PHYSICS 211B : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #4

(1) For the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 - \sum_i \hat{Q}_i \, \phi_i(t) \quad ,$$

the response to second order may be written

$$\langle \, \Psi(t) \, | \, \hat{Q}_i \, | \, \Psi(t) \, \rangle = \int\limits_{-\infty}^{\infty} \! dt' \, \chi_{ij}(t,t') \, \phi_j(t') + \int\limits_{-\infty}^{\infty} \! dt' \int\limits_{-\infty}^{\infty} \! dt'' \, \chi^{(2)}_{ijk}(t,t',t'') \, \phi_j(t') \, \phi_k(t'') + \mathcal{O}(\phi^3) \quad .$$

Find an expression for the nonlinear response tensor $\chi^{(2)}_{ijk}(t,t',t'')$ in terms of the spectral properties of $\hat{H}_0$. Some hints:

– From the above expression for $\langle Q_i(t) \rangle$ you can assume $\chi^{(2)}_{ijk}(t,t',t'') = \chi^{(2)}_{ikj}(t,t'',t')$. Your final expression should honor this symmetry.

– To obtain the linear response tensor $\chi_{ij}(t,t')$, we computed the first functional variation,

$$\begin{aligned} \frac{\delta \langle \hat{Q}_i(t) \rangle}{\delta \phi_j(t')} = \bigg\{ & - \frac{i}{\hbar} \, \langle \, \Psi(t_0) \, | \, U^\dagger(t',t_0) \, \hat{Q}_j \, \hat{U}^\dagger(t,t') \, \hat{Q}_i \, U(t,t_0) \, | \, \Psi(t_0) \, \rangle \\ & + \frac{i}{\hbar} \, \langle \, \Psi(t_0) \, | \, U^\dagger(t,t_0) \, \hat{Q}_i \, \hat{U}(t,t') \, \hat{Q}_j \, \hat{U}(t',t_0) \, | \, \Psi(t_0) \, \rangle \bigg\} \times \Theta(t-t') \, \Theta(t'-t_0) \quad , \end{aligned}$$

with $t_0 \to -\infty$, and then set $\boldsymbol{\phi} = 0$. To obtain the nonlinear response $\chi^{(2)}_{ijk}(t,t',t'')$, we must first functionally differentiate with respect to $\phi_k(t'')$. Since there are three appearances of $\hat{U}$ or $\hat{U}^\dagger$ in each of the above matrix elements, you should get *six* terms in all.

(2) Sketch the spread of particle-hole excitation frequencies, depicted for a $d=3$ Fermi gas in Fig. 9.3 of the lecture notes, in dimensions $d=2$ and $d=1$.

(3) We previously saw how the static density susceptibility of the electron gas could be written as

$$\hat{\chi}(\boldsymbol{q}) = \frac{\widehat{\sqcap}(\boldsymbol{q})}{1 + \frac{4\pi e^2}{\boldsymbol{q}^2} \, \widehat{\sqcap}(\boldsymbol{q})} \quad ,$$

where $\widehat{\sqcap}(\boldsymbol{q})$ is the polarization function. We can extend this expression to dynamical response, *viz.*

$$\hat{\chi}(\boldsymbol{q},\omega) = \frac{\widehat{\sqcap}(\boldsymbol{q},\omega)}{1 + \frac{4\pi e^2}{\boldsymbol{q}^2} \, \widehat{\sqcap}(\boldsymbol{q},\omega)} \quad .$$

Formally this may be taken as a definition of the dynamic polarization $\widehat{\sqcap}(\boldsymbol{q},\omega)$. In the *random phase approximation* (RPA), we replace $\widehat{\sqcap}(\boldsymbol{q},\omega) \to \chi^0(\boldsymbol{q},\omega)$, the noninteracting dynamic susceptibility, *i.e.*

$$\chi^0(\boldsymbol{q},t) = \frac{i}{\hbar V}\left\langle\left[\hat{n}(\boldsymbol{q},t),\hat{n}(-\boldsymbol{q},0)\right]\right\rangle\Theta(t)$$

$$\chi^0(\boldsymbol{q},\omega) = 2\int\frac{d^3k}{(2\pi)^3}\,\frac{f_{\boldsymbol{k}+\boldsymbol{q}} - f_{\boldsymbol{k}}}{\hbar\omega - \varepsilon(\boldsymbol{k}+\boldsymbol{q}) + \varepsilon(\boldsymbol{k}) + i\epsilon} \quad .$$

Using the RPA, you are invited to determine the plasmon dispersion for the two-dimensional electron gas with interactions $u(r) = e^2/r$ at $T = 0$. Some hints:

– Find the 2D Fourier transform of the interaction potential, $\hat{u}(\boldsymbol{q})$.

– Expand $\chi^0(\boldsymbol{q},\omega)$ in a series in ω^{-2}.

– Locate the pole in the RPA response function, and thereby obtain the solution $\omega(q)$ to order q^2.