PHYSICS 110A : MECHANICS 1
MIDTERM EXAMINATION SOLUTIONS

[1] A point particle of mass m moves in one space dimension with potential energy

$$U(x) = U_0 \left(\frac{x^3}{3a^3} - \frac{x}{a} \right) \quad .$$

Here U_0 and a are both positive.

(a) What are the dimensions of a and of U_0?
`[5 points]`

(b) Sketch $U(x)$, identifying the behavior at $x \to \pm\infty$, the value at $x = 0$, and the location and values of any local minima and maxima.
`[15 points]`

(c) Sketch the phase curves for $E = -\frac{2}{3} U_0$, $E = 0$, $E = \frac{2}{3} U_0$, and $E = 1.35\, U_0$. Identify which of the curves is a separatrix. Note that a given phase curve may have more than one disconnected component.
`[15 points]`

(d) Find an expression for the period of the bound orbit at $E = 0$, *i.e.* find $T(E = 0)$. Express $T(E = 0)$ fas a dimensionful quantity multiplied by a dimensionless integral.
`[15 points]`

Solution :

(a) $[a] = L$ and $[U_0] = E = ML^2/T^2$

(b) See the top panel of fig. 1. There is a local maximum at $x = -a$ and a local minimum at $x = +a$, since $U'(\pm a) = 0$.

(c) See the bottom panel of fig. 1. The phase curve for $E = \frac{2}{3} U_0$, corresponding the local maximum of $U(x)$ at $x = -a$, is a separatrix (shown in red). All phase curves for energies $E < \frac{2}{3} U_0$ are disjoint sets, *i.e.* they consist of a union of a bound periodic orbit and an unbound orbit. At energy $E = -\frac{2}{3} U_0$, corresponding to the local minimum of $U(x)$ at $x = +a$, the bound orbit has shrunken to a single fixed point. Below $E = -\frac{2}{3} U_0$ there are only unbound orbits.

(d) The turning points for the bound orbit at $E = 0$ are $x_-(0) = 0$ and $x_+(0) = \sqrt{3}\, a$, which are solutions to $E = U(x) = 0$. Thus, from $E = \frac{1}{2} mv^2 + U(x)$ we have $dt = \pm dx/v$ and

$$T(E) = \sqrt{2m} \int\limits_{x_-(E)}^{x_+(E)} \frac{dx'}{\sqrt{E - U(x')}} = \sqrt{\frac{2m}{U_0}} \int\limits_0^{\sqrt{3}\, a} \frac{dx'}{\sqrt{\frac{x'}{a} - \frac{x'^3}{3a^3}}}$$

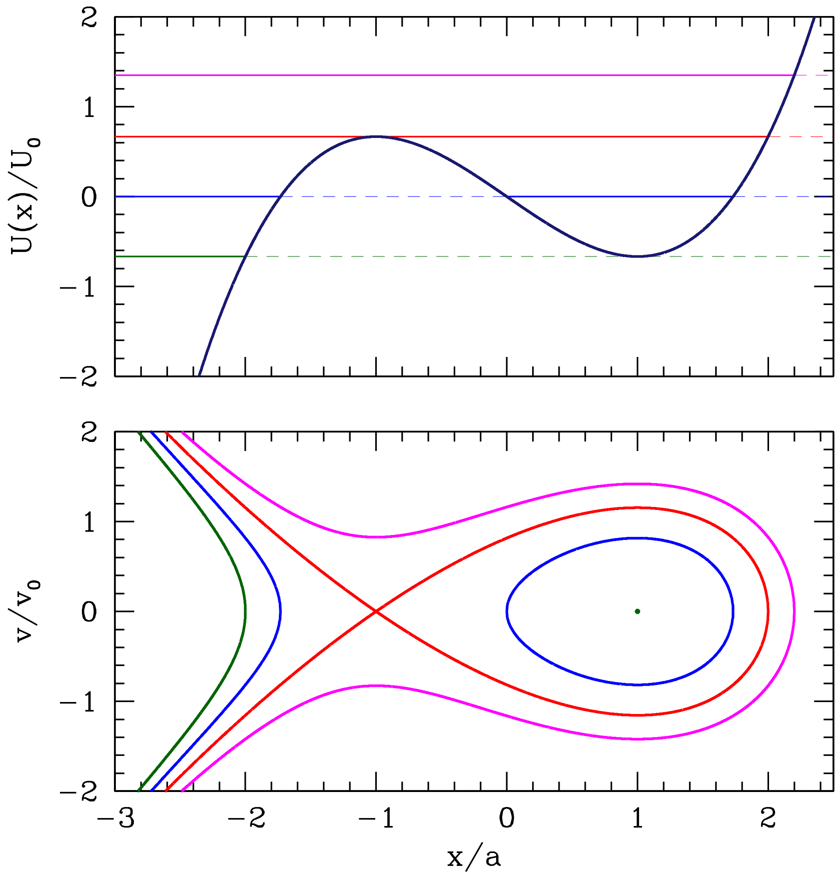

Figure 1: Top: $U(x)$ *versus* x/a. Bottom: phase curves for $E = -\frac{2}{3}U_0$ (green), $E = 0$ (blue), $E = \frac{2}{3}U_0$ (red), and $E = 1.35\,U_0$ (magenta). The velocity scale v_0 is given by $v_0 = \sqrt{2U_0/m}$.

With $s \equiv x/a$ we have

$$T(0) = \sqrt{\frac{2ma^2}{E_0}} \int\limits_0^{\sqrt{3}} \frac{ds}{\sqrt{s - \frac{1}{3}s^2}} \simeq 3.45082\sqrt{\frac{2ma^2}{E_0}} \quad ,$$

where the last result is from numerical integration.

[2] A forced, damped oscillator obeys the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega_0 t) \quad .$$

You may assume the oscillator is underdamped. Note that the forcing frequency ω_0 is identical to the natural frequency of the unforced, undamped oscillator.

(a) Write down the most general solution of this differential equation.
`[20 points]`

(b) Your solution should involve two constants. Derive two equations relating these constants to the initial position $x(0)$ and the initial velocity $\dot{x}(0)$. *You do not have to solve these equations.*
[15 points]

(c) Suppose $\omega_0 = 5.0\,\mathrm{s}^{-1}$, $\beta = 4.0\,\mathrm{s}^{-1}$, and $f_0 = 8\,\mathrm{cm\,s}^{-2}$. Suppose further you are told that $x(0) = 0$ and $x(T) = 0$, where $T = \frac{\pi}{6}\,\mathrm{s}$. Derive an expression for the initial velocity $\dot{x}(0)$.
[15 points]

Solution :

(a) The general solution with forcing $f(t) = f_0 \cos(\Omega t)$ is

$$x(t) = x_{\mathrm{h}}(t) + A(\Omega)\, f_0 \cos\big(\Omega t - \delta(\Omega)\big) \ ,$$

with

$$A(\Omega) = \Big[(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\Big]^{-1/2} \quad , \quad \delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right)$$

and

$$x_{\mathrm{h}}(t) = C\, e^{-\beta t} \cos(\nu t) + D\, e^{-\beta t} \sin(\nu t) \ ,$$

with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

In our case, $\Omega = \omega_0$, in which case $A = (2\beta\omega_0)^{-1}$ and $\delta = \frac{1}{2}\pi$. Thus, the most general solution is

$$x(t) = C\, e^{-\beta t} \cos(\nu t) + D\, e^{-\beta t} \sin(\nu t) + \frac{f_0}{2\beta\omega_0} \sin(\omega_0 t) \quad .$$

(b) We determine the constants C and D by the boundary conditions on $x(0)$ and $\dot{x}(0)$:

$$x(0) = C \qquad , \qquad \dot{x}(0) = -\beta C + \nu D + \frac{f_0}{2\beta} \quad .$$

Thus,

$$C = x(0) \qquad , \qquad D = \frac{\beta}{\nu}\, x(0) + \frac{1}{\nu}\, \dot{x}(0) - \frac{f_0}{2\beta\nu} \quad .$$

(c) From $x(0) = 0$ we obtain $C = 0$. The constant D is then determined by the condition at time $t = T = \frac{1}{6}\pi$.

Note that $\nu = \sqrt{\omega_0^2 - \beta^2} = 3.0\,\mathrm{s}^{-1}$. Thus, with $T = \frac{1}{6}\pi$, we have $\nu T = \frac{1}{2}\pi$, and

$$x(T) = D\, e^{-\beta T} + \frac{f_0}{2\beta\omega_0} \sin(\omega_0 T) \quad .$$

This determines D:

$$D = -\frac{f_0}{2\beta\omega_0}\, e^{\beta T} \sin(\omega_0 T) \quad .$$

We now can write

$$\begin{aligned}
\dot{x}(0) &= \nu D + \frac{f_0}{2\beta} \\
&= \frac{f_0}{2\beta}\left(1 - \frac{\nu}{\omega_0}\, e^{\beta T}\, \sin(\omega_0 T)\right) \\
&= \left(1 - \tfrac{3}{10}\, e^{2\pi/3}\right) \mathrm{cm/s} \quad .
\end{aligned}$$

Numerically, the value is $\dot{x}(0) \approx 0.145\,\mathrm{cm/s}$.