PHYSICS 110A : MECHANICS 1
PROBLEM SET #2 SOLUTIONS

[**1**] Using the method of partial fractions, solve the ODE

$$\frac{du}{dt} = (u-1)(u-2)(u-3)$$

for $t(u)$. Sketch the phase flow along the real u line, and the integral curves in the (t,u) plane. Show that for $u_0 < 1$ or $u_0 > 3$ that $u(t)$ flows to $u = \pm\infty$ in a *finite* time t^*, but that for $u_0 \in (1,3)$ the flow is toward the stable fixed point $u^* = 2$, which takes infinite time to reach.

Solution:

We begin with a review of the method of partial fractions as applied to our problem. We have

$$\frac{1}{(u-a)(u-b)(u-c)} = \frac{\alpha}{u-a} + \frac{\beta}{u-b} + \frac{\gamma}{u-c} \quad .$$

where

$$\alpha = \frac{1}{(a-b)(a-c)} \quad , \quad \beta = \frac{1}{(b-a)(b-c)} \quad , \quad \gamma = \frac{1}{(c-a)(c-b)} \quad .$$

Thus,

$$\frac{du}{(u-a)(u-b)(u-c)} = \alpha\, d\log(u-a) + \beta\, d\log(u-b) + \gamma\, d\log(u-c) = dt \quad .$$

Integrating, then, we have

$$(u-a)^{\alpha}\,(u-b)^{\beta}\,(u-c)^{\gamma} = (u_0-a)^{\alpha}\,(u_0-b)^{\beta}\,(u_0-c)^{\gamma}\,e^{t}$$

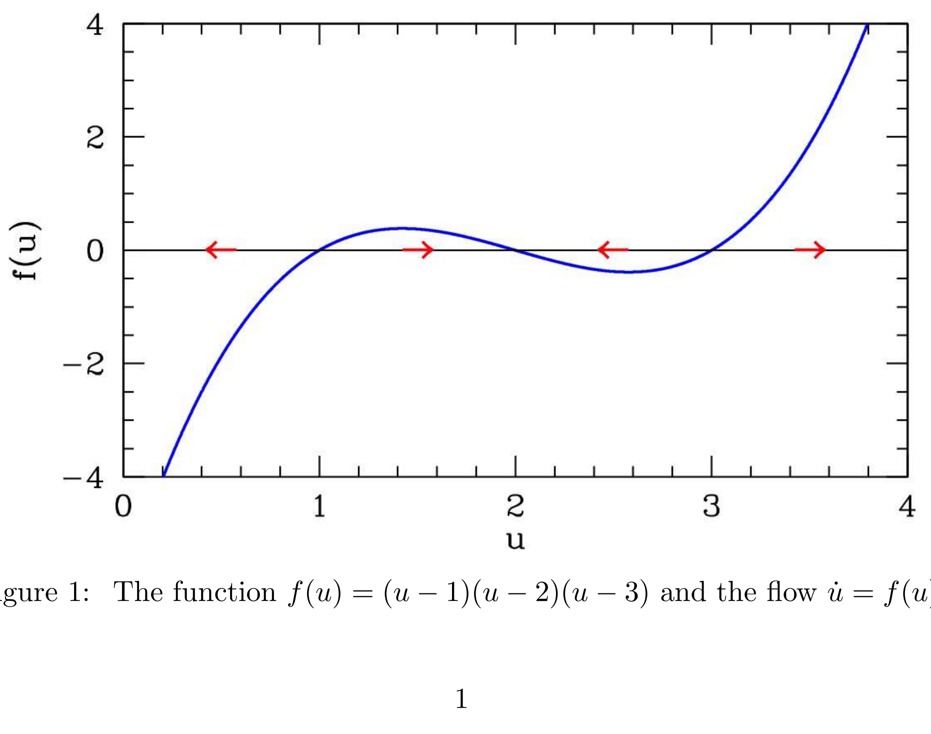

Figure 1: The function $f(u) = (u-1)(u-2)(u-3)$ and the flow $\dot{u} = f(u)$.

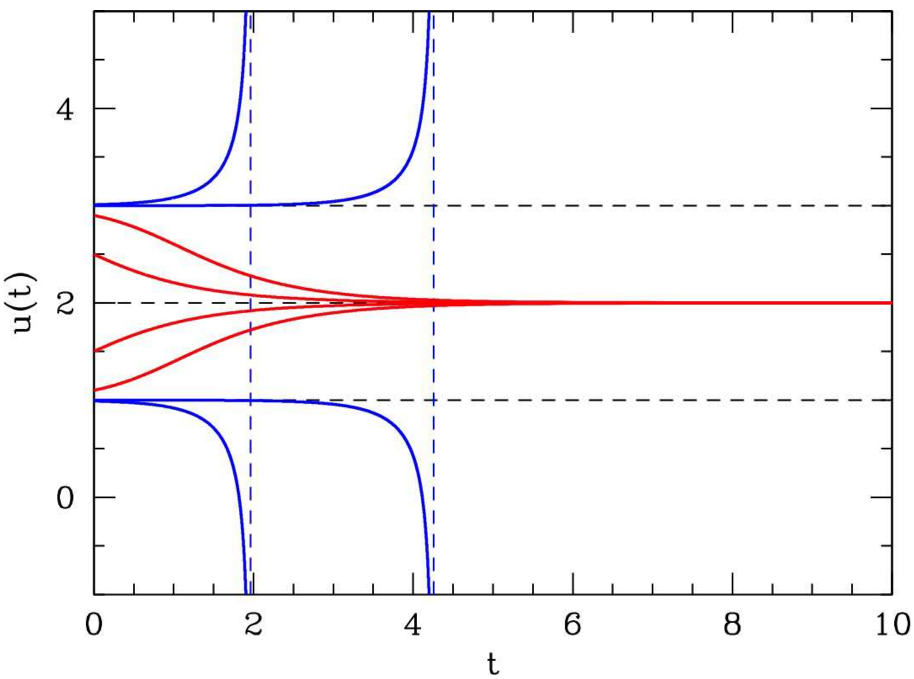

Figure 2: Integral curves for problem 1.

With $a = 1$, $b = 2$, and $c = 3$ we have $\alpha = \frac{1}{2}$, $\beta = -1$, and $\gamma = \frac{1}{2}$. Defining $v \equiv u - 2$, we then have

$$\sqrt{1 - v^{-2}} = \sqrt{1 - v_0^{-2}}\, e^t \quad ,$$

which yields

$$t(v) = \frac{1}{2} \log \left| \frac{1 - v^{-2}}{1 - v_0^{-2}} \right|$$

and

$$v(t) = \frac{v_0}{\sqrt{v_0^2 + (1 - v_0^2)\, \exp(2t)}} \quad .$$

Recall that $u = v + 2$. We see that if $|v_0| < 1$, *i.e.* if $u_0 \in (1, 3)$, then as $t \to \infty$ we have $v(t) \to 0$, which is to say $u(t) \to 2$. For $|v_0| > 1$, which is to say $u_0 < 1$ or $u_0 > 3$, the denominator vanishes at a finite time t^*, where

$$t^*(v_0) = \tfrac{1}{2} \log \left(\frac{v_0^2}{v_0^2 - 1} \right) \quad ,$$

and as $t \to t^*$ we have $v(t) \to +\infty$ for $v_0 > 0$ and $v(t) \to -\infty$ for $v_0 < 0$. The particle escapes to infinity in a finite time, because the velocity function $f(u)$ grows faster than linearly, hence, with $u_0 > 3$, for example,

$$\int_{u_0}^{\infty} \frac{du}{f(u)} < \infty$$

converges.

[2] Consider the $n = 2$ dynamical system given by

$$\frac{dx}{dt} = x - y - x^3 \qquad , \qquad \frac{dy}{dt} = rxy - y^2 \quad ,$$

where $r > 0$.

(a) Assuming $r > 1$, how many fixed points are there? Find them. *Hint: Start with the second equation.*

(b) Show that for $r < 1$ there are two more fixed points. Find them.

(c) Expanding about a fixed point (x^*, y^*), with $u_x \equiv x - x^*$ and $u_y = y - y^*$, the linearized dynamics takes the form $\dot{\boldsymbol{u}} = M\boldsymbol{u}$, where M is a 2×2 matrix. Find an expression for M at the fixed point (x^*, y^*).

(d) What are the eigenvalues of the linearized system $\dot{u} = Mu$ at $(x^*, y^*) = (0, 0)$?

Solution:

(a) Starting with the second equation, we set $\dot{y} = (rx - y)y = 0$, with solutions $y = 0$ and $y = rx$. For $y = 0$, we solve $\dot{x} = x(1 - x^2) = 0$ to obtain $x = 0$ and $x = \pm 1$. This gives us three roots, at $(0, 0)$, $(1, 0)$, and $(-1, 0)$.

(b) When $y = rx$, the first equation gives $\dot{x} = (1 - r - x^2)x = 0$, the solutions of which are $x = 0$ and $x = \pm\sqrt{1 - r}$. The first of these fixed points lies at $(0, 0)$, which we already found in part (a). The second two fixed points lie at $\pm\big(\sqrt{1 - r}, r\sqrt{1 - r}\big)$, and are real only for $r < 1$.

(c) We have $V_x(x, y) = x - y - x^3$ and $V_y(x, y) = rxy - y^2$. Thus,

$$M = \begin{pmatrix} \partial V_x/\partial x & \partial V_x/\partial y \\ \partial V_y/\partial x & \partial V_y/\partial y \end{pmatrix}_{(x^*, y^*)} = \begin{pmatrix} 1 - 3x^{*2} & -1 \\ ry^* & rx^* - 2y^* \end{pmatrix} \quad .$$

(d) At $(x^*, y^*) = (0, 0)$, we have $M = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. This gives $\mathsf{Tr}\, M = 1$ and $\mathsf{det}\, M = 0$, hence the eigenvalues are 1 and 0. Thus one unstable direction, associated with the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and eigenvalue 1, and a *fixed line*, associated with the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

[3] Consider the $n = 2$ dynamical system

$$\frac{d}{dt}\begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -U'(\phi) \end{pmatrix} \quad ,$$

where $U(\phi) = -\cos\phi + 2r\sin^2\!\phi$ with $r \geq 0$. Phase space is thus a cylinder: $(\phi, \omega) \in S^1 \times \mathbb{R}$.

(a) Show that the energy $E = \frac{1}{2}\,\omega^2 + U(\phi)$ is conserved.

(b) Show that there is a critical value r_c such that for $r < r_c$ the potential $U(\phi)$ has a single minimum at $\phi = 0$ and a single maximum at $\phi = \pm\pi$, but for $r > r_c$, there is a global minimum at $\phi = 0$, a local minimum at $\phi = \pm\pi$, and two local maxima at $\phi = \pm\phi^*(r)$. Find the value of r_c and the function $\phi^*(r)$.

(c) Sketch the potential $U(\phi)$ for $r = 0.15$. Plot the phase curves at energies $E_1 = 0$ and $E_2 = 1.5$.

(d) Sketch the potential $U(\phi)$ for $r = 0.80$. Find the separatrix energy corresponding to the energy $E^* = U_{\text{max}}$. Plot the phase curves at energies $E_1 = 0$, $E_2 = 1.2$, $E_3 = E^*$, and $E_4 = 2.2$.

Solution:

(a) With $E = \frac{1}{2}\omega^2 + U(\phi)$ we have

$$\frac{dE}{dt} = \omega\,\frac{d\omega}{dt} + \frac{dU(\phi)}{\phi}\,\frac{d\phi}{dt} = \big[\dot{\omega} + U'(\phi)\big]\omega = 0 \quad .$$

(b) We analyze the potential by finding its extrema, setting $U'(\phi) = 0$. We may write $U(\phi) = -c + 2r(1 - c^2)$ with $c \equiv \cos\phi$, hence $U'(\phi) = -(dU/dc)\,\sin\phi$, with

$$\frac{dU}{d\cos\phi} = -4rc - 1 \quad .$$

Thus, $U'(\phi) = 0$ whenever $\sin\phi = 0$ (*i.e.* the minima at $\phi = 0$ and $\phi = \pm\pi$), or $\cos\phi = c^* = -1/4r$ (*i.e.* the maxima at $\phi = \pm\cos^{-1} c^*$). The local maxima exist only if $|c^*| < 1$, which requires $r > r_c = \frac{1}{4}$, in which case $\phi^*(r) = \cos^{-1}(-1/4r)$.

(c) The plots are shown in the left panels of fig. 3.

(d) The energy of the separatrix ie $E^* = U_{\text{max}} = U\big(\phi^*(r)\big)$, thus

$$E^* = \frac{1}{4r} + 2r\Big(1 - \frac{1}{16r^2}\Big) = 2r + \frac{1}{8r} \quad .$$

For $r = 0.8$ we have $E^* = \frac{281}{160} = 1.75625$. For $E \in [-1, 1)$ there is a single closed phase curve centered at $\phi = 0$, topologically equivalent to a circle. As E exceeds $E = U(\pm\pi) = 1$, the phase curves develop another branch centered at $\phi = \pi$. (Thus $E = 1$ is also a separatrix, since the topology of the phase curves changes at that energy.) At $E = E^*$ these two disjoint portions of the phase curve merge. For $E > E^*$ there are again two disjoint phase curves, each of which winds fully around the (ϕ, ω) cylinder, one of which lies in the upper half of the cylinder ($\omega > 0$, corresponding to counterclockwise rotation of the angle ϕ), and the other in the lower half of the cylinder ($\omega < 0$, clockwise rotation of ϕ).

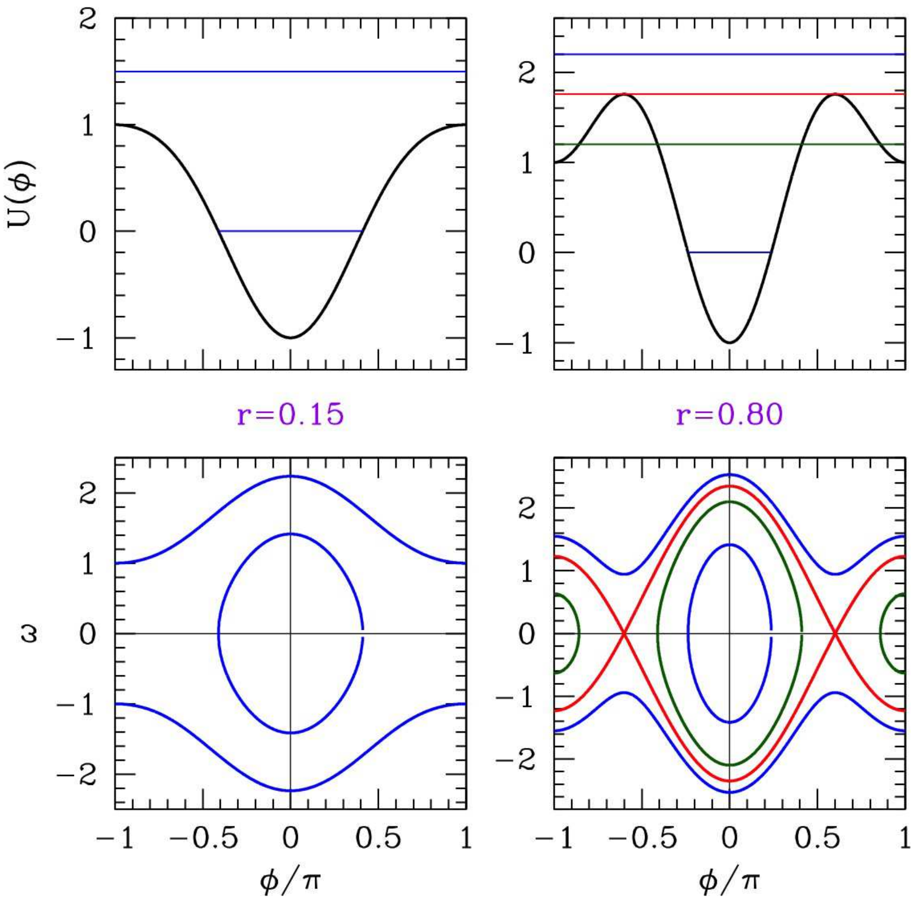

Figure 3: Potential and phase curves at representative energies for $r = 0.15$ (left panels) and $r = 0.80$ (right panels).