PHYSICS 110A : MECHANICS 1
PROBLEM SET #1 SOLUTIONS

[1] The one-component dynamical system $\dot{u} = f(u)$ is simple to solve graphically. Simply plot $f(u)$ *versus* u. The velocity $\dot{u}$ is then to the right in regions where $f(u) > 0$ and to the left in regions where $f(u) < 0$. Points where $f(u) = 0$, which are generically isolated, are *fixed points* of the dynamics, as discussed in §1.1.5 of the lecture notes.

Consider the modified logistic equation,

$$\frac{dN}{dt} = f(N) = rN\left(1 - \frac{N^2}{K^2}\right) \quad ,$$

with $r > 0$.

(a) Sketch $f(N)$ *versus* N (you may restrict your attention to $N \geq 0$) and draw arrows on the N axis in the direction of the flow $\dot{N}$.

(b) Identify all fixed points and classify them as either stable or unstable.

(c) Solve exactly for $N(t)$ subject to the initial condition $N(0) = N_0$ by using the method of partial fractions.

Solution:

(a) It's first convenient to adimensionalize the equation by defining $\nu \equiv N/K$ and $s \equiv rt$. Then one has

$$\frac{d\nu}{ds} = \nu(1 - \nu^2) \quad .$$

A graph of the vector field $f(\nu) = \nu(1 - \nu^2)$ for $\nu \geq 0$ is shown in fig. 1.

(b) There are two fixed points on the domain $\nu \in [0, \infty)$: $\nu = 0$ and $\nu = 1$ (equivalently, $N = 0$ and $N = K$). Note that $f'(\nu) = 1 - 3\nu^2$, hence $f'(0) = 1$ and $f'(1) = -2$. This means that $\nu = 0$ is an unstable fixed point, while $\nu = 1$ is a stable fixed point.

(c) We have

$$\frac{d\nu}{\nu(1 - \nu^2)} = ds \quad .$$

We now appeal to the method of partial fractions, writing $1 - \nu^2 = (1 + \nu)(1 - \nu)$ and

$$\begin{aligned}\frac{1}{\nu(1+\nu)(1-\nu)} &= \frac{A}{\nu} + \frac{B}{1+\nu} + \frac{C}{1-\nu} \\ &= \frac{A(1-\nu^2) + B\nu(1-\nu) + C\nu(1+\nu)}{\nu(1+\nu)(1-\nu)} \\ &= \frac{A + (B+C)\nu + (C-B-A)\nu^2}{\nu(1-\nu^2)} \quad .\end{aligned}$$

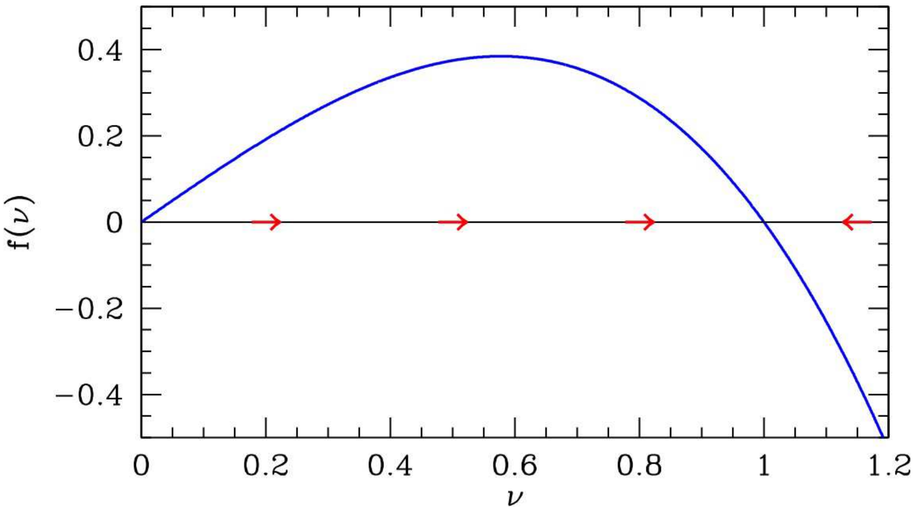

Figure 1: The function $f(\nu) = \nu(1-\nu^2)$ and the flow $\dot{\nu} = f(\nu)$.

Thus we demand $A + (B+C)\nu + (C-B-A)\nu^2 = 1$ for all ν, which yields $A = 1$, $B + C = 0$, and $C - B - A = 0$, which is to say $A = 1$, $B = -\frac{1}{2}$, and $C = \frac{1}{2}$. We thus have

$$\frac{1}{\nu(1-\nu^2)} = \frac{1}{\nu} - \frac{1}{2(1+\nu)} + \frac{1}{2(1-\nu)}$$

and

$$\begin{aligned} \frac{d\nu}{\nu(1-\nu^2)} &= d\log\nu - \tfrac{1}{2}\,d\log(1+\nu) - \tfrac{1}{2}\,d\log(1-\nu) \\ &= \tfrac{1}{2}\,d\log\left(\frac{\nu^2}{1-\nu^2}\right) = ds \quad . \end{aligned}$$

We may now integrate, obtaining

$$\frac{\nu^2}{1-\nu^2} = \frac{\nu_0^2}{1-\nu_0^2}\,\exp(2s) \quad ,$$

with $\nu_0 = \nu(0)$. This entails

$$\nu(s) = \frac{\nu_0}{\sqrt{\nu_0^2 + (1-\nu_0^2)\,\exp(-2s)}} \quad .$$

Equivalently,

$$N(t) = \frac{K N_0}{\sqrt{N_0^2 + (K^2 - N_0^2)\,\exp(-2rt)}} \quad .$$

Note that $N(t=0) = N_0$ and $N(t \to \infty) = K$, *i.e.* the flow is to the attractive (stable) fixed point.

[**2**] A particle of mass m moves in the one-dimensional potential

$$U(x) = U_0 \, \frac{x^2}{a^2} \, e^{-x/a} \quad .$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.

(b) Sketch a representative set of phase curves. Identify and classify any and all fixed points. Find the energy of each and every separatrix.

(c) Sketch all the phase curves for motions with total energy $E = \frac{2}{5} U_0$. Do the same for $E = U_0$. (Recall that $e = 2.71828\ldots$.)

(d) Derive and expression for the period T of the motion when $|x| \ll a$.

Solution:

(a) Clearly $U(x)$ diverges to $+\infty$ for $x \to -\infty$, and $U(x) \to 0$ for $x \to +\infty$. Setting $U'(x) = 0$, we obtain the equation

$$U'(x) = \frac{U_0}{a^2} \left(2x - \frac{x^2}{a} \right) e^{-x/a} = 0 \quad , \tag{1}$$

with (finite x) solutions at $x = 0$ and $x = 2a$. Clearly $x = 0$ is a local minimum and $x = 2a$ a local maximum. Note $U(0) = 0$ and $U(2a) = 4\, e^{-2}\, U_0 \approx 0.541\, U_0$.

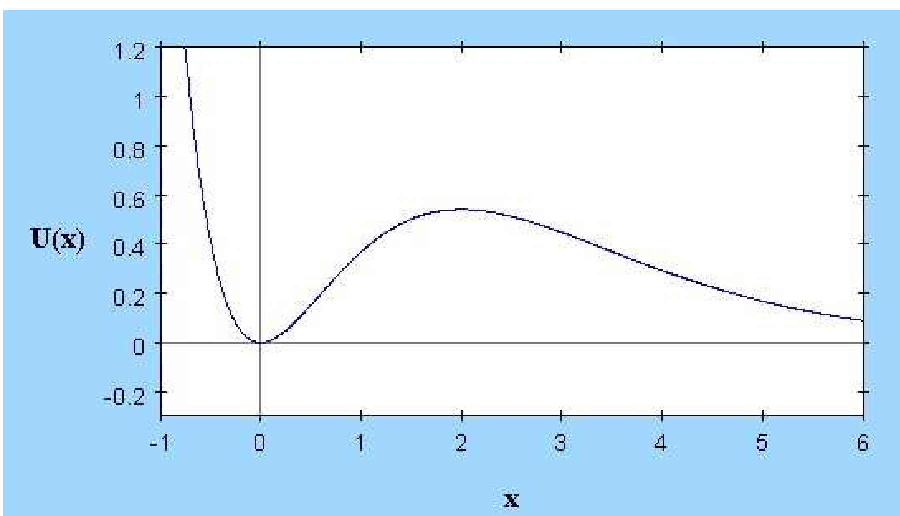

Figure 2: The potential $U(x)$. Distances are here measured in units of a, and the potential in units of U_0.

(b) Local minima of a potential $U(x)$ give rise to centers in the (x, v) plane, while local maxima give rise to saddles. In Fig. 3 we sketch the phase curves. There is a center at $(0, 0)$ and a saddle at $(2a, 0)$. There is one separatrix, at energy $E = U(2a) = 4\, e^{-2}\, U_0 \approx 0.541\, U_0$.

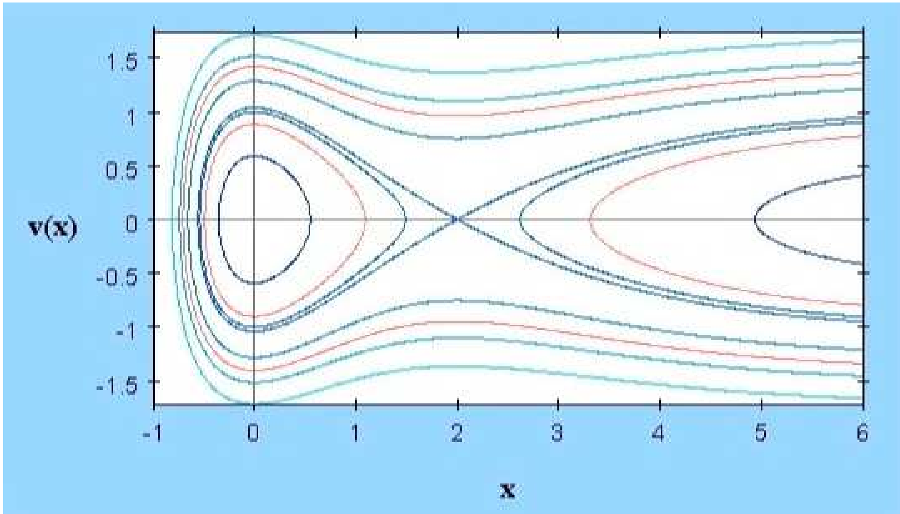

Figure 3: Phase curves for the potential $U(x)$. The red curves show phase curves for $E = \frac{2}{5}\,U_0$ (interior, disconnected red curves, $|v| < 1$) and $E = U_0$ (outlying red curve). The separatrix is the dark blue curve which forms a saddle at $(x, v) = (2, 0)$, and corresponds to an energy $E = 4\,e^{-2}\,U_0$.

(c) Even without a calculator, it is easy to verify that $4\,e^{-2} > \frac{2}{5}$. One simple way is to multiply both sides by $\frac{5}{2}\,e^2$ to obtain $10 > e^2$, which is true since $e^2 < (2.71828\ldots)^2 < 10$. Thus, the energy $E = \frac{2}{5}\,U_0$ lies below the local maximum value of $U(2a)$, which means that there are *two* phase curves with $E = \frac{2}{5}\,U_0$.

It is also quite obvious that the second energy value given, $E = U_0$, lies above $U(2a)$, which means that there is a single phase curve for this energy. One finds bound motions only for $x < 2$ and $0 \le E < U(2a)$. The phase curves corresponding to total energy $E = \frac{2}{5}\,U_0$ and $E = U_0$ are shown in Fig. 3.

(d) Expanding $U(x)$ in a Taylor series about $x = 0$, we have

$$U(x) = \frac{U_0}{a^2}\left\{x^2 - \frac{x^3}{a} + \frac{x^4}{2a^2} + \ldots\right\} \quad . \tag{2}$$

The leading order term is sufficient for $|x| \ll a$. The potential energy is then equivalent to that of a spring, with spring constant $k = 2U_0/a^2$. The period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ma^2}{2U_0}} \quad . \tag{3}$$

[**3**] A plane flying horizontally at constant speed v_0 and at height h_0 above the sea must drop a bundle of supplies to a hapless castaway on a small raft.

(a) Write Newton's second law for the bundle as it falls from the plane, assuming gravity is the only force acting on the bundle.

(b) Solve for the bundle's position as a function of time t since the release of the bundle.

(c) How far before the raft, measured horizontally, must the pilot drop the bundle if it is to hit the raft?

(d) What is this distance if $v_0 = 50\,\mathrm{m/s}$, $h_0 = 100\,\mathrm{m}$, and $g = 9.8\,\mathrm{m/s}$?

(e) Within what interval $\pm\Delta t$ of time must the pilot release the bundle if it is to land within 10 m of the raft?

Solution:

(a) Newton's second law says $\boldsymbol{F} = m\boldsymbol{a}$. There are two components we must keep track of, corresponding to horizontal (x) and vertical (z) directions. The force is $\boldsymbol{F} = -mg\hat{\boldsymbol{z}}$. Thus the mass m drops out of the equation since it is a common factor on both sides, and we have

$$\frac{d^2x}{dt^2} = 0 \qquad , \qquad \frac{d^2z}{dt^2} = -g \quad .$$

(b) Solving, we have

$$x(t) = x_0 + v_0 t \qquad , \qquad z(t) = h_0 - \tfrac{1}{2}gt^2 \quad .$$

(c) The time it takes the bundle to fall to sea level from its initial height $z(0) = h_0$ is $t^* = \sqrt{2h_0/g}$. Thus, the plane must initiate the drop a horizontal distance $x_0 = -v_0 t^* = -v_0\sqrt{2h_0/g}$ before it is directly over the raft.

(d) With the values given, we find $t^* = 4.52\,\mathrm{s}$ and $x_0 = 226\,\mathrm{m}$.

(e) In order for the bundle to arrive within a distance Δx of the raft, we set $\Delta t = \Delta x/v_0$. Thus $\Delta t = 0.2\,\mathrm{s}$.

[**4**] A mass m is constrained to move along the x-axis subject to a velocity-dependent force $F(v) = -F_0\, e^{v/V}$, where F_0 and V are constants.

(a) Find $v(t)$ if the initial velocity is $v(0) = v_0 > 0$ at time $t = 0$.

(b) At what time does the mass come instantaneously to rest?

(c) By integrating the function $v(t)$, find $x(t)$.

(d) How far does the mass travel before it starts to turn around and reverse direction?

Solution:

(a) Newton says $m\dot{v} = F(v)$, hence

$$-\frac{F_0}{m}\,dt = e^{-v/V}\,dv = -V\,de^{-v/V} \quad .$$

Integrating, we find

$$e^{-v(t)/V} - e^{-v_0/V} = \frac{F_0 t}{mV} \quad ,$$

where $v_0 = v(0)$. Solving for $v(t)$, we have

$$v(t) = -V \log\left(e^{-v_0/V} + \frac{F_0 t}{mV}\right) \quad .$$

(b) Setting $v(t^*) = 0$ we obtain

$$e^{-v_0/V} + \frac{F_0 t^*}{mV} = 1 \quad \Rightarrow \quad t^* = \frac{mV}{F_0}\left(1 - e^{-v_0/V}\right) \quad .$$

(c) We now integrate the equation $\dot{x} = v(t)$:

$$\begin{aligned} dx = v(t)\,dt &= -V \log\left(e^{-v_0/V} + \frac{F_0 t}{mV}\right) dt \\ &= -\frac{mV^2}{F_0}\,d\big(\Theta \log\Theta - \Theta\big) \quad , \end{aligned}$$

where

$$\Theta(t) = e^{-v_0/V} + \frac{F_0 t}{mV} \quad .$$

The solution is then

$$x(t) = x(0) - \frac{mV^2}{F_0}\,\Theta(t)\log\big(\Theta(t)/e\big) \quad .$$

(d) The mass is instantaneously at rest (*i.e.* $v(t^*) = 0$) when $\Theta(t^*) = 1$, which says

$$\Delta x = x(t^*) - x(0) = \frac{mV^2}{F_0} \quad .$$