Thus, if $m(\infty) > 0 > m(-\infty)$, we have normalizable sol$^{\underline{ns}}$ $\vec{\psi}_1$ and $\vec{\psi}_4$, while if $m(\infty) < 0 < m(-\infty)$, we have normalizable sol$^{\underline{ns}}$ $\vec{\psi}_2$ and $\vec{\psi}_3$. The time-dependence is

(i) $e^{ik_y y} e^{-iEt/\hbar} = e^{ik_y(y-ct)}$: up-mover, $\Upsilon = +1$

(ii) $e^{ik_y y} e^{-iEt/\hbar} = e^{ik_y(y+ct)}$: down-mover, $\Upsilon = -1$

- **Lecture 4 (Jan 14)** : **Adiabatic theorem and Berry's phase**

Consider a Hamiltonian $H(\vec{\lambda})$ dependent on a set of parameters $\vec{\lambda} = \{\lambda_1, \ldots, \lambda_K\}$, with eigenfunctions $\{\varphi_n(\vec{\lambda})\}$:

$$H(\vec{\lambda})\,|\varphi_n(\vec{\lambda})\rangle = E_n(\vec{\lambda})\,|\varphi_n(\vec{\lambda})\rangle$$

Now let $\vec{\lambda} = \vec{\lambda}(t)$ be time-dependent. The adiabatic theorem says that if $\vec{\lambda}(t)$ evolves very slowly, such that $\Delta E_n \cdot \tau \gg \hbar$, where τ is the time scale of the variation, i.e. $\tau = |\vec{\lambda}|/|\dot{\vec{\lambda}}|$, and $\Delta E_n = E_{n+1} - E_n$ is the gap between consecutive levels, then the solutions to the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H(\vec{\lambda}(t))\,|\Psi(t)\rangle$$

are proportional to the instantaneous adiabatic WFs, with

$$|\Psi_n(t)\rangle = e^{i\gamma_n(t)}\, e^{-i\hbar^{-1}\int^t dt'\, E_n(\vec{\lambda}(t'))}\, |\varphi_n(\vec{\lambda}(t))\rangle$$

with corrections which vanish exponentially in $\Delta E_n \tau/\hbar$. We recognize the $\exp\left[-i\hbar^{-1}\int^t dt'\, E_n(\vec{\lambda}(t'))\right]$ term as the dynamical phase accrued. What is $\gamma_n(t)$? Taking the time derivative and then the overlap with $\langle \varphi_n(\vec{\lambda})|$, we find

$$\frac{d\gamma_n}{dt} = i\langle \varphi_n(\vec{\lambda}(t))|\frac{d}{dt}|\varphi_n(\vec{\lambda}(t))\rangle$$
$$\equiv \vec{A}_n(\vec{\lambda}(t))\cdot\frac{d\vec{\lambda}(t)}{dt} \equiv A_n(t)$$

where

$$A^{\mu}_n(\vec{\lambda}) = i\langle \varphi_n(\vec{\lambda})|\frac{\partial}{\partial \lambda_\mu}|\varphi_n(\vec{\lambda})\rangle$$

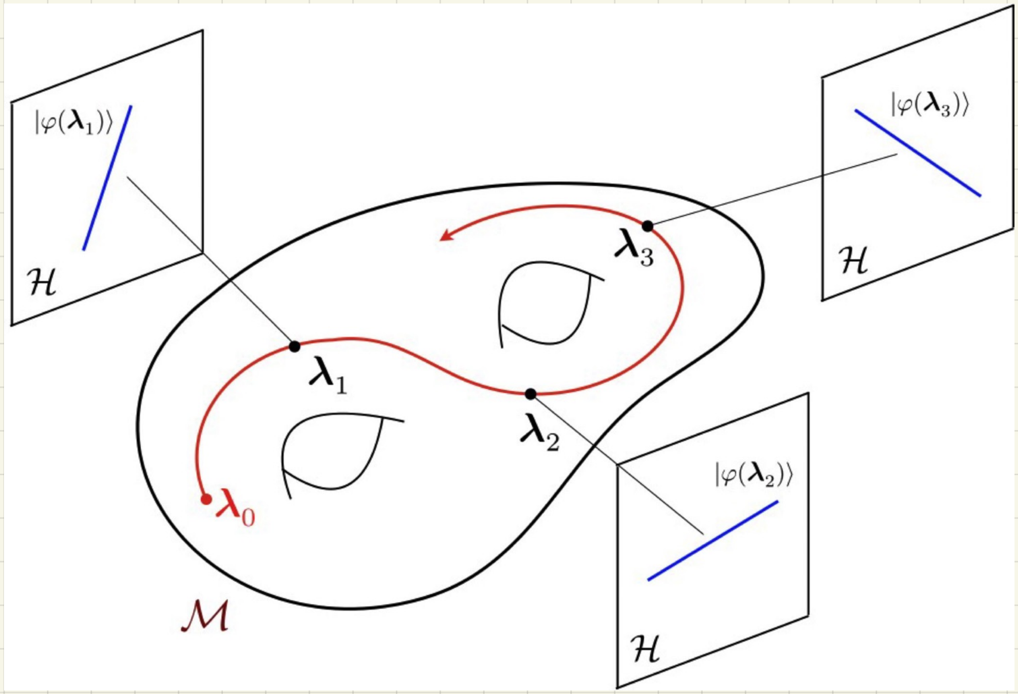
is the **Berry** (or **geometric**) **connection**. If $\vec{\lambda}(t)$ traverses a closed loop C in the space of parameters, then $|\Psi_n(t)\rangle$ will accrue a **geometric phase** (also called **Berry's phase**),

$$\gamma_n(C) = \oint_C d\vec{\lambda}\cdot\vec{A}_n(\vec{\lambda})$$

We can also eliminate the dynamical phase entirely by defining $\tilde{H}_n(\vec{\lambda}) \equiv H - E_n(\vec{\lambda})$. Then if

$$i\hbar\frac{\partial}{\partial t}|\tilde{\Psi}_n(t)\rangle = \tilde{H}_n(\vec{\lambda}(t))|\tilde{\Psi}_n(t)\rangle$$

in the adiabatic limit we have $|\tilde{\Psi}_n(t)\rangle = e^{i\gamma_n(t)}|\varphi_n(\vec{\lambda}(t))\rangle$. Note that the geometric phase is invariant under time reparameterization and depends only on the path traversed by $\vec{\lambda}$ in the parameter space manifold M.



Mathematical setting: **Hermitian line bundle** over $\mathcal{M}$.
The parameter space manifold $\mathcal{M}$ is the **base space**, and the adiabatic WFs $|\varphi_n(\vec{\lambda})\rangle$ are the **fibers**, which are projections of a **Hilbert space** $\mathcal{H}$. The adiabatic theorem furnishes us with a definition of **parallel transport** of $|\varphi_n(\vec{\lambda})\rangle$ along a curve C. The object $\vec{A}_n(\vec{\lambda})$ is the **connection** and the Berry phase $\gamma_n(C)$ is the **holonomy**, which for a closed loop does not depend on the starting point. The **curvature tensor** of the bundle is given by

$$\Omega_n^{\mu\nu}(\vec{\lambda}) = \frac{\partial A_n^{\nu}}{\partial \lambda_\mu} - \frac{\partial A_n^{\mu}}{\partial \lambda_\nu}$$

$$= i\left\langle \frac{\partial \varphi_n}{\partial \lambda_\mu} \middle| \frac{\partial \varphi_n}{\partial \lambda_\nu} \right\rangle - i\left\langle \frac{\partial \varphi_n}{\partial \lambda_\nu} \middle| \frac{\partial \varphi_n}{\partial \lambda_\mu} \right\rangle$$

Using completeness of the adiabatic basis, we may write

$$\Omega_n^{\mu\nu}(\vec{\lambda}) = i \sum_{l}{}' \frac{\langle \varphi_n | \frac{\partial H}{\partial \lambda_\mu} | \varphi_l \rangle \langle \varphi_l | \frac{\partial H}{\partial \lambda_\nu} | \varphi_n \rangle - (\mu \leftrightarrow \nu)}{\left(E_n(\vec{\lambda}) - E_l(\vec{\lambda})\right)^2}$$

$l \neq n$

Note that the connection is **gauge-covariant**, viz.

$$|\varphi_n(\vec{\lambda})\rangle \to e^{i f_n(\vec{\lambda})} |\varphi_n(\vec{\lambda})\rangle \quad \Rightarrow \quad A_n^\mu(\vec{\lambda}) \to A_n^\mu(\vec{\lambda}) - \frac{\partial f_n(\vec{\lambda})}{\partial \lambda_\mu}$$

The curvature, however, is **gauge-invariant**. Can we fix a gauge and give an unambiguous definition of the connection $A_n^\mu(\vec{\lambda})$? One way might be to choose a particular point in space $\vec{r}_0$ and demand $\langle \vec{r}_0 | \vec{\varphi}_n(\vec{\lambda}) \rangle \in \mathbb{R}_+$ for all $\vec{\lambda} \in \mathcal{M}$. But this prescription fails if there exists a value of $\vec{\lambda}$ where $\langle \vec{r}_0 | \vec{\varphi}_n(\vec{\lambda}) \rangle = 0$.

The integral of the curvature $\Omega_n^{12}(\vec{\lambda})$ over a closed two-dimensional base space is a **topological invariant**. Using Stokes' theorem, we may turn an area integral of the curvature into a line integral of the connection:

$$\int_{\mathcal{M}} d^2\lambda \, \Omega_n^{12}(\vec{\lambda}) = - \sum_i \oint_{C_i} d\vec{\lambda} \cdot \vec{A}_n(\vec{\lambda})$$

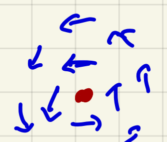
$\mathcal{M}$

where C_i encloses the i^{th} singularity of $\vec{A}_n(\vec{\lambda})$ in a counterclockwise fashion ($\mathcal{M}$ is assumed orientable). Singularities occur at points $\vec{\lambda}_i$ where $|\varphi_n(\vec{\lambda}_i)\rangle$ is ill defined (using, for example, the prescription $\langle \vec{r}_0 | \varphi_n(\vec{\lambda})\rangle \in \mathbb{R}_+$). In the vicinity of a given singularity, the connection has a vortex, behaving as

$$\vec{A}_n(\vec{\lambda}) = -q_i \, \vec{\nabla}_{\vec{\lambda}} \tan^{-1}\left(\frac{\lambda_2 - \lambda_{i,2}}{\lambda_1 - \lambda_{i,1}}\right) + \vec{A}_n^{\,reg}(\vec{\lambda})$$

which can be "unwound" by a **singular gauge transformation**,

$$|\varphi_n(\vec{\lambda})\rangle \equiv e^{iq_i \zeta(\vec{\lambda}-\vec{\lambda}_i)} |\tilde{\varphi}_n(\vec{\lambda}_i)\rangle$$



vortex

with $\zeta(\vec{\lambda}-\vec{\lambda}_i) = \tan^{-1}\left[(\lambda_2 - \lambda_{i,2})/(\lambda_1 - \lambda_{i,1})\right]$. Thus,

$$C_n \equiv \frac{1}{2\pi} \int_{\mathcal{M}} d^2\lambda \, \Omega^{12}(\vec{\lambda}) = \sum_i q_i \in \mathbb{Z}$$

In mathematical parlance, C_n is the **Chern number** of the Hermitian line bundle corresponding to the adiabatic wavefunction $|\varphi_n(\vec{\lambda})\rangle$.

- **Example : $S=\frac{1}{2}$ object in a magnetic field**

  The Hamiltonian is

$$H(\vec{B}(t)) = g\mu_B \, \vec{B}\cdot\vec{\sigma} = g\mu_B B \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

where $\vec{B}(t) = B(t)\,\hat{n}(t)$, and $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.
The adiabatic eigenfunctions are

$$|\varphi_+(\hat{n})\rangle = \begin{pmatrix} u \\ v \end{pmatrix} \quad , \quad |\varphi_-(\hat{n})\rangle = \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix}$$

where $u = \cos\frac{\theta}{2}$ and $v = \sin\frac{\theta}{2}e^{i\phi}$. Then

$$H(\vec{B})\,|\varphi_\pm(\hat{n})\rangle = \pm\, g\mu_B B$$

The connections are :

$$A_+(t) = i\langle\varphi_+|\frac{d}{dt}|\varphi_+\rangle = i(\bar{u}\dot{u} + \bar{v}\dot{v}) = -\frac{1}{2}(1-\cos\theta)\dot{\phi} = -\frac{1}{2}\dot{\omega}$$

$$A_-(t) = i\langle\varphi_-|\frac{d}{dt}|\varphi_-\rangle = i(v\dot{\bar{v}} + u\dot{\bar{u}}) = +\frac{1}{2}(1-\cos\theta)\dot{\phi} = +\frac{1}{2}\dot{\omega}$$

where $d\omega$ is the differential solid angle subtended by the path $\hat{n}(t)$. Thus $\gamma_\pm(C) = -\frac{1}{2}\omega_C$ is $\mp\frac{1}{2}$ times the solid angle subtended by $\hat{n}_C(t)$ on the Bloch sphere. We stress that $\gamma_\pm(C)$ is dependent only on the path of $\hat{n}$ itself and is time-reparameterization invariant. The components of the connections are then

$$A^\theta_\pm(\hat{n}) = 0 \quad , \quad A^\phi_\pm(\hat{n}) = \mp\frac{1}{2}(1-\cos\theta)$$

and the curvature is

$$\Omega^{\theta\phi}_\pm(\hat{n}) = \frac{\partial A^\phi_\pm}{\partial\theta} - \frac{\partial A^\theta_\pm}{\partial\phi} = \mp\frac{1}{2}\sin\theta$$

Integrating over the base space S^2,

$$C_\pm = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \Omega_\pm^{\theta\phi}(\theta,\phi) = \mp 1$$

For any rank-2 Hamiltonian $H(\vec{\lambda}) = \Delta(\vec{\lambda})\,\hat{n}(\vec{\lambda})\cdot\vec{\sigma} + E_0(\vec{\lambda})\,\mathbb{1}$ the Chern numbers of the two bands are given by

$$C_\pm = \pm\frac{1}{4\pi}\int_M d^2\lambda \; \hat{n}\cdot\frac{\partial \hat{n}}{\partial \lambda_1}\times\frac{\partial \hat{n}}{\partial \lambda_2}$$

- **Two-band models**

The base space for 2D lattice tight-binding models is the 2-torus T^2, coordinatized by (θ_1, θ_2), where

$$\vec{k} = \frac{\theta_1}{2\pi}\vec{b}_1 + \frac{\theta_2}{2\pi}\vec{b}_2$$

is the Bloch wavevector labeling the adiabatic eigenstates of the Hamiltonian $H(\vec{k})$. We may then write

$$\hat{H}(\vec{k}) = E_0(\vec{\theta}) + \Delta(\vec{\theta})\,\hat{n}(\vec{\theta})\cdot\vec{\sigma}$$

$$E_+ = E_0 + \Delta$$

$$E_- = E_0 - \Delta$$

and

$$C_\pm = \pm\frac{1}{4\pi}\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \; \hat{n}\cdot\frac{\partial \hat{n}}{\partial \theta_1}\times\frac{\partial \hat{n}}{\partial \theta_2}$$

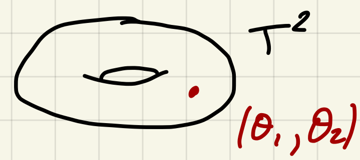
Note that $C_+ + C_- = 0$. For a Hamiltonian

$$H = d_0(\vec{\theta}) + \vec{d}(\vec{\theta})\cdot\hat{n}$$

$$\vartheta = \vartheta(\theta_1, \theta_2)$$
$$\chi = \chi(\theta_1, \theta_2)$$

we define

$$\hat{d}(\vec{\theta}) = (\sin\vartheta\cos\chi,\ \sin\vartheta\sin\chi,\ \cos\vartheta)$$



where to avoid confusion with the Bloch phases we have taken $(\theta, \phi) \to (\vartheta, \chi)$. The adiabatic WFs are

$$|\varphi_+\rangle = \begin{pmatrix} \cos\frac{1}{2}\vartheta \\ \sin\frac{1}{2}\vartheta e^{i\chi} \end{pmatrix} \quad , \quad |\varphi_-\rangle = \begin{pmatrix} -\sin\frac{1}{2}\vartheta e^{-i\chi} \\ \cos\frac{1}{2}\vartheta \end{pmatrix}$$

The singularity occurs when $\vartheta = \pi$, where χ is ill-defined. We must then find all points (θ_1, θ_2) in the BZ torus T^2 such that $\vartheta(\theta_1, \theta_2) = \pi$, and then compute the winding of $\zeta = -\chi$ as $\vec{\theta}$ winds counterclockwise around $\vec{\theta}_i$. (We have to define $\zeta = -\chi$ because the singularity is at the south pole on the Bloch sphere.) Two models:

(i) $p_x + ip_y$ superconductor:

$$H(\vec{\theta}) = \begin{pmatrix} m - 2t\cos\theta_1 - 2t\cos\theta_2 & \Delta(\sin\theta_1 - i\sin\theta_2) \\ \Delta(\sin\theta_1 + i\sin\theta_2) & -m + 2t\cos\theta_1 + 2t\cos\theta_2 \end{pmatrix}$$

$$\vec{d}(\vec{\theta}) = (\Delta\sin\theta_1,\ \Delta\sin\theta_2,\ m - 2t\cos\theta_1 - 2t\cos\theta_2)$$

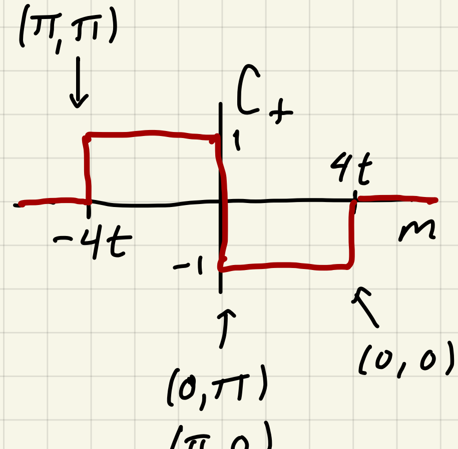
$$= |\vec{d}|(\sin\vartheta\cos\chi,\ \sin\vartheta\sin\chi,\ \cos\vartheta) \ ; \ d_0(\vec{\theta}) = 0$$

(ii) Haldane honeycomb lattice model:

$$d_0(\vec{\theta}) = -2t_2[\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 + \theta_2)]\cos\phi$$
$$d_1(\vec{\theta}) = -t_1(1 + \cos\theta_1 + \cos\theta_2)$$
$$d_2(\vec{\theta}) = t_1(\sin\theta_1 - \sin\theta_2)$$
$$d_3(\vec{\theta}) = m - 2t_2[\sin\theta_1 + \sin\theta_2 - \sin(\theta_1 + \theta_2)]\sin\phi$$

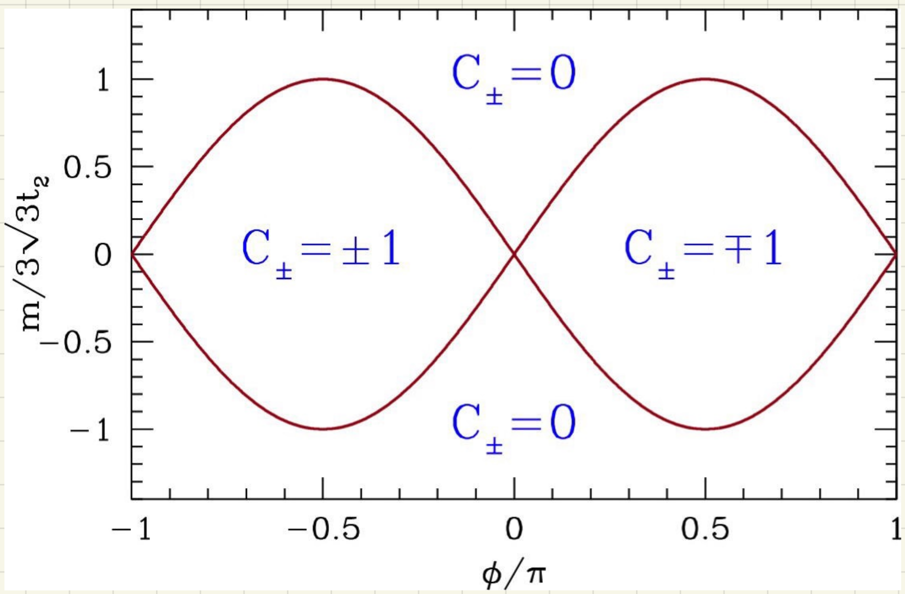
Analysis : see lecture notes §4.4.6

Results :



(i) $C_\pm(m) = \begin{cases} 0 & \text{if } m < -4t \\ \pm 1 & \text{if } m \in [-4t, 0] \\ \mp 1 & \text{if } m \in [0, 4t] \\ 0 & \text{if } m > 4t \end{cases}$

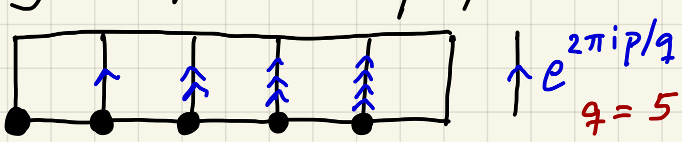
(ii) phase diagram :

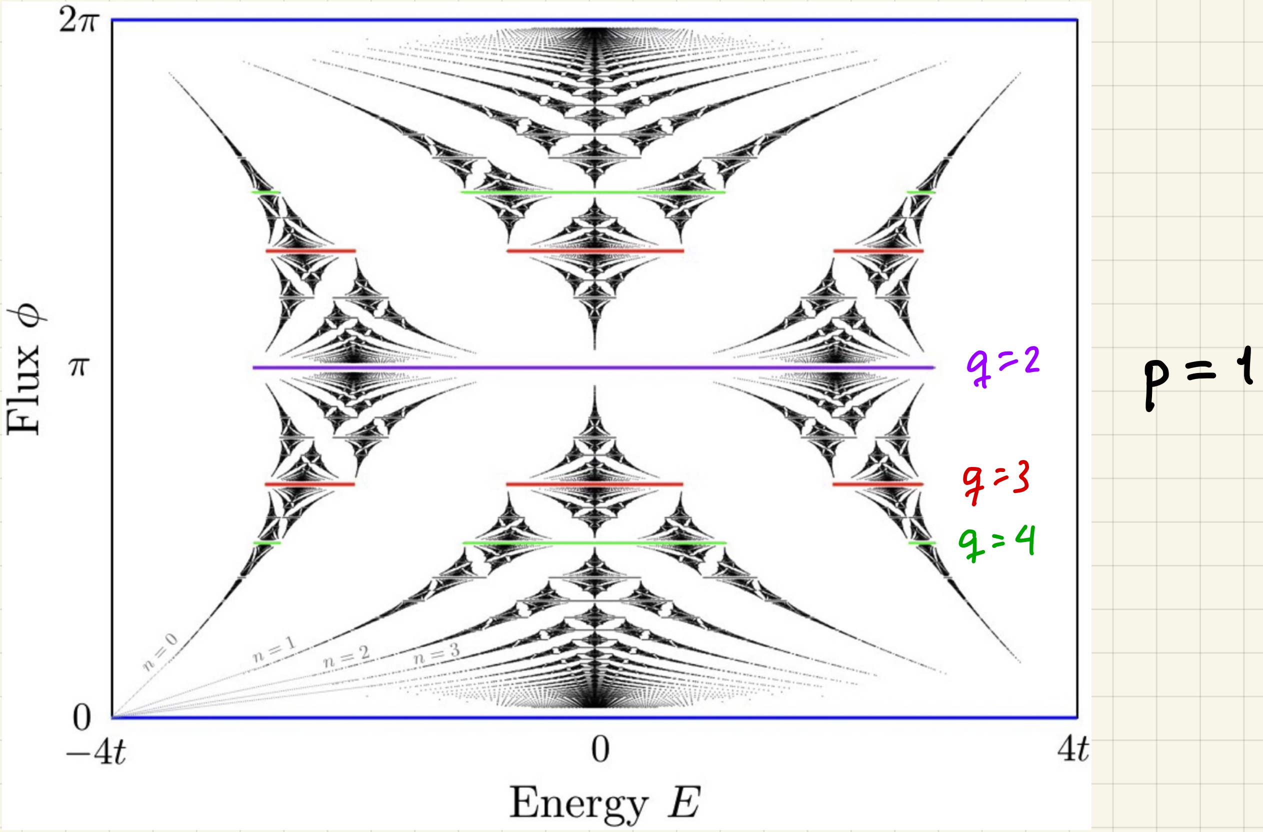


• **Chern numbers for Hofstadter's model** (square lattice) :

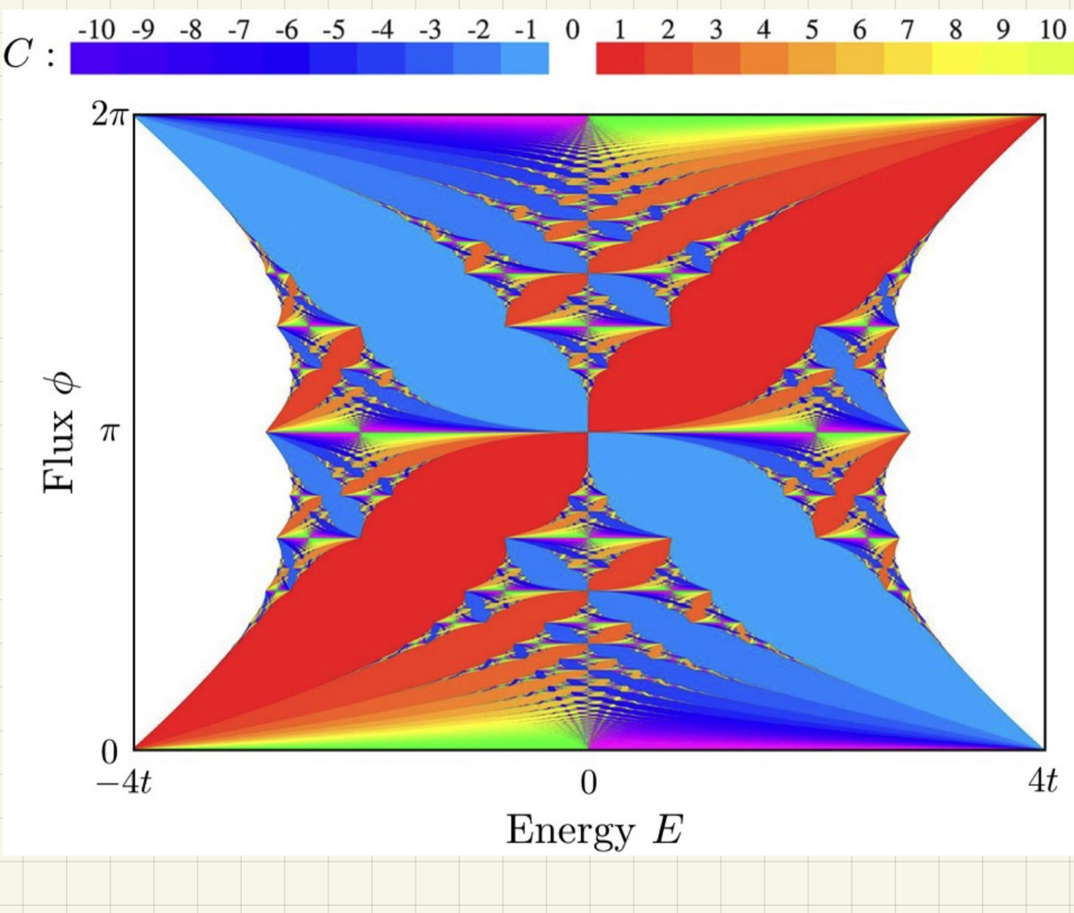
$$H = -t \begin{pmatrix} 2\cos\theta_2 & 1 & 0 & \cdots & e^{i\theta_1} \\ 1 & 2\cos\left(\theta_2 + \frac{2\pi p}{q}\right) & 1 & & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ e^{-i\theta_1} & 0 & \cdots & 1 & 2\cos\left(\theta_2 + \frac{2\pi(q-1)p}{q}\right) \end{pmatrix}$$

Recall Hofstadter's butterfly, showing the spectra vs. plaquette flux for the square lattice:

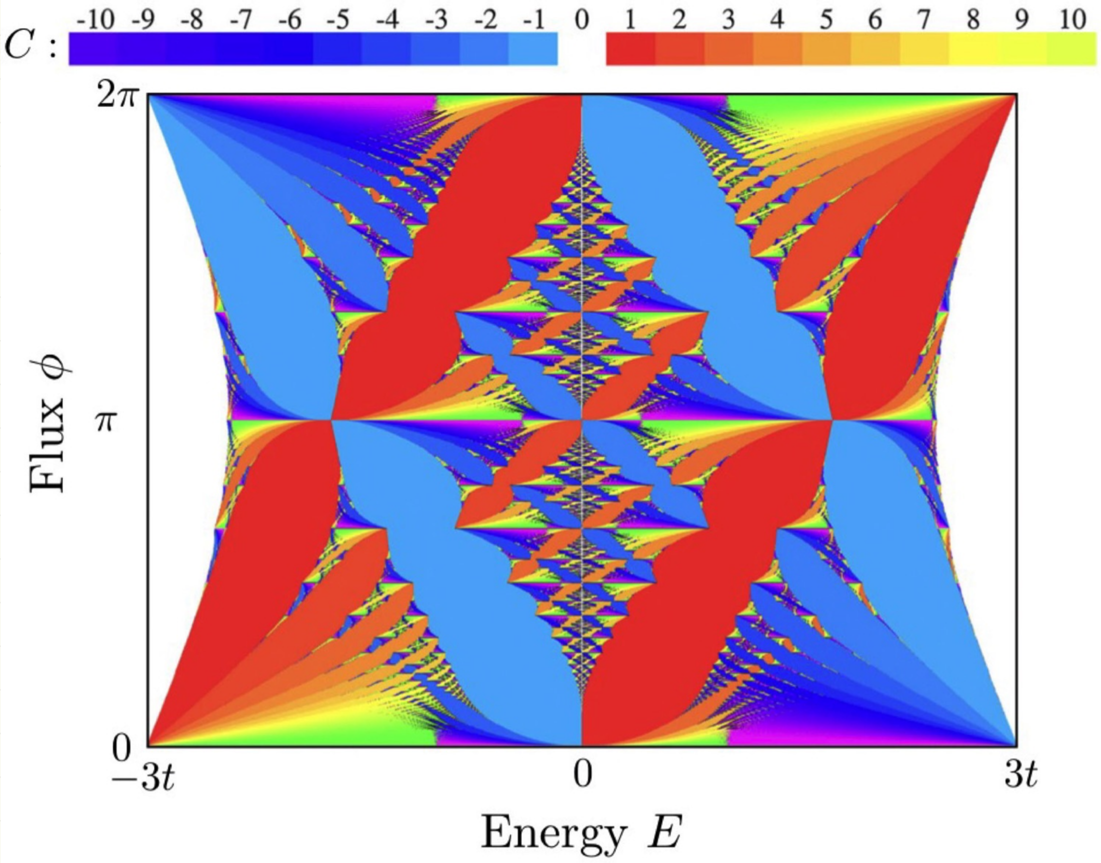




Each energy gap is associated with a Chern number. Plotting the Chern numbers in color yields the **Avron-Hofstadter butterfly**:



Honeycomb lattice Avron-Hofstadter butterfly:



- Semiclassical dynamics of Bloch wavepackets

The Hamiltonian is

$$H(t) = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}(\vec{r},t)\right)^2 + V(\vec{r})$$

with $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$. We choose a gauge in which there is no scalar potential: $\phi = 0$. We are interested in describing the semiclassical evolution of Bloch wave packets