_\pm^2} \quad .$$

Note that we may write

$$\omega_\pm^2 - \nu^2 = \tfrac{1}{2}\big[\varOmega^2 + (r-1)\,\nu^2\big] \pm \tfrac{1}{2}\sqrt{\big[\varOmega^2 + (r-1)\,\nu^2\big]^2 + 4r\nu^4} \quad .$$

and therefore we conclude $\omega_+^2 > \nu^2$ and $\omega_-^2 < \nu^2$. Thus, the masses are 180° out of phase in the high frequency mode, and are in phase in the low-frequency mode.