PHYSICS 110A : MECHANICS 1
FINAL EXAMINATION

[**1**] Provide concise but accurate answers to the following questions. Include equations and sketches where appropriate.

(a) What is a dynamical system?
`[4 points]`

(b) For a weakly damped forced harmonic oscillator, sketch the amplitude response $A(\Omega)$ and phase shift $\delta(\Omega)$ as a function of the ratio Ω/ω_0, where Ω is the forcing frequency and ω_0 is the natural frequency.
`[4 points]`

(c) What is Noether's theorem?
`[4 points]`

(d) For the geometric orbit shape $r(\phi) = r_0/\big(1 - \varepsilon\cos(\beta\phi)\big)$, what are the values of r and ϕ at periapsis ($r_{\rm p}$) and apoapsis ($r_{\rm a}$). What is the condition on β that the orbit be closed?
`[4 points]`

(e) What do we mean by 'normal modes' of small coupled oscillations? Why are they useful to study?
`[4 points]`

[**2**] Consider one-dimensional motion with the potential energy $U(x) = U_0\, a^2\, f(x)$, where

$$f(x) = \frac{e^{x/a}}{2x^2 + a^2}$$

(a) Sketch $U(x)$ as a function of x. Note that $x \in \mathbb{R}$ may take both positive and negative values. Identify the location of all minima and maxima. (It may be useful to consider the potential as a function of the dimensionless position $s = x/a$.)
`[4 points]`

(b) Sketch the phase curves in the $(x, \dot{x})$ plane. There are several different types of orbits, depending on their energy in relation to the values at the local minimum and maximum of $U(x)$. Sketch what happens at four different representative energy values, including that for the separatrix.
`[12 points]`

(c) What is the energy E^* corresponding to the separatrix?
`[4 points]`

[**3**] A point mass m rolls under the influence of gravity along a semicircular surface of radius R, as depicted in fig. 1.

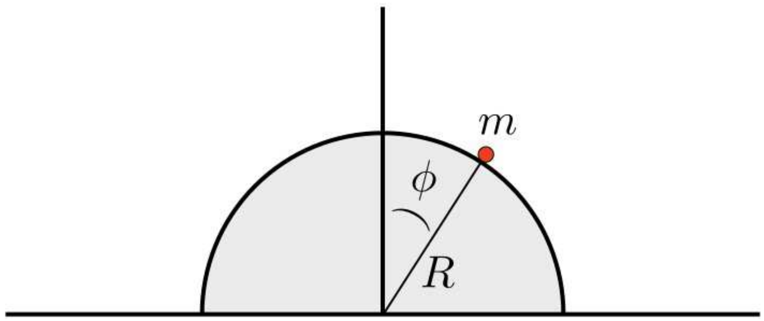

Figure 1: A mass point m rolls inside along a semicircular surface of radius R.

(a) Find the Lagrangian.
`[5 points]`

(b) Find the equations of motion.
`[5 points]`

(c) What quantities are conserved?
`[5 points]`

(d) Assume the mass starts at $\phi(0) = \phi_0$ with $\dot{\phi}(0) = 0$. At some value $\phi = \phi^*$, the centrifugal force mv^2/R starts to exceed the component of the gravitational force normal to the surface and the mass flies off. Find ϕ^*.
`[5 points]`

Aside: This is a classic problem which can be solved using the formulation of constraints, which, alas, we did not cover. However, it is even easier to solve without the constraint formalism.

[**4**] Two particles of identical masses m interact via the central potential

$$U(r) = U_0 \left\{ \left(\frac{\sigma}{r}\right)^4 - \left(\frac{\sigma}{r}\right)^2 \right\} \quad ,$$

where σ is a length scale.

(a) Sketch $U(r)$ as a function of the dimensionless variable r/σ. Find all extrema. Identify the behavior as $r \to 0$ and as $r \to \infty$.
`[5 points]`

(b) Show that a stable circular orbit exists for the relative coordinate problem provided the angular momentum ℓ is sufficiently small. Find the critical value ℓ_c above which no bound orbits exist. Define the quantity $\varepsilon \equiv 1 - (\ell/\ell_c)^2$, in which case bound orbits exist

for $0 < \varepsilon < 1$. Sketch the effective potential $U_{\text{eff}}(r)$ for the cases (i) $\ell < \ell_c$ and (ii) $\ell > \ell_c$.
[5 points]

(c) For $0 < \ell < \ell_c$ (*i.e.* $0 < \varepsilon < 1$), find the radius $r_0(\varepsilon)$ of the stable circular orbit.
[5 points]

(d) Find the frequency ω of small oscillations of the radial motion $r(t)$ about the circular orbit.
[5 points]

(e) The shape of the perturbed orbit is $r(\phi) = r_0 + \eta_0 \cos(\beta\phi)$, where η_0 is a constant determined by initial conditions and β is calculable in terms of the parameters of the problem. Find an expression for β in terms of ε.
[50 quatloos extra credit]

[5] Three identical masses m are connected by four identical springs k as depicted in the figure below. In equilibrium, the springs are all unstretched.

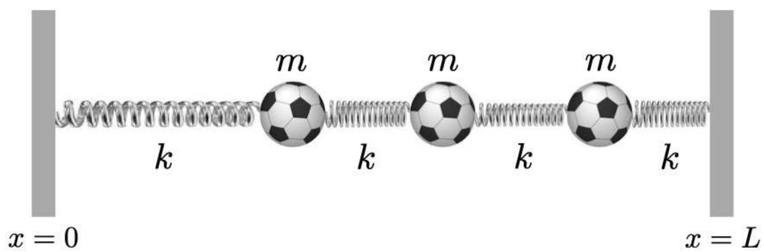

Figure 2: Three identical masses connected by four identical springs.

(a) Choose as generalized coordinates the displacements η_1, η_2, and η_3 with respect to the equilibrium positions of the masses. Write the Lagrangian.
[5 points]

(b) Find the T and V matrices (each of which is 3×3).
[5 points]

(c) Find the eigenfrequencies. You might worry that you have to solve a cubic equation, but it turns out that $P(\omega) = \det(\omega^2\mathsf{T} - \mathsf{V})$ nicely factorizes. The following identity,

$$\det \begin{pmatrix} a & c & 0 \\ c & b & c \\ 0 & c & a \end{pmatrix} = a\,(ab - 2c^2) \quad ,$$

should prove useful.
[5 points]

(d) Find the modal matrix A.
[5 points]