PHYSICS 110A : MECHANICS 1
FINAL EXAMINATION SOLUTIONS

[**1**] Provide concise but accurate answers to the following questions. Include equations and sketches where appropriate.

(a) What is a dynamical system?
`[4 points]`

A dynamical system is a set of n coupled autonomous first-order differential equations, of the form $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi}^{\mathrm{t}} = (\varphi_1, \ldots, \varphi_n)$. n is the dimension of the dynamical system.

(b) For a weakly damped forced harmonic oscillator, sketch the amplitude response $A(\Omega)$ and phase shift $\delta(\Omega)$ as a function of the ratio Ω/ω_0, where Ω is the forcing frequency and ω_0 is the natural frequency.
`[4 points]`

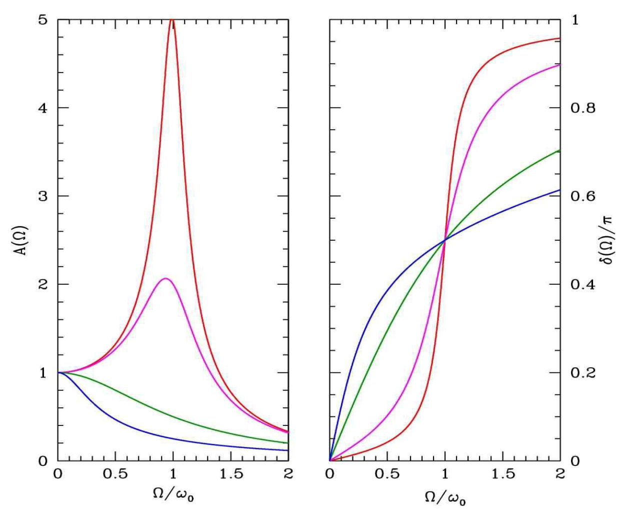

Figure 1: Amplitude and phase shift *versus* oscillator frequency (units of ω_0) for β/ω_0 values of 0.1 (red), 0.25 (magenta), 1.0 (green), and 2.0 (blue).

(c) What is Noether's theorem?
`[4 points]`

Noether's theorem says that *to every independent continuous one-parameter family of transformations which leaves the Lagrangian invariant, there corresponds an associated conserved quantity.* Specifically, if $q_\sigma \to \tilde{q}_\sigma(q, \zeta)$ is the transformation parameterized by ζ, with $\tilde{q}_\sigma(q, \zeta = 0) = q_\sigma$, then

$$\Lambda = \sum_{\sigma=1}^{n} \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta} \bigg|_{\zeta=0}$$

is conserved under the dynamics.

(d) For the geometric orbit shape $r(\phi) = r_0/\big(1 - \varepsilon\cos(\beta\phi)\big)$, what are the values of r and ϕ at periapsis ($r_{\rm p}$) and apoapsis ($r_{\rm a}$). What is the condition on β that the orbit be closed?
[4 points]

Periapsis is the distance of closest approach. This occurs for $\cos(\beta\phi) = -1$, hence $\phi_{\rm p} = (2n+1)\pi/\beta$ and $r_{\rm p} = r_0/(1+\varepsilon)$. Apoapsis occurs for $\cos(\beta\phi) = +1$, hence $\phi_{\rm a} = 2n\pi/\beta$ and $r_{\rm a} = r_0/(1-\varepsilon)$. The geometric orbit is closed provided $\beta \in \mathbb{Q}$ is a rational number.

(e) What do we mean by 'normal modes' of small coupled oscillations? Why are they useful to study?
[4 points]

The normal modes of a coupled system of small oscillations are linear combinations ξ_i of the generalized displacements η_σ which oscillate with a pure frequency ω_i, with $\eta_\sigma = \sum_i \mathsf{A}_{\sigma i}\,\xi_i$; A is the *modal matrix*. They are useful because they permit us to write the general solution to the coupled small oscillations problem as an expansion in terms of the normal modes.

[2] Consider one-dimensional motion with the potential energy $U(x) = U_0\,a^2\,f(x)$, where

$$f(x) = \frac{e^{x/a}}{2x^2 + a^2} \quad .$$

(a) Sketch $U(x)$ as a function of x. Note that $x \in \mathbb{R}$ may take both positive and negative values. Identify the location of all minima and maxima. (It may be useful to consider the potential as a function of the dimensionless position $s = x/a$.)
[4 points]

The function $U(x)/U_0$ is plotted in the top panel of fig. 2. Note that

$$f'(x) = \left(\frac{1}{a} - \frac{4x}{2x^2 + a^2}\right)\frac{e^{x/a}}{2x^2 + a^2}$$

and thus the condition $f'(x) = 0$ yields $2x^2 - 4ax + a^2 = 0$, with roots at $x_\pm = \big(1 \pm \frac{1}{\sqrt{2}}\big)a$. We also have $f(-\infty) = 0$ due to the exponential. Thus there is a global minimum $U(x) = 0$ at $x = -\infty$, a local maximum $U_0\,f(x_-)$ at $x = x_-$, and a local minimum $U_0\,f(x_+)$ at $x = x_+$.

(b) Sketch the phase curves in the $(x, \dot{x})$ plane. There are several different types of orbits, depending on their energy in relation to the values at the local minimum and maximum of $U(x)$. Sketch what happens at four different representative energy values, including that for the separatrix.
[12 points]

Conservation of energy says that $E = \frac{1}{2}mv^2 + U(x)$ is a constant, thus

$$\frac{v(x)}{v_0} = \pm\sqrt{\frac{E}{U_0} - f(x)} \quad ,$$

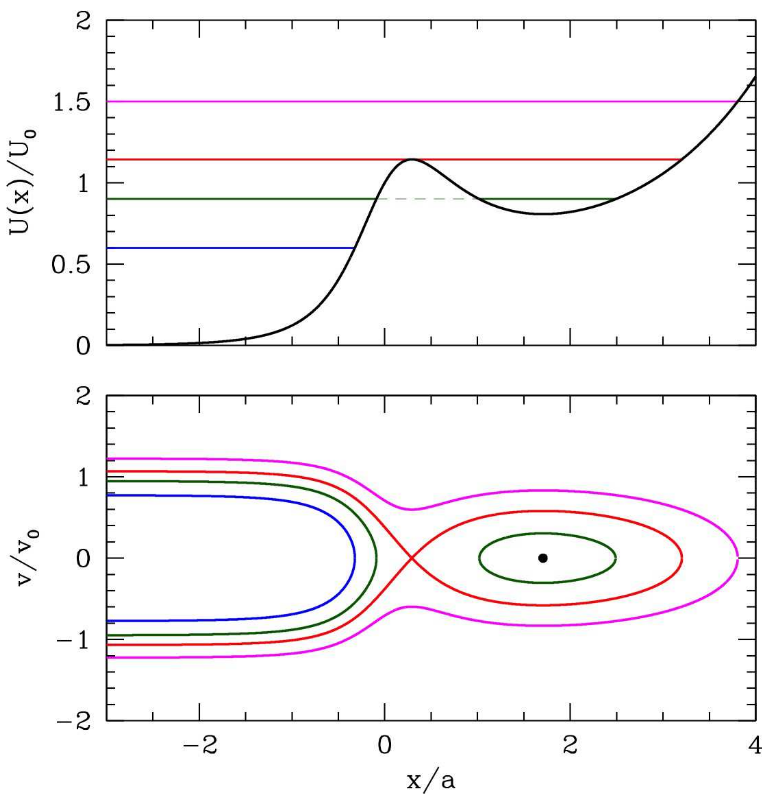

Figure 2: $U(x)$ *versus* x (top) and phase curves (bottom) for problem 2.

where $v_0 = \sqrt{2U_0/m}$. Representative phase curves at four different energies are depicted in fig. 2.

(c) What is the energy E^* corresponding to the separatrix?
`[4 points]`

The energy of the separatrix is

$$E^* = U_0\, a^2\, \frac{\exp(x_-/a)}{2x_-^2 + a^2} = U_0\, \frac{\exp\!\left(1 - \frac{1}{\sqrt{2}}\right)}{3 - 2\sqrt{2}} = 1.144017\, U_0 \quad .$$

(You were not expected to obtain the numerical coefficient.)

[3] A point mass m rolls under the influence of gravity along a semicircular surface of radius R, as depicted in fig. 3.

(a) Find the Lagrangian.
`[5 points]`

The coordinates of the mass are

$$x = R\sin\phi \qquad , \qquad y = R\cos\phi \quad .$$

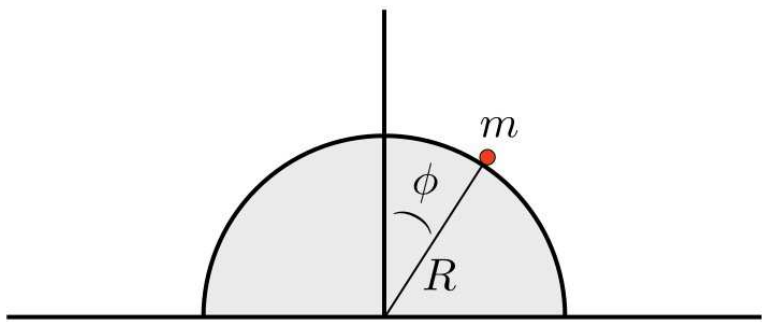

Figure 3: A mass point m rolls inside along a semicircular surface of radius R.

Thus

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = \tfrac{1}{2}mR^2\dot{\phi}^2 - mgR\cos\phi \quad .$$

(b) Find the equations of motion.
`[5 points]`

The momentum is $p_\phi = \partial L/\partial\dot{\phi} = mR^2\dot{\phi}$, and the force is $F_\phi = \partial L/\partial\phi = mgR\sin\phi$. Thus, the equation of motion is $\dot{p}_\phi = F_\phi$, *i.e.*

$$\ddot{\phi} = \omega_0^2 \sin\phi \quad ,$$

where $\omega_0 = \sqrt{g/R}$. This is the equation of an inverted pendulum.

(c) What quantities are conserved?
`[5 points]`

The only conserved quantity is the energy $E = \frac{1}{2}mR^2\dot{\phi}^2 + mgR\,\cos\phi$. Assuming the mass starts from rest at an initial angle ϕ_0, we have $E = mgR\,\cos\phi_0$.

(d) Assume the mass starts at $\phi(0) = \phi_0$ with $\dot{\phi}(0) = 0$. At some value $\phi = \phi^*$, the centrifugal force mv^2/R starts to exceed the component of the gravitational force normal to the surface and the mass flies off. Find ϕ^*.
`[5 points]`

The component of gravity $-g\hat{\boldsymbol{y}}$ normal to the surface is $-g\,\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{n}} = -g\,\cos\phi$, where the surface normal is $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}} = \sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}$. The centrifugal force is mv^2/R where

$$\tfrac{1}{2}mv^2 = mgR(\cos\phi_0 - \cos\phi) \quad \Rightarrow \quad \frac{mv^2}{R} = 2mg(\cos\phi_0 - \cos\phi) \quad .$$

Thus, we set

$$mg(\cos\phi_0 - \cos\phi) - mg\cos\phi = 0 \quad \Rightarrow \quad \cos\phi^* = \tfrac{2}{3}\cos\phi_0 \quad .$$

Thus, $\phi^* = \cos^{-1}\big(\frac{2}{3}\cos\phi_0\big)$.

Aside: This is a classic problem which can be solved using the formulation of constraints, which, alas, we did not cover. However, it is even easier to solve without the constraint formalism.

[**4**] Two particles of identical masses m interact via the central potential

$$U(r) = U_0 \left\{ \left(\frac{\sigma}{r}\right)^4 - \left(\frac{\sigma}{r}\right)^2 \right\} \quad ,$$

where σ is a length scale.

(a) Sketch $U(r)$ as a function of the dimensionless variable r/σ. Find all extrema. Identify the behavior as $r \to 0$ and as $r \to \infty$.
`[5 points]`

It is convenient to define $s \equiv r/\sigma$, in which case $U(r) = U_0 \left(s^{-4} - s^{-2}\right)$. Thus we find $U'(r) = U_0\, \sigma^{-1}\left(-4s^{-5} + 2s^{-3}\right)$. Setting $U'(r) = 0$ yields $s = \sqrt{2}$, *i.e.* $r^* = \sqrt{2}\,\sigma$. The minimum value of $U(r)$ is then $U_{\text{min}} = -\frac{1}{4}\, U_0$.

(b) Show that a stable circular orbit exists for the relative coordinate problem provided the angular momentum ℓ is sufficiently small. Find the critical value ℓ_c above which no bound orbits exist. Define the quantity $\varepsilon \equiv 1 - (\ell/\ell_c)^2$, in which case bound orbits exist for $0 < \varepsilon < 1$. Sketch the effective potential $U_{\text{eff}}(r)$ for the cases (i) $\ell < \ell_c$ and (ii) $\ell > \ell_c$.
`[5 points]`

The effective potential is

$$\begin{aligned} U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + U(r) &= U_0 \left\{ \left(\frac{\sigma}{r}\right)^4 + \left(\frac{\ell^2}{\ell_c^2} - 1\right)\left(\frac{\sigma}{r}\right)^2 \right\} \\ &= U_0 \left(s^{-4} - \varepsilon\, s^{-2} \right) \quad , \end{aligned}$$

where $\mu = \frac{1}{2} m$ and $\ell_c = \sqrt{m\sigma^2 U_0}$. When $\varepsilon < 0$ we have that $U_{\text{eff}}(r)$ is monotonically decreasing and therefore there are no bound orbits. Bound orbits require $0 < \varepsilon < 1$ which means $0 < \ell < \ell_c$. See the sketches in fig. 4.

(c) For $0 < \ell < \ell_c$ (*i.e.* $0 < \varepsilon < 1$), find the radius $r_0(\varepsilon)$ of the stable circular orbit.
`[5 points]`

Extrema of $U_{\text{eff}}(r)$ are obtained by differentiating with respect to r, or, equivalently with respect to s, and then setting the result to zero. This yields

$$-4s^{-5} + 2\varepsilon\, s^{-3} = 0 \qquad \Rightarrow \qquad s^2 = \frac{2}{\varepsilon} \quad (\varepsilon > 0) \quad .$$

Thus, $\varepsilon > 0$ in order to have a local minimum at $r_0 = \sqrt{2/\varepsilon}\,\sigma$, which is the location of the circular orbit. This requires $\ell < \ell_c = \sqrt{m\sigma^2 U_0}$.

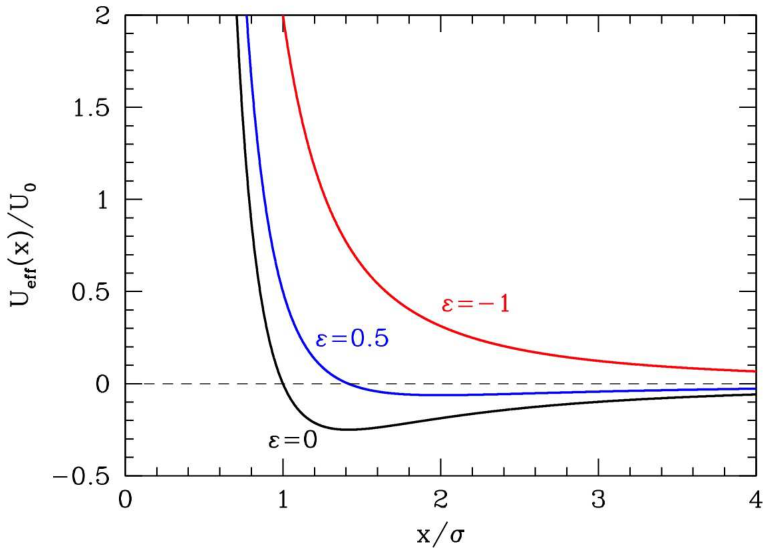

Figure 4: $U_{\rm eff}(x)$ *versus* x/σ for $\varepsilon = 0$, 0.5, and -1.

(d) Find the frequency ω of small oscillations of the radial motion $r(t)$ about the circular orbit.
[5 points]

Writing $r = r_0 + \eta$, the equations of motion are

$$\mu\ddot{\eta} = -U''_{\rm eff}(r_0)\,\eta + \mathcal{O}(\eta^2) \quad ,$$

where

$$U''_{\rm eff}(r) = U_0\,\sigma^{-2}\left(20\,s^{-6} - 6\varepsilon\,s^{-4}\right) \qquad \Rightarrow \qquad U''_{\rm eff}(r_0) = U_0\,\varepsilon^3\,\sigma^{-2} \quad ,$$

after substituting $s = r_0/\sigma = \sqrt{2/\varepsilon}$. Thus, the frequency of small radial oscillations is

$$\omega = \sqrt{\frac{U''_{\rm eff}(r_0)}{\mu}} = \sqrt{\frac{2\,\varepsilon^3\,U_0}{m\sigma^2}} \quad ,$$

with $0 < \varepsilon < 1$.

(e) The shape of the perturbed orbit is $r(\phi) = r_0 + \eta_0 \cos(\beta\phi)$, where η_0 is a constant determined by initial conditions and β is calculable in terms of the parameters of the problem. Find an expression for β in terms of ε.
[50 quatloos extra credit]

From conservation of angular momentum, we have $\dot{\phi} = \ell/\mu r_0^2$ for the circular orbit. Thus,

$$\beta = \frac{\omega}{\dot{\phi}} = \frac{\mu\,r_0^2}{\ell}\,\omega = \frac{\ell_{\rm c}}{\ell} \cdot \frac{(m/2)(2\sigma^2/\varepsilon)}{\sqrt{m\sigma^2\,U_0}} \cdot \sqrt{\frac{2\,\varepsilon^3\,U_0}{m\sigma^2}} = \sqrt{\frac{2}{\varepsilon}} \quad .$$

[5] Three identical masses m are connected by four identical springs k as depicted in the figure below. In equilibrium, the springs are all unstretched.

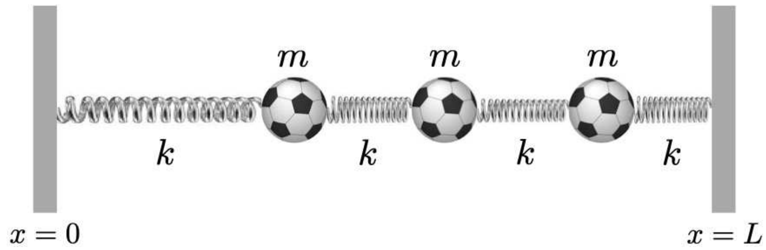

Figure 5: Three identical masses connected by four identical springs.

(a) Choose as generalized coordinates the displacements η_1, η_2, and η_3 with respect to the equilibrium positions of the masses. Write the Lagrangian.
`[5 points]`

We have

$$\begin{aligned} T &= \tfrac{1}{2}m\,\dot{\eta}_1^2 + \tfrac{1}{2}m\,\dot{\eta}_2^2 + \tfrac{1}{2}m\,\dot{\eta}_3^2 \\ U &= \tfrac{1}{2}k\,\eta_1^2 + \tfrac{1}{2}k\,(\eta_2-\eta_1)^2 + \tfrac{1}{2}k\,(\eta_3-\eta_2)^2 + \tfrac{1}{2}k\,\eta_3^2 \quad , \end{aligned}$$

and $L = T - U$.

(b) Find the T and V matrices (each of which is 3×3).
`[5 points]`

We have

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot{\eta}_\sigma\,\partial\dot{\eta}_{\sigma'}}\bigg|_{\eta=0} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \quad , \quad \mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\eta_\sigma\,\partial\eta_{\sigma'}}\bigg|_{\eta=0} = \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix} \quad .$$

(c) Find the eigenfrequencies. You might worry that you have to solve a cubic equation, but it turns out that $P(\omega) = \mathsf{det}(\omega^2\mathsf{T} - \mathsf{V})$ nicely factorizes. The following identity,

$$\mathsf{det}\begin{pmatrix} a & c & 0 \\ c & b & c \\ 0 & c & a \end{pmatrix} = a\,(ab - 2c^2) \quad ,$$

should prove useful.
`[5 points]`

We have

$$\begin{aligned} P(\omega^2) = \mathsf{det}(\omega^2\mathsf{T} - \mathsf{V}) &= k\begin{pmatrix} u-2 & -1 & 0 \\ -1 & u-2 & -1 \\ 0 & -1 & u-2 \end{pmatrix} \\ &= (u-2)\left\{(u-2)^2 - 2\right\} = (u-u_1)(u-u_2)(u-u_3) \quad , \end{aligned}$$

where $u \equiv \omega^2/\omega_0^2$, $\omega_0 \equiv \sqrt{k/m}$, and

$$u_1 = 2 - \sqrt{2} \quad , \quad u_2 = 2 \quad , \quad u_3 = 2 + \sqrt{2} \quad .$$

Thus the eigenfrequencies are

$$\omega_1 = \sqrt{2-\sqrt{2}}\,\omega_0 \quad , \quad \omega_2 = \sqrt{2}\,\omega_0 \quad , \quad \omega_3 = \sqrt{2+\sqrt{2}}\,\omega_0 \quad .$$

(d) Find the modal matrix A.
`[5 points]`

We write

$$\begin{pmatrix} u_i - 2 & -1 & 0 \\ -1 & u_i - 2 & -1 \\ 0 & -1 & u_i - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(i)} \\ \psi_2^{(i)} \\ \psi_3^{(i)} \end{pmatrix} = 0$$

and solve. We obtain the column vectors

$$\boldsymbol{\psi}^{(1)} = C_1 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^{(2)} = C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^{(3)} = C_3 \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \quad .$$

Note that the three eigenvectors are mutually orthogonal in the conventional sense, *i.e.* $\boldsymbol{\psi}^{(i)} \cdot \boldsymbol{\psi}^{(j)} = 0$ if $i \neq j$. This is because T is a multiple of the identity matrix, and thus $\boldsymbol{\psi}^{(i)} \cdot \mathsf{T} \cdot \boldsymbol{\psi}^{(j)} = 0$ is equivalent to $\boldsymbol{\psi}^{(i)} \cdot \boldsymbol{\psi}^{(j)} = 0$. For $i = j$ we have

$$\boldsymbol{\psi}^{(1)} \cdot \mathsf{T} \cdot \boldsymbol{\psi}^{(1)} = 4mC_1^2 = 1 \quad , \quad \boldsymbol{\psi}^{(2)} \cdot \mathsf{T} \cdot \boldsymbol{\psi}^{(2)} = 2mC_2^2 = 1 \quad , \quad \boldsymbol{\psi}^{(3)} \cdot \mathsf{T} \cdot \boldsymbol{\psi}^{(3)} = 4mC_3^2 = 1 \quad ,$$

and thus

$$\mathsf{A} = \frac{1}{2\sqrt{m}} \begin{pmatrix} 1 & \sqrt{2} & -1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & -1 \end{pmatrix} \quad .$$

Note that in the low frequency normal mode $i = 1$ all the masses move in phase. In the $i = 2$ normal mode, the central mass is stationary. In the $i = 3$ normal mode, the restoring force is greatest because the second and third springs have greatest compression/extension for a given mode amplitude.