Lecture 13 (Feb. 16)

Superconductivity phenomenology

Superconductivity : a phase of matter, like metal, insulator, ferromagnet, etc. Basic features of SC phase :

- **vanishing electrical resistance** :
  The DC resistivity vanishes in zero external magnetic field. Verified in some cases to 10^{-15} of ρ_{normal}. The AC resistivity $\rho(\omega, T, H=0)$ vanishes for $T < T_c$ below a frequency $\omega_c = 2\Delta/\hbar$, where Δ is the gap in the electronic energy spectrum. The superconducting state is a condensate of electron **pairs**, and breaking a pair results in two fermionic **quasiparticles**, each with energy $\geq \Delta$.

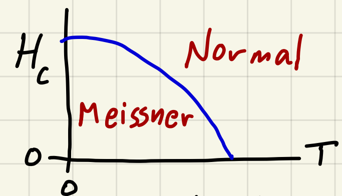
- **flux expulsion** :
  Magnetic fields can penetrate only within a distance λ of the surface of a bulk superconductor, where λ is the **London penetration depth**. Note that a perfect conductor, in which $\sigma = \infty$, must have $\vec{E} = 0$ lest $\vec{j} = \sigma \vec{E}$ diverge, but then $\vec{\nabla} \times \vec{E} = -c^{-1} \partial_t \vec{B} = 0 \Rightarrow \partial_t \vec{B} = 0$, so the field lines are <u>frozen</u>. But in a superconductor the field lines are <u>expelled</u>. If the superconductor is not simply connected,

flux quantized in units of $\phi_L = hc/2e = 2.07\times10^{-7}\,\mathrm{G\,cm}^2$ can thread the holes.

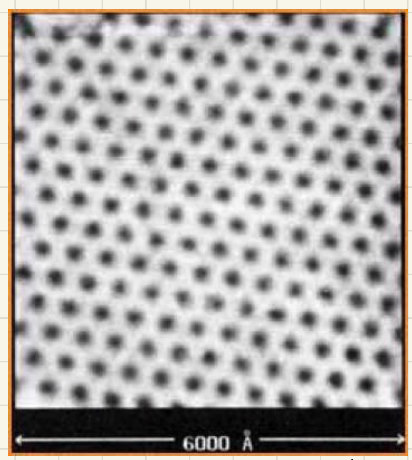
- **critical fields**:

Flux expulsion, known as the **Meissner effect**, pertains only for $T<T_c$ and for $H<H_c(T)$, where

$$H_c(T) \simeq H_c(0)\left(1-\frac{T^2}{T_c^2}\right)$$



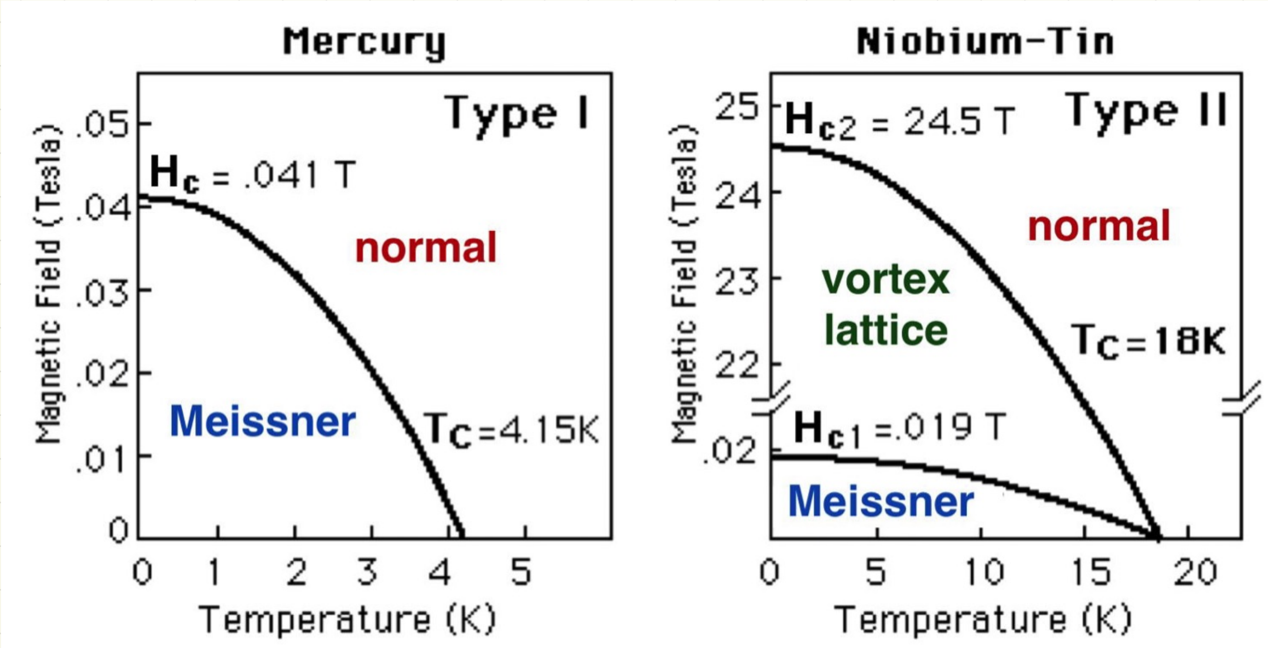
$H_c(T)$ is called the **critical field**. For most elemental **type-I** materials (e.g., Hg, Sn, Nb, Pb) one has $H_c(0) \lesssim 1\,\mathrm{kG}$. In **type-II** materials there are two critical fields, with $H<H_{c1}$ the Meissner phase and $H_{c1}<H<H_{c2}$ the **mixed phase** where quantized **vortices** with flux ϕ_L penetrate the system. For $H>H_{c2}$ there is uniform flux penetration and the system is normal. The **upper critical field** H_{c2} is set by the condition that the vortex cores, whose width is given by the **coherence length** ξ, start to overlap. Thus $H_{c2} = \phi_L/2\pi\xi^2$. Vortex lines can be pinned by disorder, and themselves may exist in different phases (solid, glass, liquid = normal). Typically the ratio H_{c2}/H_{c1} is given by



vortex lattice in $NbSe_2$
$H = 1\,T$, $T = 1.8\,K$

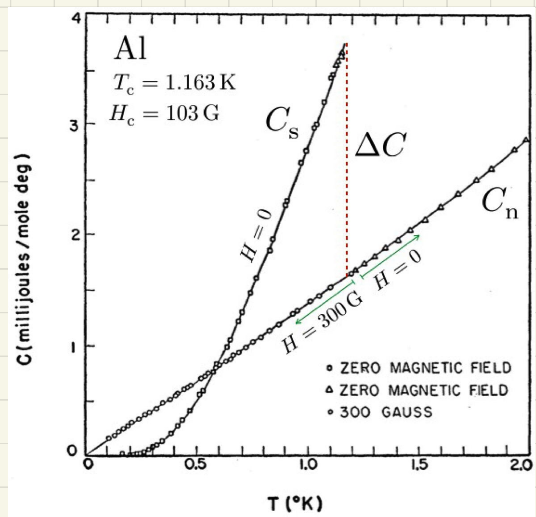
$$H_{c2} = \sqrt{2}\,\kappa\, H_{c1}$$

where $\kappa = \lambda/\xi$. Type-II materials require $H_{c2} > H_{c1}$, i.e. $\kappa > \sqrt{2}$, and typically pertains in alloys like Nb-Sn.



- **Specific heat jump:**

The specific heat in metals is given by $C_V/V \equiv c_V = \frac{\pi^2}{3} k_B^2 T g(\varepsilon_F)$. In a superconductor, after subtracting off the phonon contribution AT^3, one finds $c_V^{elec} \propto e^{-\Delta/k_B T}$. There is also a jump in c_V right at $T = T_c$, the magnitude of which is about 3x as large as $c_V(T_c^+)$. Consistent with a second order phase transition and $\alpha = 0$. As we shall see, this behavior is explained by a simple Landau theory.



C_p in Al at $H = 0$ and $H = 300$ G. $H_c = 100$ G.

- **Persistent currents:**

  Consider a metallic ring in its Meissner state with quantized trapped flux $n\phi_L$ ($n \in \mathbb{Z}$). When the field H is lowered to $H = 0$, the trapped flux remains, and there is a **persistent current** which flows in the ring. In thick rings, such currents have been demonstrated to exist undiminished for years. They decay via tunneling, and their lifetimes astronomically long.

- **Tunneling and Josephson effect:**

  The SC energy gap can be measured using electron tunneling between a SC and a normal metal, or between two SCs. In the case of a weak link between two SCs, current can flow at zero bias, i.e. the **Josephson effect.**

– Thermodynamics of Superconductors

Let f = Helmholtz free energy density. Then

$$df = -s\,dT + \frac{1}{4\pi}\vec{H}\cdot d\vec{B}$$

which says $f = f(T, \vec{B})$. Here $\vec{B}$ is the **magnetic field**, but what is under direct experimental control is the **magnetizing field** $\vec{H}$, since $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ and $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ where ρ and $\vec{j}$ are the free charge and current densities. Note

$$s = -\left(\frac{\partial f}{\partial T}\right)_{\vec{B}} \quad , \quad \vec{H} = 4\pi\left(\frac{\partial f}{\partial \vec{B}}\right)_T$$

Recall $\vec{B} = \vec{H} + 4\pi\vec{M} \equiv \mu\vec{H}$ where $\vec{M}$ is the magnetization density. Since we have no direct control over $\vec{B}$, we make a Legendre transformation to the Gibbs free energy density $g(T,\vec{H})$, viz.

$$g(T,\vec{H}) = f(T,\vec{B}) - \frac{1}{4\pi}\vec{B}\cdot\vec{H}$$

$$dg = -s\,dT - \frac{1}{4\pi}\vec{B}\cdot d\vec{H}$$

and thus $s = -\partial g/\partial T|_{\vec{H}}$ and $\vec{B} = -4\pi\,\partial g/\partial\vec{H}|_T$. If our sample is isotropic, then

$$g(T,\vec{H}) = g(T,0) - \frac{1}{4\pi}\int_0^H dH'\,B(H')$$

In a normal metal, $\mu \simeq 1$ and $B \simeq H$, hence

$$g_n(T,H) = g_n(T,H=0) - \frac{H^2}{8\pi}$$

But in the Meissner phase of a superconductor, $B = 0$, and we then have

$$g_s(T,H) = g_s(T,0)$$

For a type-I material, the free energies cross at $H = H_c$, i.e.

$$g_s(T,0) = g_n(T,0) - \frac{H_c^2}{8\pi}$$

and we identify $-H_c^2/8\pi$ as the **condensation energy**

density. We now have

$$g_s(T,H) - g_n(T,H) = \frac{1}{8\pi}\left(H^2 - H_c^2(T)\right)$$

from which we conclude that the SC state is the thermodynamically stable one for $H < H_c(T)$. Differentiate wrt T to now obtain

$$S_s(T,H) - S_n(T,H) = \frac{1}{4\pi} H_c(T)\,\frac{dH_c(T)}{dT} < 0$$

since $H_c(T)$ is a decreasing function. The entropy difference is independent of the magnetizing field $\vec{H}$. The latent heat $l = T\Delta s$ vanishes at the transition because $\Delta s = 0$, but the specific heat is discontinuous:

$$c_s(T_c, H=0) - c_n(T_c, H=0) = \frac{T_c}{4\pi}\left(\frac{dH_c(T)}{dT}\right)^2_{T_c} > 0$$

Phenomenologically, we had $H_c'(T_c) \approx -2H_c(0)/T_c$, hence

$$\Delta c \equiv c_s(T_c, 0) - c_n(T_c, 0) \approx \frac{H_c^2(0)}{\pi T_c}$$

For general $T < T_c$, then

$$\Delta c(T,H) = \frac{T}{8\pi}\,\frac{d^2 H_c^2(T)}{dT^2}$$

wrong near $T=0$

$$\simeq \frac{T H_c^2(0)}{2\pi T_c^2}\left\{3\left(\frac{T}{T_c}\right)^2 - 1\right\}$$

from which we identify $\gamma \simeq H_c^2(0)/2\pi T_c^2$, with $c_n(T) = \gamma T$.

Note also $\Delta C(T_c, 0)/C_n(T_c, 0) \simeq 2$. Within the microscopic BCS theory, one finds

$$H_c(T) = H_c(0)\left\{1 - \alpha\left(\frac{T}{T_c}\right)^2 + \mathcal{O}\left(e^{-\Delta/k_B T}\right)\right\}$$

with $\alpha \approx 1.07$. Thus $H_c^{BCS}(0) = \left(2\pi\gamma T_c^2/\alpha\right)^{1/2}$.

- **London Theory**

This is a two fluid model of superconductors. We write

$$n = n_n + n_s$$
$$\vec{j} = \vec{j}_n + \vec{j}_s = -e\left(n_n \vec{v}_n + n_s \vec{v}_s\right)$$

The normal fluid is dissipative, hence $\vec{j}_n = \sigma_n \vec{E}$. The superfluid is ballistic, with

$$m\frac{d\vec{v}_s}{dt} = -e\vec{E} \quad , \quad \frac{d\vec{j}_s}{dt} = \frac{n_s e^2}{m}\vec{E}$$

Add magnetic field:

$$\begin{aligned}\frac{d\vec{v}_s}{dt} &= -\frac{e}{m}\left(\vec{E} + \frac{\vec{v}_s}{c}\times\vec{B}\right) \\ &= \frac{\partial\vec{v}_s}{\partial t} + (\vec{v}_s\cdot\vec{\nabla})\vec{v}_s \\ &= \frac{\partial\vec{v}_s}{\partial t} + \vec{\nabla}\left(\tfrac{1}{2}\vec{v}_s^{\,2}\right) - \vec{v}_s\times(\vec{\nabla}\times\vec{v}_s)\end{aligned}$$

We conclude

$$\frac{\partial\vec{v}_s}{\partial t} + \frac{e}{m}\vec{E} + \vec{\nabla}\left(\tfrac{1}{2}\vec{v}_s^{\,2}\right) = \vec{v}_s\times\left(\vec{\nabla}\times\vec{v}_s - \frac{e\vec{B}}{mc}\right)$$

Now take the curl and use $\vec{\nabla}\times\vec{E} = -c^{-1}\partial_t\vec{B}$ to find

$$\frac{\partial\vec{Q}}{\partial t} = \vec{\nabla}\times(\vec{v}_s\times\vec{Q})$$

where

$$\vec{Q} = \vec{\nabla}\times\vec{v}_s - \frac{e\vec{B}}{mc}$$

Thus if $\vec{Q}=0$ it must remain so. We assume that $\vec{Q}=0$ holds in equilibrium, whence

$$\vec{\nabla}\times\vec{v}_s = \frac{e\vec{B}}{mc} \quad\Rightarrow\quad \vec{\nabla}\times\vec{j}_s = -\frac{n_s e^2}{mc}\vec{B}$$

The latter equation entails the Meissner effect, since taking the curl gives

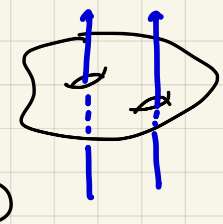
$$\nabla^2\vec{B} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\frac{4\pi}{c}\vec{\nabla}\times\vec{j} = \frac{4\pi n_s e^2}{mc^2}\vec{B} \equiv \lambda_L^{-2}\vec{B}$$

Here $\lambda_L = (mc^2/4\pi n_s e^2)^{1/2}$ is the **London penetration depth**. Since $\vec{B}=\vec{\nabla}\times\vec{A}$, we may write

$$\vec{j}_s = -\frac{c}{4\pi\lambda_L^2}\vec{A}$$

provided an appropriate choice of gauge for $\vec{A}$ is made. Since $\vec{\nabla}\cdot\vec{j}_s=0$ holds in steady state, we are led to conclude that $\vec{\nabla}\cdot\vec{A}=0$ is the proper gauge (a.k.a. Coulomb gauge). Still this allows for a "little gauge transformation" $\vec{A}\to\vec{A}+\vec{\nabla}\chi$ provided $\nabla^2\chi=0$. For a simply connected body, $\vec{j}_s\cdot\hat{n}\big|_{\partial\Omega}=0$

along its boundaries, hence $\hat{n}\cdot\vec{\nabla}\chi\big|_{\partial\Omega}=0$ on the boundaries, which says χ is determined everywhere up to a global constant. If the SC is multiply connected, then the condition $\hat{n}\cdot\vec{\nabla}\chi\big|_{\partial\Omega}=0$ allows for non-constant solutions. Let $\mathcal{D}$ be a hole in the SC. Then



$$\oint_{\partial\mathcal{D}} d\vec{\ell}\cdot\vec{A} = \int_{\mathcal{D}} dS\,\hat{n}\,\vec{B} = \Phi_{\mathcal{D}} \;=\; \text{flux through } \mathcal{D}$$

Inside the SC, we may write $\vec{A}=\vec{\nabla}\chi$ (with or without a supercurrent), hence $\Phi_{\mathcal{D}}=\oint_{\mathcal{D}} d\chi=\Delta\chi$. F. London argued that if the gauge transformation $\vec{A}\to\vec{A}+\vec{\nabla}\chi$ were then associated with a charge e object, then $\Phi_{\mathcal{D}}=n\phi_0$ where $\phi_0=hc/e$ = Dirac flux quantum. Onsager corrected this to $\Phi_{\mathcal{D}}=n\phi_L$ where $\phi_L=hc/e^*=hc/2e$ = London flux quantum on the basis that the condensate is composed of electron pairs. This has been confirmed by experiment.

DeGennes argued thusly: the free energy is

$$F=\underbrace{\int d^3x\, f_0}_{E_{\text{cond}}}+E_{\text{kinetic}}+E_{\text{field}}$$

$$E_{\text{kinetic}}=\int d^3x\,\tfrac{1}{2}mn_s\vec{v}_s^{\,2}(\vec{x})=\int d^3x\,\frac{m}{2n_se^2}\vec{j}_s^{\,2}(\vec{x})$$

$$E_{\text{field}}=\int d^3x\,\frac{\vec{B}^2(\vec{x})}{8\pi}$$

In steady state, $\vec{\nabla}\times\vec{B}=4\pi c^{-1}\vec{j}_s$, hence

$$F = F_0 + \int d^3x \left\{ \frac{\vec{B}^2}{8\pi} + \frac{\lambda_L^2}{8\pi} (\vec{\nabla} \times \vec{B})^2 \right\}$$

whence

$$\frac{\delta F}{\delta \vec{B}(\vec{x})} = 0 \Rightarrow \vec{B} - \lambda_L^2 \vec{\nabla} \times \vec{B} = 0$$

field operator (bosonic)

- **Ginzburg – Landau theory**

In ^{4}He, the order parameter is $\Psi(\vec{x}) = \langle \psi(\vec{x}) \rangle$.

$\Psi \neq 0 \Leftrightarrow$ Bose – Einstein condensation. Fermions cannot condense!

Rather, the order parameter of an s-wave superconductor is

$$\Psi(\vec{x}) = \langle \underbrace{\psi_\uparrow(\vec{x}) \psi_\downarrow(\vec{x})}_{\text{composite operator with BE statistics}} \rangle$$