PHYSICS 110A : MECHANICS 1
PROBLEM SET #7 SOLUTIONS

[**1**] A point mass m slides inside a hoop of radius R and mass M, which itself rolls without slipping on a horizontal surface, as depicted in fig. 1.

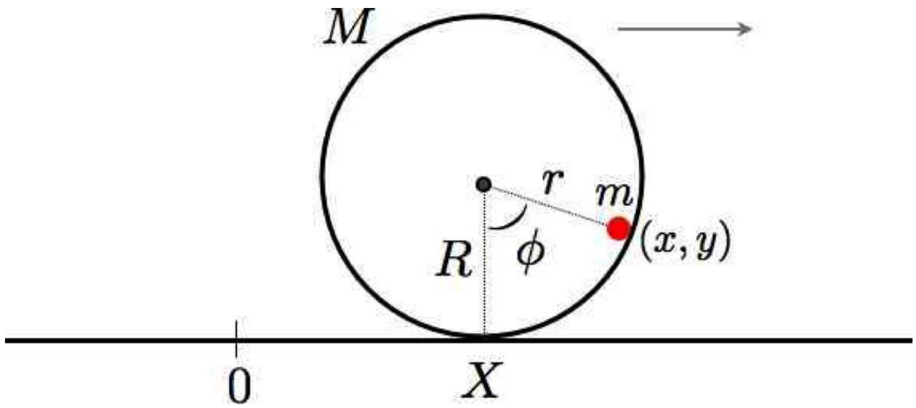

Figure 1: A mass point m rolls inside a hoop of mass M and radius R which rolls without slipping on a horizontal surface.

Choose as general coordinates (X, ϕ, r), where X is the horizontal location of the center of the hoop, ϕ is the angle the mass m makes with respect to the vertical ($\phi = 0$ at the bottom of the hoop), and r is the distance of the mass m from the center of the hoop. Since the mass m slides inside the hoop, there is a constraint:

$$G(X, \phi, r) = r - R = 0 \ .$$

Nota bene: The kinetic energy of the moving hoop, including translational and rotational components (but not including the mass m), is $T_{\text{hoop}} = M\dot{X}^2$ (*i.e.* twice the translational contribution alone).

(a) Find the Lagrangian $L(X, \phi, r, \dot{X}, \dot{\phi}, \dot{r}, t)$.

<u>Solution</u> : The Cartesian coordinates and velocities of the mass m are

$$x = X + r\sin\phi \qquad \dot{x} = \dot{X} + \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$
$$y = R - r\cos\phi \qquad \dot{y} = -\dot{r}\cos\phi + r\dot{\phi}\sin\phi$$

The Lagrangian is then

$$L = \overbrace{(M + \tfrac{1}{2}m)\dot{X}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + m\dot{X}(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)}^{T} - \overbrace{mg(R - r\cos\phi)}^{U}$$

Note that we are not allowed to substitute $r = R$ and hence $\dot{r} = 0$ in the Lagrangian *prior* to obtaining the equations of motion. Only *after* the generalized momenta and forces are computed are we allowed to do so.

(b) Find all the generalized momenta p_σ, the generalized forces F_σ, and the forces of constraint Q_σ.

Solution : The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{X}\,\sin\phi$$
$$p_X = \frac{\partial L}{\partial \dot{X}} = (2M+m)\dot{X} + m\dot{r}\,\sin\phi + mr\dot{\phi}\,\cos\phi$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + mr\dot{X}\,\cos\phi$$

The generalized forces and the forces of constraint are

$$F_r = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + m\dot{X}\dot{\phi}\,\cos\phi + mg\cos\phi \qquad Q_r = \lambda\,\frac{\partial G}{\partial r} = \lambda$$
$$F_X = \frac{\partial L}{\partial X} = 0 \qquad Q_X = \lambda\,\frac{\partial G}{\partial X} = 0$$
$$F_\phi = \frac{\partial L}{\partial \phi} = m\dot{X}\dot{r}\,\cos\phi - m\dot{X}\dot{\phi}\,\sin\phi - mgr\sin\phi \qquad Q_\phi = \lambda\,\frac{\partial G}{\partial \phi} = 0\ .$$

The equations of motion are

$$\dot{p}_\sigma = F_\sigma + Q_\sigma\ .$$

At this point, we can legitimately invoke the constraint $r = R$ and set $\dot{r} = 0$ in all the p_σ and F_σ.

(c) Derive expressions for all conserved quantities.

Solution : There are two conserved quantities, which each derive from continuous invariances of the Lagrangian *which respect the constraint*. The first is the total momentum p_X:

$$F_X = 0 \quad \Longrightarrow \quad P \equiv p_X = \text{constant}\ .$$

The second conserved quantity is the Hamiltonian, which in this problem turns out to be the total energy $E = T + U$. Incidentally, we can use conservation of P to write the energy in terms of the variable ϕ alone. From

$$\dot{X} = \frac{P}{2M+m} - \frac{mR\,\cos\phi}{2M+m}\,\dot{\phi}\ ,$$

we obtain

$$\begin{aligned} E &= \tfrac{1}{2}(2M+m)\dot{X}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 + mR\dot{X}\dot{\phi}\,\cos\phi + mgR(1-\cos\phi) \\ &= \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2}mR^2\left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\dot{\phi}^2 + mgR(1-\cos\phi)\ , \end{aligned}$$

where we've defined the dimensionless ratio $\alpha \equiv m/2M$. It is convenient to define the quantity

$$\Omega^2 \equiv \left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\dot\phi^2 + 2\omega_0^2(1-\cos\phi)\ ,$$

with $\omega_0 \equiv \sqrt{g/R}$. Clearly Ω^2 is conserved, as it is linearly related to the energy E:

$$E = \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2} mR^2\Omega^2\ .$$

(d) Derive a differential equation of motion involving the coordinate $\phi(t)$ alone. *I.e.* your equation should not involve r, X, or the Lagrange multiplier λ.

Solution : From conservation of energy,

$$\frac{d(\Omega^2)}{dt} = 0 \quad\Longrightarrow\quad \left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\ddot\phi + \left(\frac{\alpha\,\sin\phi\,\cos\phi}{1+\alpha}\right)\dot\phi^2 + \omega_0^2\,\sin\phi = 0\ ,$$

again with $\alpha = m/2M$. Incidentally, one can use these results in eqns. and to eliminate $\dot\phi$ and $\ddot\phi$ in the expression for the constraint force, $Q_r = \lambda = \dot p_r - F_r$. One finds

$$\begin{aligned}\lambda &= -mR\,\frac{\dot\phi^2 + \omega_0^2\,\cos\phi}{1+\alpha\sin^2\!\phi} \\ &= -\frac{mR\omega_0^2}{(1+\alpha\sin^2\!\phi)^2}\left\{(1+\alpha)\left(\frac{\Omega^2}{\omega_0^2} - 4\,\sin^2(\tfrac{1}{2}\phi)\right) + (1+\alpha\sin^2\!\phi)\,\cos\phi\right\}\ .\end{aligned}$$

This last equation can be used to determine the angle of detachment, where λ vanishes and the mass m falls off the inside of the hoop. This is because the hoop can only supply a repulsive normal force to the mass m. This was worked out in detail in my lecture notes on constrained systems.

[2] A uniformly dense ladder of mass m and length 2ℓ leans against a block of mass M, as shown in Fig. 2. Choose as generalized coordinates the horizontal position X of the right end of the block, the angle θ the ladder makes with respect to the floor, and the coordinates (x, y) of the ladder's center-of-mass. These four generalized coordinates are not all independent, but instead are related by a certain set of constraints.

Recall that the kinetic energy of the ladder can be written as a sum $T_{\rm CM} + T_{\rm rot}$, where $T_{\rm CM} = \frac{1}{2}m(\dot x^2 + \dot y^2)$ is the kinetic energy of the center-of-mass motion, and $T_{\rm rot} = \frac{1}{2}I\dot\theta^2$, where I is the moment of inertial. For a uniformly dense ladder of length 2ℓ, $I = \frac{1}{3}m\ell^2$.

(a) Write down the Lagrangian for this system in terms of the coordinates X, θ, x, y, and their time derivatives.

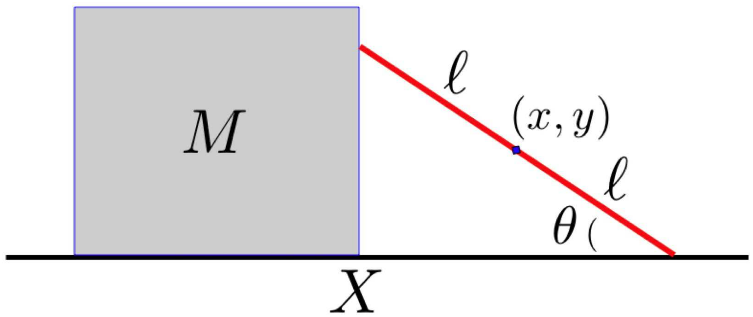

Figure 2: A ladder of length 2ℓ leaning against a massive block. All surfaces are frictionless..

Solution : We have $L = T - U$, hence

$$L = \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} I \dot{\theta}^2 - mgy \ .$$

(b) Write down all the equations of constraint.

Solution : There are two constraints, corresponding to contact between the ladder and the block, and contact between the ladder and the horizontal surface:

$$\begin{aligned} G_1(X, \theta, x, y) &= x - \ell \cos\theta - X = 0 \\ G_2(X, \theta, x, y) &= y - \ell \sin\theta = 0 \ . \end{aligned}$$

(c) Write down all the equations of motion.

Solution : Two Lagrange multipliers, λ_1 and λ_2, are introduced to effect the constraints. We have for each generalized coordinate q_σ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \equiv Q_\sigma \ ,$$

where there are $k = 2$ constraints. We therefore have

$$\begin{aligned} M\ddot{X} &= -\lambda_1 \\ m\ddot{x} &= +\lambda_1 \\ m\ddot{y} &= -mg + \lambda_2 \\ I\ddot{\theta} &= \ell \sin\theta \, \lambda_1 - \ell \cos\theta \, \lambda_2 \ . \end{aligned}$$

These four equations of motion are supplemented by the two constraint equations, yielding six equations in the six unknowns $\{X, \theta, x, y, \lambda_1, \lambda_2\}$.

(d) Find all conserved quantities.

Solution : The Lagrangian and all the constraints are invariant under the transformation

$$X \to X + \zeta \quad , \quad x \to x + \zeta \quad , \quad y \to y \quad , \quad \theta \to \theta \ .$$

The associated conserved 'charge' is

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = M\dot{X} + m\dot{x} \ .$$

Using the first constraint to eliminate x in terms of X and θ, we may write this as

$$\Lambda = (M+m)\dot{X} - m\ell \sin\theta\,\dot{\theta} \ .$$

The second conserved quantity is the total energy E. This follows because the Lagrangian and all the constraints are independent of t, and because the kinetic energy is homogeneous of degree two in the generalized velocities. Thus,

$$\begin{aligned} E &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 + mgy \\ &= \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}\Big(I + m\ell^2 - \tfrac{m}{M+m}\, m\ell^2 \sin^2\theta\Big)\,\dot{\theta}^2 + mg\ell\sin\theta \ , \end{aligned}$$

where the second line is obtained by using the constraint equations to eliminate x and y in terms of X and θ.

(e) What is the condition that the ladder detaches from the block? You do not have to solve for the angle of detachment! Express the detachment condition in terms of any quantities you find convenient.

Solution : The condition for detachment from the block is simply $\lambda_1 = 0$, *i.e.* the normal force vanishes.

Further analysis : It is instructive to work this out in detail (though this level of analysis was not required for the exam). If we eliminate x and y in terms of X and θ, we find

$$\begin{aligned} x &= X + \ell\cos\theta & \qquad y &= \ell\sin\theta \\ \dot{x} &= \dot{X} - \ell\sin\theta\,\dot{\theta} & \qquad \dot{y} &= \ell\cos\theta\,\dot{\theta} \\ \ddot{x} &= \ddot{X} - \ell\sin\theta\,\ddot{\theta} - \ell\cos\theta\,\dot{\theta}^2 & \qquad \ddot{y} &= \ell\cos\theta\,\ddot{\theta} - \ell\sin\theta\,\dot{\theta}^2 \ . \end{aligned}$$

We can now write

$$\lambda_1 = m\ddot{x} = m\ddot{X} - m\ell\sin\theta\,\ddot{\theta} - m\ell\cos\theta\,\dot{\theta}^2 = -M\ddot{X} \ ,$$

which gives

$$(M+m)\ddot{X} = m\ell\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big) \ ,$$

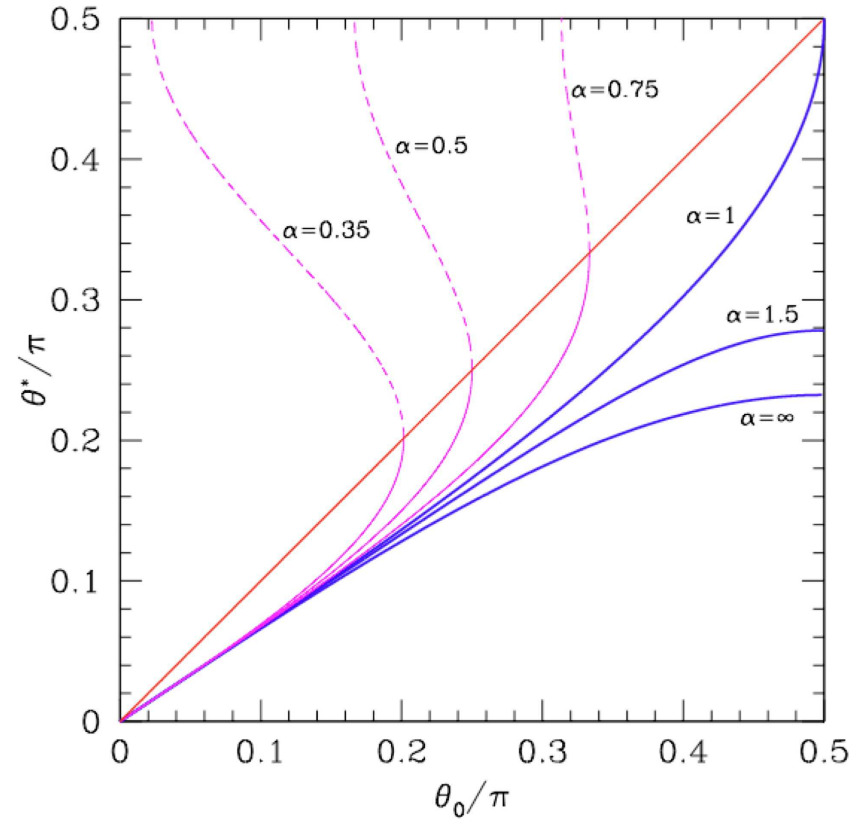

Figure 3: Plot of θ^* *versus* θ_0 for the ladder-block problem. Allowed solutions, shown in blue, have $\alpha \geq 1$, and thus $\theta^* \leq \theta_0$. Unphysical solutions, with $\alpha < 1$, are shown in magenta. The line $\theta^* = \theta_0$ is shown in red.

and hence

$$Q_x = \lambda_1 = -\frac{Mm}{m+m}\,\ell\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) .$$

We also have

$$\begin{aligned} Q_y = \lambda_2 &= mg + m\ddot{y} \\ &= mg + m\ell\big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\big) . \end{aligned}$$

We now need an equation relating $\ddot{\theta}$ and $\dot{\theta}$. This comes from the last of the equations of motion:

$$\begin{aligned} I\ddot{\theta} &= \ell\sin\theta\,\lambda_1 - \ell\cos\theta\lambda_2 \\ &= -\tfrac{Mm}{M+m}\,\ell^2\big(\sin^2\!\theta\,\ddot{\theta} + \sin\theta\cos\theta\,\dot{\theta}^2\big) - mg\ell\cos\theta - m\ell^2\big(\cos^2\!\theta\,\ddot{\theta} - \sin\theta\cos\theta\,\dot{\theta}^2\big) \\ &= -mg\ell\cos\theta - m\ell^2\Big(1 - \tfrac{m}{M+m}\,\sin^2\!\theta\Big)\,\ddot{\theta} + \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 . \end{aligned}$$

Collecting terms proportional to $\ddot{\theta}$, we obtain

$$\big(I + m\ell^2 - \tfrac{m}{M+m}\,\sin^2\!\theta\big)\,\ddot{\theta} = \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 - mg\ell\,\cos\theta\ .$$

We are now ready to demand $Q_x = \lambda_1 = 0$, which entails

$$\ddot{\theta} = -\frac{\cos\theta}{\sin\theta}\,\dot{\theta}^2\ .$$

Substituting this into eqn. , we obtain

$$\big(I + m\ell^2\big)\,\dot{\theta}^2 = mg\ell\,\sin\theta\ .$$

Finally, we use this result to substitute for $\dot{\theta}^2$ to obtain an equation for the detachment angle θ^*:

$$E - \frac{\Lambda^2}{2(M+m)} = mg\ell\sin\theta_0 = \left(3 - \frac{m}{M+m}\cdot\frac{m\ell^2}{I+m\ell^2}\,\sin^2\!\theta^*\right)\cdot\tfrac{1}{2}mg\ell\,\sin\theta^*\ .$$

If our initial conditions are that the system starts from rest[1] with an angle of inclination θ_0, then the detachment condition becomes

$$\begin{aligned}\sin\theta_0 &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\left(\frac{m}{M+m}\right)\left(\frac{m\ell^2}{I+m\ell^2}\right)\sin^3\!\theta^* \\ &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\,\alpha^{-1}\,\sin^3\!\theta^*\ ,\end{aligned}$$

where

$$\alpha \equiv \left(1 + \frac{M}{m}\right)\left(1 + \frac{I}{m\ell^2}\right)\ .$$

Note that $\alpha \geq 1$, and that when $M/m = \infty$[2], we recover $\theta^* = \sin^{-1}\big(\tfrac{2}{3}\sin\theta_0\big)$. For finite α, the ladder detaches at a larger value of θ^*. A sketch of θ^* *versus* θ_0 is provided in Fig. 3. Note that, provided $\alpha \geq 1$, detachment always occurs for some unique value θ^* for each θ_0.

[1]'Rest' means that the initial velocities are $\dot{X} = 0$ and $\dot{\theta} = 0$, and hence $\Lambda = 0$ as well.
[2]I must satisfy $I \leq m\ell^2$.