PHYSICS 110A : MECHANICS 1
PROBLEM SET #6 SOLUTIONS

[**1**] A point mass m slides frictionlessly, under the influence of gravity, along a massive ring of radius a and mass M. The ring is affixed by horizontal springs to two fixed vertical surfaces, as depicted in Fig. 1. All motion is within the plane of the figure.

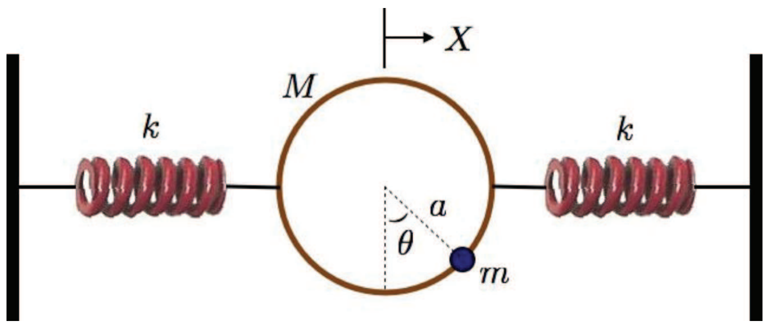


Figure 1: A point mass m slides frictionlessly along a massive ring of radius a and mass M, which is affixed by horizontal springs to two fixed vertical surfaces.

(a) Choose as generalized coordinates the horizontal displacement X of the center of the ring with respect to equilibrium, and the angle θ a radius to the mass m makes with respect to the vertical (see Fig. 1). You may assume that at $X = 0$ the springs are both unstretched. Find the Lagrangian $L(X, \theta, \dot{X}, \dot{\theta}, t)$.

(b) Derive the equations of motion.

(c) Find the eigenfrequencies for the small oscillations of this system. You may find it convenient to define $\Omega^2 \equiv 2k/M$, $\nu^2 \equiv g/a$, and $r \equiv m/M$.

Solution:

(a) The coordinates of the mass point are

$$x = X + a\sin\theta \quad , \quad y = -a\cos\theta \ .$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2} m \big(\dot{X} + a\cos\theta\,\dot{\theta}\big)^2 + \tfrac{1}{2} m a^2 \sin^2\!\theta\,\dot{\theta}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2} m a^2 \dot{\theta}^2 + m a\cos\theta\,\dot{X}\,\dot{\theta} \ . \end{aligned}$$

The potential energy is

$$U = kX^2 - mga\cos\theta \ .$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2} m a^2 \dot{\theta}^2 + m a\cos\theta\,\dot{X}\,\dot{\theta} - kX^2 + mga\cos\theta \ .$$

(b) The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} \ ,$$

for each generalized coordinate q_σ. For X we have

$$(M+m)\ddot{X} + ma\cos\theta\,\ddot{\theta} - ma\sin\theta\,\dot{\theta}^2 = -2kX \ .$$

For θ,

$$ma^2\,\ddot{\theta} + ma\cos\theta\ddot{X} = -mga\sin\theta \ .$$

(c) Linearizing the equations of motion, we have

$$\begin{pmatrix} (M+m)\omega^2 - 2k & ma\omega^2 \\ ma\omega^2 & ma^2\omega^2 - mga \end{pmatrix} \begin{pmatrix} \hat{X}(\omega) \\ \hat{\theta}(\omega) \end{pmatrix} = 0 \quad .$$

Setting the determinant to zero, we arrive at the quadratic equation

$$\omega^4 - \big(\varOmega^2 + (1+r)\nu^2\big)\omega^2 + \nu^2\varOmega^2 = 0 \quad ,$$

and the eigenfrequencies are given by

$$\omega_\pm^2 = \tfrac{1}{2}\big(\varOmega^2 + (1+r)\,\nu^2\big) \pm \tfrac{1}{2}\sqrt{\big(\varOmega^2 - (1+r)\nu^2\big)^2 + 4r\varOmega^2\nu^2} \quad .$$

[2] [José and Saletan problem 3.11] Consider a three-dimensional one-particle system whose potential energy in cylindrical polar coordinates $\{\rho, \phi, z\}$ is of the form $V(\rho, k\phi + z)$, where k is a constant.

(a) Find a symmetry of the Lagrangian and use Noether's theorem to obtain the constant of the motion associated with it.

(b) Write down at least one other constant of the motion.

(c) Obtain an explicit expression for the vector field $d\boldsymbol{\varphi}/dt = \boldsymbol{V}(\boldsymbol{\varphi})$, where

$$\boldsymbol{\varphi} = \begin{pmatrix} \rho \\ \phi \\ z \\ \dot{\rho} \\ \dot{\phi} \\ \dot{z} \end{pmatrix} \quad ,$$

and use it to verify that the functions found in (a) and (b) are indeed constants of the motion.

Solution:

(a) We have

$$L = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) - V(\rho, k\phi + z) \ .$$

Consider now the one-parameter family of coordinate transformations,

$$\begin{aligned}\tilde\phi(\lambda) &\equiv \phi + \lambda \\ \tilde z(\lambda) &\equiv z - \lambda k \ .\end{aligned}$$

Clearly

$$k\tilde\phi + \tilde z = k\phi + z \ ,$$

hence L does not vary with λ, and therefore

$$Q = \sum_\sigma \frac{\partial L}{\partial \dot q_\sigma}\,\frac{\partial \tilde q_\sigma}{\partial \lambda}\bigg|_{\lambda=0} = \ m\rho^2\dot\phi - mk\dot z$$

is conserved: $\dot Q = 0$.

(b) Since $\frac{\partial L}{\partial t} = 0$, the Hamiltonian H is conserved. And since the kinetic energy is homogeneous of degree two in the generalized velocities $\{\dot\rho, \dot\phi, \dot z\}$, the Hamiltonian is simply the total energy: $H = T + E$. Thus,

$$E = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) + V(\rho, k\phi + z)$$

is conserved: $\dot E = 0$.

(c) The 'dynamical vector field' Δ is simply the total time derivative, expressed in terms of derivatives with respect to coordinates and velocities:

$$\begin{aligned}\Delta = \frac{d}{dt} &= \dot q_\sigma\,\frac{\partial}{\partial q_\sigma} + \ddot q_\sigma\,\frac{\partial}{\partial \dot q_\sigma} \\ &= \dot\rho\,\frac{\partial}{\partial\rho} + \dot\phi\,\frac{\partial}{\partial\phi} + \dot z\,\frac{\partial}{\partial z} + \ddot\rho\,\frac{\partial}{\partial\dot\rho} + \ddot\phi\,\frac{\partial}{\partial\dot\phi} + \ddot z\,\frac{\partial}{\partial\dot z}\end{aligned}$$

The generalized accelerations follow from the equations of motion,

$$\begin{aligned}m\ddot\rho &= -\frac{\partial V}{\partial\rho} \\ \frac{d}{dt}\big(m\rho^2\dot\phi\big) &= -\frac{\partial V}{\partial\phi} = -k\,\frac{\partial V}{\partial z} \\ m\ddot z &= -\frac{\partial V}{\partial z} \ ,\end{aligned}$$

which yield

$$\ddot\rho = -\frac{1}{m}\,\frac{\partial V}{\partial\rho} \quad , \quad \ddot\phi = -\frac{k}{m\rho^2}\,\frac{\partial V}{\partial z} - \frac{2\,\dot\rho\,\dot\phi}{\rho} \quad , \quad \ddot z = -\frac{1}{m}\,\frac{\partial V}{\partial z} \ .$$

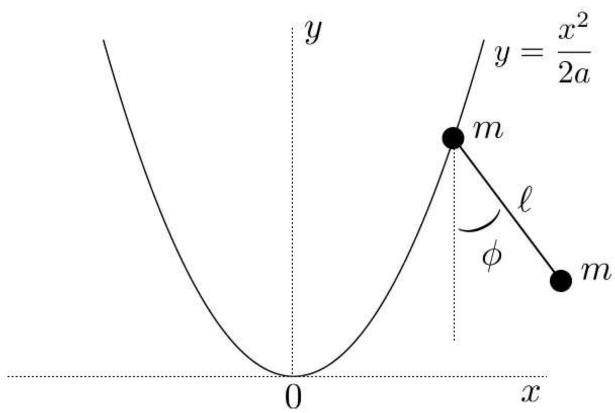
Therefore, we have

$$
\begin{aligned}
\Delta Q &= \Delta\left(m\rho^2\dot\phi - mk\dot z\right)\\
&= \dot\rho\cdot 2m\rho\dot\phi + \ddot\phi\cdot m\rho^2 + \ddot z\cdot(-mk)\\
&= 2m\rho\,\dot\rho\,\dot\phi + \left(-\frac{k}{m\rho^2}\,\frac{\partial V}{\partial z} - \frac{2\,\dot\rho\,\dot\phi}{\rho}\right)\cdot m\rho^2 + \left(-\frac{1}{m}\,\frac{\partial V}{\partial z}\right)\cdot(-mk) = 0\ .
\end{aligned}
$$

We also have

$$
\begin{aligned}
\Delta E &= \Delta\,\tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) + \Delta\,V(\rho, k\phi + z)\\
&= \dot\rho\,\frac{\partial V}{\partial\rho} + \dot\phi\,\frac{\partial V}{\partial\phi} + \dot z\,\frac{\partial V}{\partial z} + m\dot\rho\,\ddot\rho + m\rho^2\dot\phi\,\ddot\phi + m\dot z\,\ddot z\\
&= \dot\rho\,\frac{\partial V}{\partial\rho} + \dot\phi\cdot k\,\frac{\partial V}{\partial z} + \dot z\,\frac{\partial V}{\partial z} + m\dot\rho\cdot\left(-\frac{1}{m}\,\frac{\partial V}{\partial\rho}\right)\\
&\qquad + m\rho^2\dot\phi\cdot\left(-\frac{k}{m\rho^2}\,\frac{\partial V}{\partial z} - \frac{2\,\dot\rho\,\dot\phi}{\rho}\right) + m\dot z\cdot\left(-\frac{1}{m}\,\frac{\partial V}{\partial z}\right) = 0\ .
\end{aligned}
$$

[3] A mass m moves frictionlessly under the influence of gravity along the curve $y = x^2/2a$. Attached to the mass is a massless rigid rod of length ℓ, at the end of which is an identical mass m. The rod is constrained to swing in the (x, y) plane, as depicted in the figure below.



(a) Choose as generalized coordinates x and ϕ. Find the kinetic energy T and potential energy U.

(b) Find the conjugate momenta p_x and p_ϕ.

(c) Find the conjugate forces F_x and F_ϕ.

(d) Write the Lagrangian for small oscillations, when $x^2 \ll a^2$ and $\phi \ll \pi$.

(e) Find the coupled equations of motion for small oscillations.

(f) Find the eigenfrequencies for the two modes of oscillation.

Solution:

(a) The coordinates of the mass on the curve are $(x_1\,,\,y_1) = (x\,,\,x^2/2a)$. Note $\dot{y} = (x/a)\,\dot{x}$. The coordinates for the hanging mass are $(x_2\,,\,y_2) = (x + \ell\sin\phi\,,\,x^2/2a - \ell\cos\phi)$. The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \\ &= m\bigg(1 + \frac{x^2}{a^2}\bigg)\dot{x}^2 + \tfrac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\Big(\cos\phi + \frac{x}{a}\,\sin\phi\Big)\,\dot{x}\,\dot{\phi} \quad . \end{aligned}$$

The potential energy is

$$U = mg(y_1 + y_2) = \frac{mg}{a}\,x^2 - mg\ell\cos\phi \quad .$$

(b) We have $L = T - U$ and

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = 2m\bigg(1 + \frac{x^2}{a^2}\bigg)\dot{x} + m\ell\Big(\cos\phi + \frac{x}{a}\,\sin\phi\Big)\,\dot{\phi} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m\ell^2\dot{\phi} + m\ell\Big(\cos\phi + \frac{x}{a}\,\sin\phi\Big)\,\dot{x} \quad . \end{aligned}$$

(c) We have

$$\begin{aligned} F_x &= \frac{\partial L}{\partial x} = \frac{2mx}{a^2}\,\dot{x}^2 + \frac{m\ell}{a}\,\sin\phi\,\dot{x}\,\dot{\phi} - \frac{2mg}{a}\,x \\ F_\phi &= \frac{\partial L}{\partial \phi} = m\ell\Big(-\sin\phi + \frac{x}{a}\,\cos\phi\Big)\,\dot{x}\dot{\phi} - mg\ell\sin\phi \quad . \end{aligned}$$

(d) For small $|x|$ and $|\phi|$, the Lagrangian is a quadratic form in the generalized coordinates and velocities, *viz.*

$$\widetilde{L} = m\dot{x}^2 + \tfrac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\,\dot{x}\,\dot{\phi} - \frac{mg}{a}\,x^2 - \tfrac{1}{2}mgl\phi^2 + mgl \quad .$$

(e) Now we have

$$p_x = 2m\dot{x} + m\ell\dot{\phi} \qquad , \qquad p_\phi = m\ell^2\dot{\phi} + m\ell\dot{x}$$

and

$$F_x = -\frac{2mg}{a}\,x \qquad , \qquad F_\phi = -mg\ell\,\phi \quad .$$

The equations of motion are therefore

$$\begin{aligned} 2m\ddot{x} + m\ell\ddot{\phi} &= -\frac{2mg}{a}\,x \\ m\ell\ddot{x} + m\ell^2\ddot{\phi} &= -mgl\,\phi \quad . \end{aligned}$$

(f) Define the dimensionless quantity $u \equiv x/\ell$. Then the equations of motion are

$$\begin{aligned} 2\ddot{u} + \ddot{\phi} &= -2\varOmega_1^2\, u \\ \ddot{u} + \ddot{\phi} &= -\varOmega_2^2\, \phi \quad , \end{aligned}$$

where $\varOmega_1 \equiv (g/a)^{1/2}$ and $\varOmega_2 \equiv (g/\ell)^{1/2}$. Writing $u(t) = u_0\, e^{-i\omega t}$ and $\phi(t) = \phi_0 e^{-i\omega t}$, we have the coupled algebraic equations

$$\begin{pmatrix} 2(\omega^2 - \varOmega_1^2) & \omega^2 \\ \omega^2 & \omega^2 - \varOmega_2^2 \end{pmatrix} \begin{pmatrix} u_0 \\ \phi_0 \end{pmatrix} = 0 \quad .$$

Setting the determinant to zero, we obtain a quadratic equation in ω^2,

$$\omega^4 - 2(\varOmega_1^2 + \varOmega_2^2)\omega^2 + \varOmega_1^2\varOmega_2^2 = 0 \quad ,$$

with solutions

$$\omega_\pm^2 = \varOmega_1^2 + \varOmega_2^2 \pm \sqrt{\varOmega_1^4 + \varOmega_2^4} \quad .$$