**PHYSICS 110A : MECHANICS 1**
**PROBLEM SET #2**

[**1**] Using the method of partial fractions, solve the ODE

$$\frac{du}{dt} = (u-1)(u-2)(u-3)$$

for $t(u)$. Sketch the phase flow along the real $u$ line, and the integral curves in the $(t,u)$ plane. Show that for $u_0 < 1$ or $u_0 > 3$ that $u(t)$ flows to $u = \pm\infty$ in a *finite* time $t^*$, but that for $u_0 \in (1,3)$ the flow is toward the stable fixed point $u^* = 2$, which takes infinite time to reach.

[**2**] Consider the $n = 2$ dynamical system given by

$$\frac{dx}{dt} = x - y - x^3 \qquad , \qquad \frac{dy}{dt} = rxy - y^2 \quad ,$$

where $r > 0$.

(a) Assuming $r > 1$, how many fixed points are there? Find them. *Hint: Start with the second equation.*

(b) Show that for $r < 1$ there are two more fixed points. Find them.

(c) Expanding about a fixed point $(x^*, y^*)$, with $u_x \equiv x - x^*$ and $u_y = y - y^*$, the linearized dynamics takes the form $\dot{\boldsymbol{u}} = M\boldsymbol{u}$, where $M$ is a $2 \times 2$ matrix. Find an expression for $M$ at the fixed point $(x^*, y^*)$.

(d) What are the eigenvalues of the linearized system $\dot{u} = Mu$ at $(x^*, y^*) = (0,0)$?

[**3**] Consider the $n = 2$ dynamical system

$$\frac{d}{dt}\begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -U'(\phi) \end{pmatrix} \quad ,$$

where $U(\phi) = -\cos\phi + 2r\sin^2\!\phi$ with $r \geq 0$. Phase space is thus a cylinder: $(\phi, \omega) \in S^1 \times \mathbb{R}$.

(a) Show that the energy $E = \frac{1}{2}\omega^2 + U(\phi)$ is conserved.

(b) Show that there is a critical value $r_c$ such that for $r < r_c$ the potential $U(\phi)$ has a single minimum at $\phi = 0$ and a single maximum at $\phi = \pm\pi$, but for $r > r_c$, there is a global minimum at $\phi = 0$, a local minimum at $\phi = \pm\pi$, and two local maxima at $\phi = \pm\phi^*(r)$. Find the value of $r_c$ and the function $\phi^*(r)$.

(c) Sketch the potential $U(\phi)$ for $r = 0.15$. Plot the phase curves at energies $E_1 = 0$ and $E_2 = 1.5$.

(d) Sketch the potential $U(\phi)$ for $r = 0.80$. Find the separatrix energy corresponding to the energy $E^* = U_{\max}$. Plot the phase curves at energies $E_1 = 0$, $E_2 = 1.2$, $E_3 = E^*$, and $E_4 = 2.2$.