PHYSICS 110A : MECHANICS 1
PROBLEM SET #4 SOLUTIONS

[1] An electrical circuit consists of a resistor R and a capacitor C connected in series to an emf $V(t)$.

(a) Write down the differential equation for the charge $Q(t)$ on one of the capacitor plates.

(b) Solve the homogeneous equation for $Q(t)$, *i.e.* find $Q(t)$ when $V(t) = 0$ subject to arbitrary initial value of $Q(0)$.

(c) Solve for the current $I(t)$ flowing in the circuit when $V(t) = V_0\,\Theta(t)$. Assume $Q(0) = 0$.

(d) Solve for $I(t)$ when $V(t) = V_0\,\sin(\Omega t)\,\Theta(t)$ and $Q(0)$=0.

For parts (c) and (d), you should use the Green's function formalism in the time domain. The following integral may prove useful:

$$\int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{e^{-i\omega s}}{1-i\omega\tau} = \frac{1}{\tau}\,e^{-s/\tau}\,\Theta(s) \quad .$$

Solution :

(a) Kirchoff's law applied to the circuit yields $RI + C^{-1}Q = V(t)$, *i.e.*

$$\dot{Q} + \frac{Q}{RC} = \frac{V(t)}{R} \quad .$$

We identify $RC \equiv RC$ as the time constant.

(b) Solving when $V(t) = 0$, we obtain

$$Q(t) = Q(0)\,e^{-t/RC} \quad .$$

(c) Taking the Fourier transform of the equation in part (a), we have

$$\hat{Q}(\omega) = \frac{C\hat{V}(\omega)}{1-i\omega RC} \quad .$$

Thus, using the integral given in the problem statement, we have

$$Q(t) = Q(0)\,e^{-t/RC} + \frac{1}{R}\int\limits_0^t dt'\,V(t')\,e^{-(t-t')/RC} \quad .$$

For $V(t') = V_0\Theta(t')$ and $Q(0) = 0$, we have

$$Q(t) = CV_0\left(1 - e^{-t/RC}\right)\Theta(t) \quad .$$

The current is then

$$I(t) = \dot{Q}(t) = \frac{V_0}{R}\, e^{-t/RC}\, \Theta(t) \quad .$$

Another approach is to use an integrating factor:

$$e^{-t/RC}\,\frac{d}{dt}\Big[Q(t)\, e^{t/RC}\Big] = \frac{V(t)}{R} \qquad \Rightarrow \qquad \frac{d}{dt}\Big[Q(t)\, e^{t/RC}\Big] = \frac{V(t)}{R}\, e^{t/RC} \quad .$$

Now integrate:

$$\begin{gathered}\int\limits_0^t \! dt'\, \frac{d}{dt'}\Big[Q(t')\, e^{t'/RC}\Big] = \frac{1}{R}\int\limits_0^t \! dt'\, V(t')\, e^{t/RC} = \frac{V_0}{R}\int\limits_0^t \! dt'\, e^{t/RC} \\ Q(t)\, e^{t/RC} - Q(0) = CV_0\left(e^{t/RC} - 1\right)\Theta(t) \quad ,\end{gathered}$$

which yields

$$Q(t) = Q(0)\, e^{-t/RC} + CV_0\left(1 - e^{-t/RC}\right)\Theta(t) \quad .$$

(d) We have

$$\begin{aligned} Q(t) &= \frac{V_0}{R}\int\limits_0^t \! dt'\; \sin(\Omega t')\, e^{-(t-t')/\tau} \\ &= \frac{V_0}{R}\, e^{-t/\tau}\, \mathsf{Im}\!\int\limits_0^t \! dt'\; e^{(\tau^{-1}+i\Omega)\, t'} \\ &= CV_0\, \mathsf{Im}\left\{\frac{1}{1+i\Omega\tau}\left[e^{i\Omega t} - e^{-t/\tau}\right]\right\} \\ &= CV_0\left\{\frac{\sin(\Omega t) - \Omega t\, \cos(\Omega t)}{1+\Omega^2\tau^2} + \frac{\Omega\tau}{1+\Omega^2\tau^2}\, e^{-t/\tau}\right\} \quad . \end{aligned}$$

[2] Do either of the following:

(a) A forced, damped harmonic oscillator obeys the equation of motion

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\, e^{-\gamma t}\, \Theta(t) \quad .$$

Compute $x(t)$ assuming $x(0) = \dot{x}(0) = 0$.

(b) A forced, damped harmonic oscillator obeys the equation of motion

$$\Big(\frac{d}{dt} + \alpha\Big)\Big(\frac{d}{dt} + \beta\Big) x = f_0\, e^{-\gamma t}\, \Theta(t) \quad .$$

Compute $x(t)$ assuming $x(0) = \dot{x}(0) = 0$.

Solution :

(a) The Green's function is

$$G(s) = \nu^{-1}\, e^{-\beta s}\, \sin(\nu s)\, \Theta(s) \quad .$$

Thus,

$$\begin{aligned}
x(t) &= \frac{f_0}{\nu} \int_0^t \! dt'\, e^{-\beta(t-t')}\, \sin\big(\nu(t-t')\big)\, e^{-\gamma t'} \\
&= \frac{f_0}{\nu}\, e^{-\beta t}\, \mathsf{Im}\left\{ e^{i\nu t} \int_0^t \! dt'\, e^{\beta t'}\, e^{-i\nu t'}\, e^{-\gamma t'} \right\} \\
&= \frac{f_0}{\nu}\, \mathsf{Im}\bigg\{ \frac{e^{i\nu t}}{\beta - \gamma - i\nu} \Big[e^{(\beta-\gamma-i\nu)t} - 1 \Big] \bigg\} \\
&= \frac{f_0}{\nu} \left\{ \frac{\nu\, e^{-\gamma t} - \big[\nu \cos(\nu t) + (\beta-\gamma)\sin(\nu t)\big]\, e^{-\beta t}}{(\beta-\gamma)^2 + \nu^2} \right\} \quad .
\end{aligned}$$

(b) In fact the two equations are equivalent provided we identify $2\beta = \alpha + \tilde{\beta}$ and $\omega_0^2 = \alpha\tilde{\beta}$, where $\tilde{\beta}$ is our temporarily renamed β from part (b). But let's try a different solution. At the level of differential operators, we have

$$\frac{d}{dt} + \alpha = e^{-\alpha t}\, \frac{d}{dt}\, e^{\alpha t} \quad ,$$

which means

$$\Big(\frac{d}{dt} + \alpha\Big)\Big(\frac{d}{dt} + \beta\Big) x = e^{-\alpha t}\, \frac{d}{dt}\Big[e^{\alpha t}\, (\dot{x} + \beta x)\Big] \quad .$$

Thus, our second order ODE may be recast as

$$\frac{d}{dt}\Big[e^{\alpha t}\, (\dot{x} + \beta x)\Big] = f_0\, e^{(\alpha-\gamma)t}\, \Theta(t) \quad .$$

Now replace t in the above equation by t' and integrate over the interval $t' \in [0,t]$, resulting in

$$e^{\alpha t}\Big(\dot{x}(t) + \beta x(t)\Big) - \Big(\dot{x}(0) + \beta x(0)\Big) = f_0 \int_0^t \! dt'\, e^{(\alpha-\gamma)t'} = \frac{f_0}{\alpha - \gamma}\Big(e^{(\alpha-\gamma)t} - 1\Big) \quad .$$

This is equivalent to

$$\begin{aligned}
e^{-\beta t}\, \frac{d}{dx}\Big[e^{\beta t}\, x(t)\Big] &= \dot{x}(t) + \beta x(t) \\
&= \Big(\beta x(0) + \dot{x}(0)\Big)\, e^{-\alpha t} + \frac{f_0}{\alpha - \gamma}\Big(e^{-\gamma t} - e^{-\alpha t}\Big) \quad .
\end{aligned}$$

Now multiply both sides by $e^{\beta t}$, send $t \to t'$, and then integrate over $t' \in [0, t]$, yielding

$$e^{\beta t}\, x(t) - x(0) = \Big(\beta x(0) + \dot{x}(0)\Big)\, \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha} + \frac{f_0}{\alpha - \gamma} \left[\frac{e^{(\beta-\gamma)t} - 1}{\beta - \gamma}\right] - \frac{f_0}{\alpha - \gamma} \left[\frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha}\right] \quad ,$$

which says

$$x(t) = \left(\frac{\beta\, e^{-\alpha t} - \alpha\, e^{\beta t}}{\beta - \alpha}\right) x(0) + \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}\, \dot{x}(0)$$

$$\qquad - \frac{f_0}{(\beta - \alpha)(\gamma - \beta)(\alpha - \gamma)} \Big[(\gamma - \beta)\, e^{-\alpha t} + (\alpha - \gamma)\, e^{-\beta t} + (\beta - \alpha)\, e^{-\gamma t}\Big] \quad .$$

Using L'Hospital's rule, one can check that the RHS remains finite in the limits $\alpha \to \beta$, $\beta \to \gamma$, and $\gamma \to \alpha$. With the initial conditions $x(0) = \dot{x}(0) = 0$ we have

$$x(t) = -\frac{f_0}{(\beta - \alpha)(\gamma - \beta)(\alpha - \gamma)} \Big[(\gamma - \beta)\, e^{-\alpha t} + (\alpha - \gamma)\, e^{-\beta t} + (\beta - \alpha)\, e^{-\gamma t}\Big] \quad .$$