Boulevard of Broken Symmetries*

Adriaan M.J. Schakel
Institut für Theoretische Physik
Freie Universität Berlin
Arnimallee 14, 14195 Berlin
e-mail: schakel@physik.fu-berlin.de

February 1, 2008

*Dedicated to the memory of Anna M. Konings.

Abstract

Effective theories of quantum liquids (superconductors and superfluids of various types) are derived starting from microscopic models at the absolute zero of temperature. Special care is taken to assure Galilei invariance. The effective theories are employed to investigate the quantum numbers carried by the topological defects present in the phases with spontaneously broken symmetries. Due to topological terms induced by quantum fluctuations, these numbers are sometimes found to be fractional. The zero-temperature effective theories are further used to study the quantum critical behavior of the liquid-to-insulator transition which these systems undergo as the applied magnetic field, the amount of impurities, or the charge carrier density varies. The classical, finite-temperature phase transitions to the normal state are discussed from the point of view of dual theories, where the defects of the original formulation become the elementary excitations. A connection with bosonization is pointed out.

PACS: 05.30.Fk, 05.30.Jp, 64.60.Ht, 64.60.Cn

Contents

List of Figures

List of Tables

Preface

The title of this report, Boulevard of Broken Symmetries, is chosen to indicate that the (condensed matter) systems we will be discussing in these pages have a spontaneously broken symmetry in common*. The notion of a spontaneously broken symmetry is one of the paradigms of modern physics. It plays a central role in today's understanding of many startling phenomena known in condensed matter physics, statistical physics, high-energy physics, and cosmology. A spontaneously broken symmetry indicates the presence of a global symmetry which is not apparent in the state the system is in. The symmetry is not lost, but implemented in a nontrivial way. If the symmetry involved is a continuous one (and the dimensionality is larger than two), the breaking is accompanied by the occurrence of gapless modes. Due to their gaplessness, they are the dominant elements in an effective description of the system with broken symmetry valid at low energy and small momentum.

This will be the subject of the first chapter. The topic has a long history, what is relatively new is how the effective description can be reconciled with Galilei invariance—one of the symmetries governing the nonrelativistic world of condensed matter physics. More specific, we will derive the effective theories at the absolute zero of temperature of classical hydrodynamics, of a superconductor both in the weak-coupling as well as in the strong-coupling limit, of superfluid ^{3}He, and of a bosonic superfluid. We also consider the behavior of a superconductor and the bosonic superfluid at finite temperature. Throughout this report we employ the powerful apparatus of quantum field theory, mostly in the functional-integral formulation. In this approach, the effective theories are obtained by integrating out certain degrees of freedom contained in the microscopic description of the quantum system under consideration.

The occurrence of gapless modes is not the only manifestation of a broken continuous symmetry. A closely related one is the appearance of topological defects. These objects, which are often of paramount importance to understand the physical properties of a system with a broken symmetry, can have peculiar quantum numbers associated with them. This will be the subject of Chap. 2. We shall consider a superfluid ^{3}He film, one-dimensional metals, a two-dimensional model exhibiting an exotic mechanism which leads to superconductivity, and free electrons confined to move in a plane. Again using the functional-integral approach to quantum field theory, we shall see how

*The true story behind this title is the following. Every late summer during the eighties, Amsterdam was visited by a traveling theater group. They built up their tents in a central park of the city surrounded by the Concert Building and the major museums (and also by the high school this author attended). The name of this enterprise, marking the end of the summer and its fantasies, was called "Boulevard of Broken Dreams"

so-called topological terms arise in the effective theories after the fermionic degrees of freedom contained in the microscopic models are integrated out. These terms are the cause of the peculiar quantum numbers which are expected to arise in these systems.

The next chapter, Chap. 3, deals with so-called dual theories. Duality is one of the hottest topics in contemporary condensed matter physics. Quite generally, a dual description of a system refers to a description in terms of a different set of variables than the original set. The systems we will consider in this chapter—a superfluid ^{4}He film, a superconducting film, and a bulk superconductor, all at finite temperature—have vortex solutions in common. In the original formulation, these topological defects are described as singular objects. In the dual theories discussed in this chapter, these objects are instead described in a nonsingular fashion by field theory. The dual theories are employed to discuss the (equilibrium) phase transition these systems undergo as a function of temperature. It will also be pointed out in this chapter that the concept of duality in two dimensions is closely related to bosonization—a powerful computational tool in this reduced dimensionality.

In the last chapter, Chap. 4, we discuss quantum phase transitions. These are phase transitions, taking place close to the absolute zero of temperature, which are driven not by temperature, but by some other parameter in the system. In contrast to equilibrium transitions at finite temperature, time is important in quantum phase transitions. This naturally leads to the use of quantum field theory to describe these. Recent experiments on quantum phase transitions in various two-dimensional systems have raised some very interesting questions. It is generally believed that the concept of duality will play a decisive role in answering these. We will be concerned in this chapter with 2nd-order quantum phase transitions. In addition to a diverging correlation length, which is common for 2nd-oder equilibrium transitions at finite temperature, these transitions also have a diverging correlation time. We will discuss in detail the universality class defined by repulsively interacting bosons which undergo a so-called superfluid-to-Mott-insulating phase transition. We will consider the system both in the absence and in the presence of disorder. We then continue to describe the quantum phase transition which a quantum-Hall liquid undergoes in the absence of disorder as the applied magnetic field varies. We shall discuss scaling theory and consider some relevant experiments.

Acknowledgments

The author wishes to express his gratitude to Professor H. Kleinert for many interesting discussions and helpful advise.

This work was performed as part of a scientific network supported by the European Science Foundation, an association of 62 European national funding agencies (see network's URL, http://defect.unige.ch/defect.html).

Notation

We adopt Feynman's notation and denote a spacetime point by $x = x_\mu = (t, \mathbf{x})$, $\mu = 0, 1, \cdots, d$, with d the number of space dimensions, while the energy k_0 and momentum $\mathbf{k}$ of a particle will be denoted by $k = k_\mu = (k_0, \mathbf{k})$. The time derivative $\partial_0 = \partial/\partial t$ and the gradient ∇ are sometimes combined in a single vector $\tilde{\partial}_\mu = (\partial_0, -\nabla)$. The tilde on ∂_μ is to alert the reader for the minus sign appearing in the spatial components of this vector. We define the scalar product $k \cdot x = k_\mu x_\mu = k_\mu g_{\mu\nu} k_\nu = k_0 t - \mathbf{k} \cdot \mathbf{x}$, with $g_{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1)$ and use Einstein's summation convention. Because of the minus sign in the definition of the vector $\tilde{\partial}_\mu$ it follows that $\tilde{\partial}_\mu a_\mu = \partial_0 a_0 + \nabla \cdot \mathbf{a}$, with a_μ an arbitrary vector.

Integrals over spacetime are denoted by

$$\int_x = \int_{t,\mathbf{x}} = \int \mathrm{d}t \, \mathrm{d}^d x,$$

while those over energy and momentum by

$$\int_k = \int_{k_0,\mathbf{k}} = \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathrm{d}^d k}{(2\pi)^d}.$$

When no integration limits are indicated, the integrals are assumed to be over all possible values of the integration variables.

Natural units $\hbar = c = k_\mathrm{B} = 1$ are adopted throughout unless explicitly stated.

Chapter 1

Nonrelativistic Effective Theories

In this chapter we shall derive effective theories governing the low-energy, small-momentum behavior of some nonrelativistic systems. The systems considered here have in common a spontaneously broken global symmetry. On account of Goldstone's theorem such a breakdown is always accompanied by gapless modes. If the original symmetry G is spontaneously broken to some subgroup H, the Goldstone modes parameterize the coset space G/H. Being gapless, these modes are the dominant excitations at low energy and small momentum and for this reason the relevant degrees of freedom to build effective theories from.

In a Lorenz-invariant theory, massive elementary excitations have a spectrum of the form $E^2(\mathbf{k}) \sim \mathbf{k}^2 + m^2$, with $\mathbf{k}$ the momentum and m the mass of the excitation. In this case, a gapless excitation is easily described by taking the limit $m \to 0$ in the massive theory. In a Galilei-invariant theory, however, it is a priori not clear how to describe a gapless mode. Here, elementary excitations of a free theory have a spectrum of the form $E(\mathbf{k}) \sim \mathbf{k}^2/2m$, so that the limit $m \to 0$ cannot be taken. Although elements of the solution to this problem were present in the literature (see, e.g., [1, 2]), it was not until a paper by Greitner, Wilczek, and Witten [3] that the issue was settled. They considered the nonrelativistic Goldstone mode of a spontaneously broken global U(1) symmetry generated by the total particle number. Their general considerations based on symmetry principles were powerful enough to fully determine the effective theory.

In the following, we shall compute the effective theory describing the nonrelativistic Goldstone modes as they arise in an ideal classical fluid (Sec. 1.1), in a BCS superconductor (Sec. 1.3), in superfluid ^{3}He-a (Sec. 1.5), and in a weakly interacting Bose gas (Sec. 1.6). In addition, we investigate the strong-coupling limit of the pairing theory (Sec. 1.4) and the high-temperature limit of a weakly interacting Bose gas (Sec. 1.7) as well as that of the BCS theory (Sec. 1.8).

1.1 Effective Theory of Hydrodynamics

The hydrodynamics of an ideal, classical fluid was already well understood in the 19th century. The case of isentropic flow, for which the entropy per unit mass is constant, is particularly simple. The pressure P is then a function of the mass density ρ only, and the flow is automatically a potential flow. A feature of such a fluid is that it supports unattenuated sound waves, i.e., propagating density oscillations. The waves are unattenuated because viscosity and thermal conductivity, which usually serve to dissipate the energy of a propagating mode, are absent. More important to our present considerations is that sound waves are gapless. It would be gratifying to identify them as the Goldstone mode of a broken continuous symmetry since that would explain their gaplessness. We will argue in this section that such an identification is indeed possible.

To describe the hydrodynamics of an isentropic fluid, we use Eckart's variational principle [4] and start with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\rho\mathbf{v}^2 - \rho e + \phi[\partial_0\rho + \nabla\cdot(\rho\mathbf{v})], \tag{1.1}$$

where $\mathbf{v}$ is the velocity field, ρ the mass density, and e the internal energy per unit mass. For isentropic flow e is a function of ρ alone. The first and second term in (1.1) represent the kinetic and potential energy density, respectively. The variable ϕ is a Lagrange multiplier introduced to impose the conservation of mass:

$$\partial_0\rho + \nabla\cdot(\rho\mathbf{v}) = 0; \tag{1.2}$$

its dimension is $[\phi] = \mathrm{m}^2\mathrm{s}^{-1}$. The variation of (1.1) with respect to $\mathbf{v}$ yields the equation

$$\mathbf{v} = \nabla\phi. \tag{1.3}$$

It shows that, indeed, isentropic flow is automatically a potential flow and it also identifies the Lagrange multiplier ϕ as the velocity potential. (How to incorporate vortices in this variational approach will be discussed in the next section.) With (1.3), the Lagrangian (1.1) becomes

$$\mathcal{L} = -\rho[\partial_0\phi + \tfrac{1}{2}(\nabla\phi)^2 + e], \tag{1.4}$$

where we performed integrations by part. A second field equation can be obtained by varying the Lagrangian with respect to ρ. This yields the Bernoulli equation

$$\partial_0\phi + \tfrac{1}{2}(\nabla\phi)^2 + h = 0, \tag{1.5}$$

with $h = \partial(\rho e)/\partial\rho$ the specific enthalpy. From this definition of h, one can easily derive the thermodynamic relation for isentropic flow

$$\nabla h = \frac{1}{\rho}\nabla P, \tag{1.6}$$

with $P = \rho^2\partial e/\partial\rho$ the pressure. On taking the gradient of (1.5) and using (1.6) one obtains Euler's equation

$$\partial_0\mathbf{v} + \tfrac{1}{2}\nabla\mathbf{v}^2 + \frac{1}{\rho}\nabla P = 0 \tag{1.7}$$

governing the flow of the fluid.

We next wish to investigate the symmetry content of the theory. Classical hydrodynamics has the following invariances:

(i) Invariance under spacetime translations, $x_\mu \to x_\mu + \epsilon_\mu$, with ϵ_μ a constant vector.

(ii) Invariance under global translations of the velocity potential, $\phi \to \phi + \alpha$, with α a constant.

(iii) Invariance under Galilei boosts,

$$\begin{array}{ll} t \to t' = t, & \mathbf{x} \to \mathbf{x}' = \mathbf{x} - \mathbf{u}t; \\ \partial_0 \to \partial_0' = \partial_0 + \mathbf{u}\cdot\nabla, & \nabla \to \nabla' = \nabla, \end{array} \tag{1.8}$$

with $\mathbf{u}$ a constant velocity.

According to Noether's theorem, symmetries imply conservation laws. Associated with the above invariances we obtain the following conservation laws:

(i) $\tilde{\partial}_\mu t_{\mu\nu} = 0$, where $t_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ is the energy-momentum tensor, with $t_{0i} = p_i$ the momentum density and $t_{00} = \mathcal{H}$ the Hamiltonian;

(ii) $\tilde{\partial}_\mu g_\mu = 0$, with g_0 the mass density and g_i the mass current;

(iii) $\tilde{\partial}_\mu g_{\mu j} = 0$, with $g_{0j} = -g_0 x_j + t\, p_j$ and $g_{ij} = g_i x_j - t\, t_{ij}$ the corresponding charge densities and currents. Physically, the conservation $dG_{0i}/dt = 0$ of the charges $G_{0i} = \int_{\mathbf{x}} g_{0i}$ means that the center of mass of the fluid, $\mathbf{X} = \int_{\mathbf{x}} \mathbf{x}\rho/M$, with $M = \int_{\mathbf{x}} \rho$ the total mass, moves with constant velocity,

$$M\frac{d\mathbf{X}}{dt} = \int_{\mathbf{x}} \mathbf{p}. \tag{1.9}$$

Here, the right-hand side denotes the total momentum of the fluid.

From the Lagrangian (1.4) we obtain as explicit form for the various charge densities and currents [5]:

$$g_0 = -\frac{\partial \mathcal{L}}{\partial \partial_0 \phi} = \rho \tag{1.10}$$

$$g_i = -\frac{\partial \mathcal{L}}{\partial \partial_i \phi} = \rho v_i \tag{1.11}$$

$$p_j (= t_{0j}) = -\frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\partial_j \phi = \rho v_j \tag{1.12}$$

$$t_{ij} = \mathcal{L}\delta_{ij} - \frac{\partial \mathcal{L}}{\partial \partial_i \phi}\partial_j \phi = P\delta_{ij} + \rho v_i v_j \tag{1.13}$$

$$\mathcal{H} (= t_{00}) = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi}\partial_0 \phi - \mathcal{L} = \frac{\rho}{2}\mathbf{v}^2 + \rho e \tag{1.14}$$

$$t_{i0} = \frac{\partial \mathcal{L}}{\partial \partial_i \phi}\partial_0 \phi = (\mathcal{H} + P)v_i. \tag{1.15}$$

Time derivatives $\partial_0\phi$ have been eliminated through the field equation (1.5), so that, for example, $\mathcal{L}$ in the last equation is replaced with

$$\mathcal{L} \to \rho h - \rho e = \left(\rho\frac{\partial}{\partial\rho} - 1\right)(\rho e) = P. \tag{1.16}$$

A few remarks are in order. First, the Hamiltonian $\mathcal{H}$ is the sum of the kinetic and potential energy density, as required. Second, the equivalence of the mass current $\mathbf{g}$ and the momentum density $\mathbf{p}$, which is a hallmark of Galilei invariance, is satisfied by the theory. Finally, the set of equations (1.10)–(1.14) constitutes all the equations of hydrodynamics. This brings us to the conclusion that the Lagrangian (1.4) encodes all the relevant information for the description of an isentropic fluid.

We next turn to the description of sound waves. We restrict ourselves to waves of small amplitude. These generate only small deviations in the mass density $\bar{\rho}$ and pressure $\bar{P}$ of the fluid at rest, so that we can expand the Lagrangian (1.4) in powers of $\rho - \bar{\rho} = \tilde{\rho}$, with $|\tilde{\rho}| << \bar{\rho}$:

$$\mathcal{L} = -(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2)(\bar{\rho} + \tilde{\rho}) - \bar{e}\bar{\rho} - \bar{h}\tilde{\rho} - \tfrac{1}{2}\bar{h}'\tilde{\rho}^2 + \mathcal{O}(\tilde{\rho}^3). \tag{1.17}$$

The derivative $(')$ is with respect to ρ and is to be evaluated at $\rho = \bar{\rho}$. Since for the system at rest ϕ is constant, it follows from (1.5) that $\bar{h} = 0$. If we denote the thermodynamic derivative $\partial P/\partial\rho$ by c^2, which has the dimension of a velocity squared, the coefficient of the quadratic term in $\tilde{\rho}$ can be written as

$$\bar{h}' = \frac{1}{\bar{\rho}}\bar{P}' = \frac{c_0^2}{\bar{\rho}}. \tag{1.18}$$

Apart from an irrelevant constant term $(-\bar{e}\bar{\rho})$ the Lagrangian becomes to this order

$$\mathcal{L} = -(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2)(\bar{\rho} + \tilde{\rho}) - \frac{c_0^2}{2\bar{\rho}}\tilde{\rho}^2. \tag{1.19}$$

We next eliminate $\tilde{\rho}$ by substituting

$$\tilde{\rho} = -\frac{\bar{\rho}}{c_0^2}(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2), \tag{1.20}$$

which follows from expanding the field equation (1.5). Physically, this equation reflects Bernoulli's principle: in regions of rapid flow, the mass density $\rho = \bar{\rho} + \tilde{\rho}$ and therefore the pressure is low. It also shows that the expansion in $\tilde{\rho}$ is one in derivatives. The higher-order terms that have been neglected correspond to higher-order derivatives. At low energy and small momentum, these additional terms can be ignored. After eliminating $\tilde{\rho}$, we obtain a Lagrangian governing the velocity potential ϕ [6]:

$$\mathcal{L}_{\text{eff}} = -\bar{\rho}(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2) + \frac{\bar{\rho}}{2c_0^2}(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2)^2. \tag{1.21}$$

This is precisely the effective theory of a nonrelativistic (Abelian) Goldstone mode obtained in Ref. [3] using general arguments. Apart from an irrelevant constant it is identical to a proposal by Takahashi [2] which was also based on symmetry principles.

The field equation one obtains for the velocity potential ϕ from (1.21) is nonlinear:

$$\bar{\rho}(\partial_0^2\phi + \tfrac{1}{2}\partial_0\mathbf{v}^2) - \rho c_0^2\nabla\cdot\mathbf{v} + \tfrac{1}{2}\bar{\rho}(\partial_0\mathbf{v}^2 + \mathbf{v}\cdot\nabla\mathbf{v}^2) = 0. \tag{1.22}$$

The information contained in this equation cannot be more than the conservation of mass because ϕ was initially introduced in (1.1) as a Lagrange multiplier precisely to enforce this conservation law. Indeed, remembering that the combination $\bar{\rho}(\partial_0\phi+\frac{1}{2}\mathbf{v}^2)$ denotes c_0^2 times $\tilde{\rho}(x) = \rho(x) - \bar{\rho}$, with $\bar{\rho}$ the constant mass density of the fluid at rest, we see that (1.22) reproduces (1.2) in this approximation. To simplify (1.22), we replace $\bar{\rho}$ in the first and last term with the full mass density ρ (which is justified to this order) to arrive at the known [7], but unfamiliar field equation

$$\partial_0^2\phi - c_0^2\nabla^2\phi = -\partial_0\mathbf{v}^2 - \tfrac{1}{2}\mathbf{v}\cdot\nabla\mathbf{v}^2 \tag{1.23}$$

of sound waves. If we ignore the nonlinear terms, it becomes the more familiar wave equation

$$\partial_0^2\phi - c_0^2\nabla^2\phi = 0, \tag{1.24}$$

implying a gapless linear spectrum, and identifying c, which was introduced via the thermodynamic derivative $\partial P/\partial\rho = c^2$, as the sound velocity.

The combination $\partial_0\phi + \frac{1}{2}(\nabla\phi)^2$ appearing in the description of a nonrelativistic gapless field is dictated by Galilei invariance. To obtain the transformation property of the velocity potential ϕ under a Galilei boost (1.8) we note that since $\nabla\phi$ is a velocity field, $\nabla\phi(x) \to \nabla'\phi'(x') = \nabla\phi(x) - \mathbf{u}$, with $x' = (t, \mathbf{x} - \mathbf{u}t)$. This gives as transformation rule for ϕ

$$\phi(x) \to \phi'(x') = \phi(x) - \mathbf{u}\cdot\mathbf{x} + f(t), \tag{1.25}$$

with $f(t)$ a yet undetermined function of time. To determine $f(t)$ we note that the factor $-\partial_0\phi$ in the Lagrangian (1.4) is the chemical potential per unit mass. Indeed, using the standard definition $\mu = \partial\mathcal{H}/\partial\rho$, we find

$$\mu = \tfrac{1}{2}\mathbf{v}^2 + h = -\partial_0\phi, \tag{1.26}$$

where in the second equality we used the field equation (1.5). This identification fixes the transformation rule of $-\partial_0\phi$:

$$-\partial_0\phi(x) \to -\partial_0'\phi'(x') = -\partial_0\phi(x) - \mathbf{u}\cdot\mathbf{v}(x) + \tfrac{1}{2}\mathbf{u}^2 \tag{1.27}$$

and in combination with (1.25), yields for $f(t)$

$$\partial_0 f(t) = \tfrac{1}{2}\mathbf{u}^2, \quad \text{or} \quad f(t) = \tfrac{1}{2}\mathbf{u}^2 t \tag{1.28}$$

up to an irrelevant constant. It is easily checked that both the combination $\partial_0\phi + \frac{1}{2}(\nabla\phi)^2$ appearing in the effective theory (1.21) as well as the field equation (1.23) are invariant under Galilei boosts. So, contrary to what is sometimes stated in the literature [8], sound waves are invariant under Galilei boosts. The linearized wave equation (1.24) is, of course, not invariant because essential nonlinear terms are omitted.

Greitner, Wilczek, and Witten arrived at the effective Lagrangian (1.21) by requiring that the effective theory of sound waves can only be constructed with the help of the Galilei-invariant combination $\partial_0\phi + \frac{1}{2}(\nabla\phi)^2$, and that it should give the dispersion relation $E^2(\mathbf{k}) = c^2\mathbf{k}^2$.

From the effective Lagrangian (1.21) one can again calculate the various Noether charge densities and currents [2]. They are, as might be expected, of the same form as the exact expressions (1.10)–(1.14), but now with the approximations

$$\rho \simeq \bar{\rho} - \frac{\bar{\rho}}{c_0^2}(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2) \tag{1.29}$$

as follows from Eq. (1.20),

$$\mathcal{H} \simeq \frac{\rho}{2}\mathbf{v}^2 + \frac{c_0^2}{2\bar{\rho}}(\rho - \bar{\rho})^2, \tag{1.30}$$

and

$$P \simeq -\bar{\rho}(\partial_0\phi + \tfrac{1}{2}\mathbf{v}^2) \simeq c_0^2(\rho - \bar{\rho}). \tag{1.31}$$

This last equation is consistent with the expression one obtains from directly expanding the pressure: $P(\rho) = \bar{P} + \tilde{\rho}\bar{P}'$ since $\bar{P} = 0$ and $\bar{P}' = c_0^2$.

To recapitulate, we have derived an effective theory describing a gapless mode starting from the Lagrangian (1.4) which entails the complete hydrodynamics of an isentropic fluid. We now wish to argue that this gapless mode is a Goldstone mode associated with a spontaneously broken symmetry. A first indication in favor of such an interpretation follows because the Hamiltonian $\mathcal{H}$ displays a property typical for a system with broken symmetry, namely that it is a function not of the velocity potential itself, but of the gradient of the field. The energy is minimal if ϕ is uniform throughout the sample, i.e., there is rigidity [9].

Usually the broken symmetry can be identified by the general property that the Goldstone mode is translated under the broken symmetry operations. (There may be other effects too, but the translation is always present.) Here, this general characteristic does not uniquely identify the broken symmetry because ϕ is translated under two symmetry operations. According to the transformation rule (1.25) with $f(t)$ given in (1.28), we have that under a Galilei boost

$$\delta^G_{\mathbf{u}}\phi(x) := \phi'(x) - \phi(x) = -\mathbf{u}\cdot\mathbf{x} + t\mathbf{u}\cdot\nabla\phi(x), \tag{1.32}$$

where we took the transformation parameter $\mathbf{u}$ infinitesimal small so that quadratic and higher powers in $\mathbf{u}$ may be ignored. The first term at the right-hand side shows that the velocity potential is translated under a Galilei boost. The second symmetry under which ϕ is translated is generated by the total particle number, or, equivalently, by the total mass $M = \int_{\mathbf{x}} \rho$. To see this we first compute from (1.4) the conjugate momentum π_ϕ of ϕ,

$$\pi_\phi = \frac{\partial\mathcal{L}}{\partial\partial_0\phi} = -\rho, \tag{1.33}$$

implying that ϕ and ρ are canonically conjugate [10]:

$$\{\phi(t,\mathbf{x}), \rho(t,\mathbf{x}')\} = -\delta(\mathbf{x} - \mathbf{x}'), \tag{1.34}$$

where $\{\,,\}$ denotes the Poisson bracket. We then use a central result of classical field theory stating that the charge Q of a continuous symmetry is the generator of the corresponding transformations of the fields, $\chi(x) \to \chi'(x)$. More specifically, for an infinitesimal transformation $\delta^Q_\alpha \chi(x) = \chi'(x) - \chi(x)$ one has

$$\delta^Q_\alpha \chi(x) = -\alpha\{\chi(x), Q\}, \tag{1.35}$$

with α the transformation parameter. Equation (1.34) thus implies that under the global U(1) symmetry generated by the total mass, ϕ is indeed translated

$$\delta^M_\alpha \phi(x) = -\alpha\{\phi(x), M\} = \alpha. \tag{1.36}$$

The point is that the generators of Galilei boosts $G_{0j} = \int_{\mathbf{x}}(-x_j g_0 + t p_j)$ also contain the mass density $\rho = g_0$. It is therefore impossible to distinguish a broken Galilei invariance from a broken mass symmetry by considering the algebra alone.

Let us at this point pause for a moment and consider the case of a superfluid. It is well established that in the normal-to-superfluid phase transition, the global U(1) symmetry generated by the total mass is spontaneously broken. This comes about because in a superfluid, which is a quantum system, many particles Bose-Einstein condense in a single quantum state. The coherence allows us to describe the system by a complex field

$$\psi(x) = \sqrt{\rho(x)/m}\, \mathrm{e}^{im\varphi(x)/\hbar} \tag{1.37}$$

normalized such that $|\psi|^2$ yields the particle number density ρ/m of the condensate. To underscore the quantum nature of a superfluid, Planck's constant has been made explicit. The field φ, which turns out to describe the Goldstone mode of the broken global U(1) symmetry, is a phase field and therefore compact. The field ψ is frequently referred to as the condensate wavefunction, even in the modern literature. In our view, this is somewhat misleading. As has been stressed by Feynman [11], ψ is a classical field describing the coherent behavior of many condensed particles in the same way as the classical field of electrodynamics describes the behavior of many photons in a single state. For these classical fields there is no probability interpretation as is required for wavefunctions [12].

It has been argued by Feynman [11] that the ψ-field is governed by a nonrelativistic $|\psi|^4$-theory defined by the Lagrangian

$$\mathcal{L}_\psi = i\hbar\psi^*\partial_0\psi - \frac{\hbar^2}{2m}|\nabla\psi|^2 - \frac{c_0^2}{2\bar\rho}(m|\psi|^2 - \bar\rho)^2. \tag{1.38}$$

The potential energy has its minimum along a circle away from the origin at $|\psi|^2 = \bar\rho/m$, implying a spontaneous breakdown of the global U(1) symmetry. The Lagrangian (1.38) with (1.37) reduces to the one given in (1.19) when the term $-\hbar^2(\nabla\rho)^2/8m^2\rho$ is ignored. Using the expression for the pressure, cf. (1.16)

$$P = \left[\rho\left(\frac{\partial}{\partial\rho} - \partial_i\frac{\partial}{\partial\partial_i\rho}\right) - 1\right](\rho e), \tag{1.39}$$

we find that it gives the contribution $-\hbar^2(\nabla^2\rho)/4m^2$ to the pressure—the so-called quantum pressure. The reason for calling it this way is that it is the only place where

Planck's constant appears in the equations. To the order in which we are working, it is consistent to ignore this term. Because the field equation for ψ derived from (1.38) has the *form* of a nonlinear Schrödinger equation, we will refer to ψ as a Schrödinger field. We trust however that the reader realizes that it is a classical field unrelated to a Schrödinger wavefunction.

Let us compare the transformation properties of the Schrödinger field with that of the velocity potential ϕ of an isentropic fluid. Under a Galilei boost, $\psi(x)$ transforms as [12]

$$\psi(x) \to \psi'(x') = \exp[i(-\mathbf{u}\cdot\mathbf{x} + \tfrac{1}{2}\mathbf{u}^2 t)m/\hbar]\,\psi(x). \tag{1.40}$$

With $m\varphi/\hbar$ denoting the phase of the Schrödinger field, we see that φ transforms in the same way as does the velocity potential.

Using that the canonical conjugate of the ψ-field is $\pi_\psi = i\hbar\psi^*$, we easily derive the Poisson bracket

$$\{\psi(x), M\} = -i(m/\hbar)\psi(x). \tag{1.41}$$

This shows that the total mass M generates phase transformations on the ψ-field: $\psi(x) \to \psi'(x) = \exp(i\alpha m/\hbar)\psi(x)$. The phase φ of the Schrödinger field is consequently translated under the symmetry, just like the velocity potential ϕ. This transformation property identifies φ as the Goldstone mode of the broken U(1) mass symmetry. The Poisson bracket (1.41) also implies that φ and ρ are canonical conjugate [9], cf. (1.34)

$$\{\varphi(t,\mathbf{x}), \rho(t,\mathbf{x}')\} = -\delta(\mathbf{x}-\mathbf{x}'). \tag{1.42}$$

A similar relation holds for superconductors. On quantizing, the Poisson bracket is replaced by a commutator. The Heisenberg uncertainty relation that results for the conjugate pair has recently been demonstrated experimentally [13].

As remarked above, a necessary condition for the spontaneous breakdown of the global U(1) symmetry is the presence of a condensate. Such an intrinsic quantum phenomenon, requiring many particles in a single state, has no analog in a classical setting. Hence, the U(1) symmetry cannot be broken in classical hydrodynamics. This leaves us with the second possibility, namely that of a spontaneously broken Galilei invariance. The breakdown is a result of the presence of a finite mass density. This can be inferred [2] from considering the transformation of the momentum density $\mathbf{p}(x)$ under a Galilei boost:

$$\delta^G_{\mathbf{u}} p_i(x) = -u_j\{p_i(x), G_{0j}(t)\} = -u_i\rho(x) + t\mathbf{u}\cdot\nabla p_i(x), \tag{1.43}$$

or with $\delta^G_{\mathbf{u}} = u_j\delta^G_j$

$$\delta^G_j p_i(x) = -\rho(x)\delta_{ji} + t\partial_j p_i(x). \tag{1.44}$$

If the mass density ρ is finite, the right-hand side is nonzero, which is a symmetry-breaking condition.

1.2 Including Vortices

There is an essential difference between the spontaneous breakdown of the Galilei symmetry and that of the global U(1) symmetry. Although both symmetries are Abelian,

the latter is a compact symmetry, whereas the Galilei group is noncompact. More specifically, the transformation parameter α of the U(1) group has a finite domain ($0 \leq \alpha < 2\pi$), while the domain of $\mathbf{u}$, the transformation parameter of the Galilei group, is infinite. As a result, the velocity potential of classical hydrodynamics cannot be represented as the phase of a complex field. An immediate physical manifestation of the difference is that a system with broken U(1) invariance supports topologically stable vortices, whereas a system with broken Galilei invariance does not. This is not to say that vortices are absent in the latter case, it merely states that their stability is not guaranteed by topological conservation laws. Closely connected to this is that the circulation is not quantized in classical hydrodynamics, which is known to exist in superfluids. Yet, the circulation is conserved also in isentropic fluids. This is again not for topological, but for dynamical reasons, the conservation being proven by invoking Euler's equation (1.7) as was first done for an ideal, incompressible fluid by Helmholtz [14] and generalized to a compressible fluid by Thomson [15].

The easiest way to observe vortices in a classical fluid is to punch a hole in the bottom of the vessel containing the fluid. As the fluid pours out a vortex is formed in the remaining fluid—a phenomenon daily observed by people unplugging a sinkhole. Often, as happens in, for example, superfluid ^{4}He, the core of a vortex is in the normal state so that the symmetry is restored there. In the present context this would mean that inside the vortex core, the fluid mass density ρ is zero. This is indeed what is observed: the vortex core consists of air, therefore no fluid is present and $\rho = 0$ there.

In the eye of a tropical cyclone—another example of a vortex, nature does its best to restore the Galilei symmetry, record low atmospheric pressures being measured there. (A complete restoration would imply the absence of air corresponding to zero pressure.)

It is customary to incorporate vortices in a potential flow via the introduction of so-called Clebsch potentials [16]. We will not follow this route, but instead use the powerful principle of defect gauge symmetry developed by Kleinert [17, 18, 19]. In this approach, one introduces a so-called vortex gauge field $\phi^{\mathrm{P}}_\mu = (\phi^{\mathrm{P}}_0, \boldsymbol{\phi}^{\mathrm{P}})$ in the Lagrangian via minimally coupling to the Goldstone field:

$$\tilde{\partial}_\mu \phi \to \tilde{\partial}_\mu \phi + \phi^{\mathrm{P}}_\mu, \tag{1.45}$$

with $\tilde{\partial}_\mu = (\partial_0, -\nabla)$ and

$$\nabla \times \boldsymbol{\phi}^{\mathrm{P}} = -2\boldsymbol{\omega}, \tag{1.46}$$

so that $\nabla \times \mathbf{v} = 2\boldsymbol{\omega}$ yields (twice) the vorticity $\boldsymbol{\omega}$ of the vortex. The combination $\tilde{\partial}_\mu \phi + \phi^{\mathrm{P}}_\mu$ is invariant under the local gauge transformation

$$\phi(x) \to \phi(x) + \alpha(x); \qquad \phi^{\mathrm{P}}_\mu \to \phi^{\mathrm{P}}_\mu - \tilde{\partial}_\mu \alpha(x), \tag{1.47}$$

with ϕ^{P}_μ playing the role of a gauge field. The left-hand side of (1.46) may be thought of as defining the "magnetic field" associated with the vortex gauge field $\mathbf{B}^{\mathrm{P}} = \nabla \times \boldsymbol{\phi}^{\mathrm{P}}$.

For illustrative purposes, let us consider an external, static vortex with circulation Γ located along a line L, which may be closed or infinitely long [14]. Then, $\boldsymbol{\omega} = \frac{1}{2}\Gamma \boldsymbol{\delta}(L)$, where $\boldsymbol{\delta}(L)$ is a delta function on the line L,

$$\delta_i(L) = \int_L \mathrm{d}y_i \, \delta(\mathbf{x} - \mathbf{y}). \tag{1.48}$$

This model with a static, external vortex may be thought of as describing the steady flow in the presence of a vortex pinned to a fixed impurity. The field equation for ϕ obtained after the substitution (1.45) reads

$$\partial_0 \mu + c_0^2 \nabla \cdot \mathbf{v} = \partial_0 \mathbf{v}^2 + \tfrac{1}{2} \mathbf{v} \cdot \nabla \mathbf{v}^2 - \mathbf{v} \cdot \mathbf{E}^{\mathrm{P}} \tag{1.49}$$

with $\mu = -(\partial_0 \phi + \phi_0^{\mathrm{P}})$ the chemical potential and $\mathbf{v} = \nabla\phi - \boldsymbol{\phi}^{\mathrm{P}}$ the velocity of the flow in the presence of the vortex. The last term gives a coupling of the velocity field to the "electric field" associated with ϕ_μ^{P},

$$\mathbf{E}^{\mathrm{P}} = -\nabla \phi_0^{\mathrm{P}} - \partial_0 \boldsymbol{\phi}^{\mathrm{P}}. \tag{1.50}$$

Note that the field equation (1.49) is invariant under local vortex gauge transformations. Ignoring the higher-order terms and choosing the gauge $\phi_0^{\mathrm{P}} = 0$, we obtain as equation for the flow in the presence of a static vortex:

$$\nabla \cdot \mathbf{v} = 0, \quad \text{or} \quad \nabla \cdot (\nabla\phi - \boldsymbol{\phi}^{\mathrm{P}}) = 0, \tag{1.51}$$

which is solved by

$$\phi(\mathbf{x}) = -\int_{\mathbf{y}} G(\mathbf{x}-\mathbf{y}) \nabla \cdot \boldsymbol{\phi}^{\mathrm{P}}(\mathbf{y}). \tag{1.52}$$

Here, $G(\mathbf{x})$ is the Green function of the Laplace operator

$$G(\mathbf{x}) = \int_{\mathbf{k}} \frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2} = \frac{1}{4\pi|\mathbf{x}|}. \tag{1.53}$$

Straightforward manipulations then yield the well-known Biot-Savart law for the velocity field in the presence of a static vortex [14, 17]

$$\mathbf{v}_{\mathrm{v}}(\mathbf{x}) = \frac{\Gamma}{4\pi} \int_L \mathrm{d}\mathbf{y} \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}, \tag{1.54}$$

where the integration is along the vortex. This exemplifies the viability of the vortex gauge principle as an alternative to describe vortices in a potential flow.

Let us continue to study the dynamics of vortices—a subject that has recently received considerable attention in the literature [20]. Because the vortex motion is determined by the flow itself, the vortex can no longer be considered as external. We shall see that the nonlinear part of the field equation (1.49) becomes relevant here.

In the absence of external forces, the vortex moves with a constant velocity, $\mathbf{v}_L$ say. The flow in the presence of a moving vortex can be obtained from the static solution (1.54) by replacing the coordinate $\mathbf{x}$ with $\mathbf{x} - \mathbf{v}_L t$. This implies that

$$\partial_0 \mathbf{v}_{\mathrm{v}}(\mathbf{x} - \mathbf{v}_L t) = -\mathbf{v}_L \cdot \nabla \mathbf{v}_{\mathrm{v}}(\mathbf{x} - \mathbf{v}_L t). \tag{1.55}$$

Since the solution $\mathbf{v}_{\mathrm{v}}$ is curl-free outside the vortex core, the right-hand side may there be written as $-\nabla(\mathbf{v}_L \cdot \mathbf{v}_{\mathrm{v}})$. To study sound waves in the presence of a moving vortex, we write the velocity field as $\mathbf{v}(x) = \mathbf{v}_{\mathrm{v}}(\mathbf{x} - \mathbf{v}_L t) + \nabla\tilde{\phi}(x)$, with $\tilde{\phi}$ describing small

variations around the moving vortex solution. Equation (1.55) then requires that we write for the chemical potential in (1.49)

$$\mu(x) = \mathbf{v}_L \cdot \mathbf{v}_{\rm v}(\mathbf{x} - \mathbf{v}_L t) - \partial_0 \tilde{\phi}(x). \tag{1.56}$$

This leads to the linearized field equation [21]

$$\partial_0^2 \tilde{\phi}(x) - c_0^2 \nabla^2 \tilde{\phi}(x) = -\mathbf{v}_{\rm v}(\mathbf{x}) \cdot \nabla \partial_0 [2\tilde{\phi}(t, \mathbf{x}) - \tilde{\phi}(t, 0)], \tag{1.57}$$

describing sound waves in the presence of a moving vortex. In deriving (1.57) we again used the gauge $\phi_0^{\rm P} = 0$, and approximated $\mathbf{v}_{\rm v}(\mathbf{x} - \mathbf{v}_L t)$ by $\mathbf{v}_{\rm v}(\mathbf{x})$. To linear order in $\tilde{\phi}$ this is allowed since the vortex, being driven by the sound wave, has a velocity

$$\mathbf{v}_L(t) = \nabla \tilde{\varphi}(t, \mathbf{x} = \mathbf{v}_L t) \approx \nabla \tilde{\varphi}(t, 0). \tag{1.58}$$

We also neglected a term quadratic in $\mathbf{v}_{\rm v}$ which is justified because this velocity is much smaller than the sound velocity outside the vortex core [21]. The first term at the right-hand side in (1.57) stems from the nonlinear term $\partial_0 \mathbf{v}^2$ in the general field equation (1.49). Thus the nonlinearity of sound waves becomes detectable. Equation (1.57) can be used as a basis to study the scattering of phonons of a free moving vortex [20].

So far we have contrasted the spontaneous breakdown of the Galilei invariance (caused by a finite mass density) and that of the global U(1) symmetry (caused by a nonzero condensate). A superfluid, however, has a finite mass density as well as a nonzero condensate. Both symmetries are therefore broken and we expect two different Goldstone modes to be present. This is indeed what is observed in superfluid ^{4}He. The system supports besides first sound, which are the usual density waves associated with the spontaneously broken Galilei invariance, also second sound, or entropy waves. The latter mode depends crucially on the presence of the condensate and is the Goldstone mode associated with the spontaneously broken global U(1) symmetry generated by the total mass. At the transition point, the second sound velocity vanishes, whereas the first sound velocity remains finite. This is as expected since only the condensate vanishes at the superfluid phase transition; the total mass density remains finite.

1.3 Anderson-Bogoliubov Mode

In this section we study the so-called Anderson-Bogoliubov mode [22, 23] of a neutral superconductor. This mode is known to be the gapless Goldstone mode associated with the spontaneously broken global U(1) symmetry generated by the total mass. Given the general arguments of Ref. [3] we expect that the effective theory of a superconductor is exactly of the form (1.21) we obtained for sound waves in classical hydrodynamics, but with ϕ replaced by a compact field.

Our starting point is the famous microscopic model of Bardeen, Cooper, and Schrieffer (BCS) defined by the Lagrangian [24]

$$\begin{aligned} \mathcal{L} &= \psi_\uparrow^*[i\partial_0 - \xi(-i\nabla)]\psi_\uparrow + \psi_\downarrow^*[i\partial_0 - \xi(-i\nabla)]\psi_\downarrow - \lambda_0 \psi_\uparrow^* \, \psi_\downarrow^* \, \psi_\downarrow \, \psi_\uparrow \\ &:= \mathcal{L}_0 + \mathcal{L}_{\rm i}, \end{aligned} \tag{1.59}$$

where $\mathcal{L}_{\rm i} = -\lambda_0 \psi_\uparrow^* \psi_\downarrow^* \psi_\downarrow \psi_\uparrow$ is a contact interaction term, representing the effective, phonon mediated, attraction between electrons with coupling constant $\lambda_0 < 0$, and $\mathcal{L}_0$ is the remainder. In (1.59), the field $\psi_{\uparrow(\downarrow)}$ is an anticommuting field describing the electrons with mass m and spin up (down); $\xi(-i\nabla) = \epsilon(-i\nabla) - \mu_0$, with $\epsilon(-i\nabla) = -\nabla^2/2m$, is the kinetic energy operator with the chemical potential μ_0 subtracted.

The Lagrangian (1.59) is invariant under global U(1) transformations. Under such a transformation, the electron fields pick up an additional phase factor

$$\psi_\sigma \to \mathrm{e}^{i\alpha}\psi_\sigma \tag{1.60}$$

with $\sigma = \uparrow, \downarrow$ and α a constant. Notwithstanding its simple form, the microscopic model (1.59) is a good starting point to describe BCS superconductors. The reason is that the interaction term allows for the formation of Cooper pairs which below a critical temperature condense. This results in a nonzero expectation value of the field Δ describing the Cooper pairs, and a spontaneous breakdown of the global U(1) symmetry. This in turn gives rise to the gapless Anderson-Bogoliubov mode which—after incorporating the electromagnetic field—lies at the root of most startling properties of superconductors [25].

To obtain the effective theory of this mode, the fermionic degrees of freedom have to be integrated out. To this end we introduce Nambu's notation and rewrite the Lagrangian (1.59) in terms of a two-component field

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^* \end{pmatrix} \qquad \psi^\dagger = (\psi_\uparrow^*, \psi_\downarrow). \tag{1.61}$$

In this notation, $\mathcal{L}_0$ becomes

$$\mathcal{L}_0 = \psi^\dagger \begin{pmatrix} i\partial_0 - \xi(-i\nabla) & 0 \\ 0 & i\partial_0 + \xi(-i\nabla) \end{pmatrix} \psi, \tag{1.62}$$

where we explicitly employed the anticommuting character of the electron fields and neglected terms which are a total derivative. The partition function,

$$Z = \int \mathrm{D}\psi^\dagger \mathrm{D}\psi \exp\left(i \int_x \mathcal{L} \right), \tag{1.63}$$

must for our purpose be written in a form bilinear in the electron fields. This is achieved by rewriting the quartic interaction term as a functional integral over auxiliary fields Δ and Δ^*:

$$\begin{aligned} &\exp\left(-i\lambda_0 \int_x \psi_\uparrow^* \psi_\downarrow^* \psi_\downarrow \psi_\uparrow \right) = \\ &\int \mathrm{D}\Delta^* \mathrm{D}\Delta \exp\left[-i \int_x \left(\Delta^* \psi_\downarrow \psi_\uparrow + \psi_\uparrow^* \psi_\downarrow^* \Delta - \frac{1}{\lambda_0} \Delta^* \Delta \right) \right], \end{aligned} \tag{1.64}$$

where, as always, an overall normalization factor is omitted. Classically, Δ merely abbreviates the product of two electron fields

$$\Delta = \lambda_0 \psi_\downarrow \psi_\uparrow. \tag{1.65}$$

It would therefore be more appropriate to give Δ two spin labels $\Delta_{\downarrow\uparrow}$. Since $\psi_\uparrow$ and $\psi_\downarrow$ are anticommuting fields, Δ is antisymmetric in these indices. Physically, it describes the Cooper pairs of the superconducting state.

By employing (1.64), we can cast the partition function in the desired bilinear form:

$$Z = \int \mathrm{D}\psi^\dagger \mathrm{D}\psi \int \mathrm{D}\Delta^* \mathrm{D}\Delta \, \exp\left(\frac{i}{\lambda_0}\int_x \Delta^*\Delta\right) \times \exp\left[i\int_x \psi^\dagger \begin{pmatrix} i\partial_0 - \xi(-i\nabla) & -\Delta \\ -\Delta^* & i\partial_0 + \xi(-i\nabla) \end{pmatrix} \psi\right]. \tag{1.66}$$

Changing the order of integration and performing the Gaussian integral over the Grassmann fields, we obtain

$$Z = \int \mathrm{D}\Delta^* \mathrm{D}\Delta \, \exp\left(iS_{\mathrm{eff}}[\Delta^*,\Delta] + \frac{i}{\lambda_0}\int_x \Delta^*\Delta\right), \tag{1.67}$$

where S_{eff} is the one-loop effective action which, using the identity Det(A) = exp[Tr ln(A)], can be cast in the form

$$S_{\mathrm{eff}}[\Delta^*,\Delta] = -i\,\mathrm{Tr}\ln\begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\Delta \\ -\Delta^* & p_0 + \xi(\mathbf{p}) \end{pmatrix}, \tag{1.68}$$

where $p_0 = i\partial_0$ and $\xi(\mathbf{p}) = \epsilon(\mathbf{p}) - \mu_0$, with $\epsilon(\mathbf{p}) = \mathbf{p}^2/2m$. The trace Tr appearing here needs some explanation. Explicitly, it is defined as

$$S_{\mathrm{eff}} = -i\mathrm{Tr}\ \ln\left[K(p,x)\right] = -i\mathrm{tr}\ln\left[K(p,x)\delta(x-y)\big|_{y=x}\right], \tag{1.69}$$

where the trace tr is the usual one over discrete indices. We abbreviated the matrix appearing in (1.68) by $K(p,x)$ so as to cover the entire class of actions of the form

$$S = \int_x \psi^\dagger(x) K(p,x) \psi(x). \tag{1.70}$$

The delta function in (1.69) arises because $K(p,x)$ is obtained as a second functional derivative of the action

$$\frac{\delta^2 S}{\delta\psi^\dagger(x)\,\delta\psi(x)} = K(p,x)\,\delta(x-y)\big|_{y=x}, \tag{1.71}$$

each of which gives a delta function. Since the action has only one integral $\int_x$ over spacetime, one delta function remains. Because it is diagonal, it may be taken out of the logarithm and (1.69) can be written as

$$\begin{aligned} S_{\mathrm{eff}} &= -i\mathrm{tr}\int_x \ln\left[K(p,x)\right]\delta(x-y)\big|_{y=x} \\ &= -i\mathrm{tr}\int_x\int_k \mathrm{e}^{ik\cdot x}\,\ln\left[K(p,x)\right]\mathrm{e}^{-ik\cdot x}. \end{aligned} \tag{1.72}$$

In the last step, we used the integral representation of the delta function:

$$\delta(x) = \int_k \mathrm{e}^{-ik\cdot x}, \tag{1.73}$$

shifted the exponential function $\exp(ik \cdot y)$ to the left, which is justified because the derivative p_μ does not operate on it, and, finally, set y_μ equal to x_μ. We thus see that the trace Tr in (1.69) stands for the trace over discrete indices as well as the integration over spacetime and over energy and momentum. The integral $\int_k$ arises because the effective action calculated here is a one-loop result with k_μ the loop energy and momentum.

The integrals in (1.72) cannot in general be evaluated in closed form because the logarithm contains energy-momentum operators and spacetime-dependent functions in a mixed order. To disentangle the integrals resort has to be taken to a derivative expansion [26] in which the logarithm is expanded in a Taylor series. Each term contains powers of the energy-momentum operator p_μ which acts on every spacetime-dependent function to its right. All these operators are shifted to the left by repeatedly applying the identity

$$f(x)p_\mu g(x) = (p_\mu - i\tilde{\partial}_\mu)f(x)g(x), \tag{1.74}$$

where $f(x)$ and $g(x)$ are arbitrary functions of spacetime and the derivative $\tilde{\partial}_\mu = (\partial_0, -\nabla)$ acts *only* on the next object to the right. One then integrates by parts, so that all the p_μ's act to the left where only a factor $\exp(ik \cdot x)$ stands. Ignoring total derivatives and taking into account the minus signs that arise when integrating by parts, one sees that all occurrences of p_μ (an operator) are replaced with k_μ (an integration variable). The exponential function $\exp(-ik \cdot x)$ can at this stage be moved to the left where it is annihilated by the function $\exp(ik \cdot x)$. The energy-momentum integration can now in principle be carried out and the effective action be cast in the form of an integral over a local density $\mathcal{L}_{\rm eff}$:

$$S_{\rm eff} = \int_x \mathcal{L}_{\rm eff}. \tag{1.75}$$

This is in a nutshell how the derivative expansion works.

In the mean-field approximation, the functional integral (1.67) is approximated by the saddle point:

$$Z = \exp\left(iS_{\rm eff}[\Delta^*_{\rm mf}, \Delta_{\rm mf}] + \frac{i}{\lambda_0}\int_x \Delta^*_{\rm mf}\Delta_{\rm mf}\right), \tag{1.76}$$

where $\Delta_{\rm mf}$ is the solution of mean-field equation

$$\frac{\delta S_{\rm eff}}{\delta\Delta^*(x)} = -\frac{1}{\lambda_0}\Delta. \tag{1.77}$$

If we assume the system to be spacetime independent so that $\Delta_{\rm mf}(x) = \bar{\Delta}$, Eq. (1.77) yields the celebrated BCS gap [24] equation:

$$\begin{aligned}\frac{1}{\lambda_0} &= -i\int_k \frac{1}{k_0^2 - E^2(k) + i\eta} \\ &= -\frac{1}{2}\int_{\mathbf{k}} \frac{1}{E(\mathbf{k})},\end{aligned} \tag{1.78}$$

where η is an infinitesimal positive constant that is to be set to zero at the end of the calculation, and

$$E(\mathbf{k}) = \sqrt{\xi^2(\mathbf{k}) + |\bar{\Delta}|^2} \tag{1.79}$$

is the spectrum of the elementary fermionic excitations. If this equation yields a solution with $\bar{\Delta} \neq 0$, the global U(1) symmetry (1.60) is spontaneously broken since

$$\bar{\Delta} \to \mathrm{e}^{2i\alpha}\bar{\Delta} \neq \bar{\Delta} \tag{1.80}$$

under this transformation. The factor 2 in the exponential function arises because Δ, describing the Cooper pairs, is built from two electron fields. It satisfies Landau's definition of an order parameter as its value is zero in the symmetric, disordered state and nonzero in the state with broken symmetry. It directly measures whether the U(1) symmetry is spontaneously broken.

In the case of a spacetime-independent system, the effective action (1.68) is readily evaluated. Writing

$$\begin{aligned}\begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\bar{\Delta} \\ -\bar{\Delta}^* & p_0 + \xi(\mathbf{p}) \end{pmatrix} = &\begin{pmatrix} p_0 - \xi(\mathbf{p}) & 0 \\ 0 & p_0 + \xi(\mathbf{p}) \end{pmatrix} \\ &- \begin{pmatrix} 0 & \bar{\Delta} \\ \bar{\Delta}^* & 0 \end{pmatrix},\end{aligned} \tag{1.81}$$

and expanding the second logarithm in a Taylor series, we recognize the form

$$\begin{aligned}S_{\mathrm{eff}}[\bar{\Delta}^*, \bar{\Delta}] = &-i\,\mathrm{Tr}\ln\begin{pmatrix} p_0 - \xi(\mathbf{p}) & 0 \\ 0 & p_0 + \xi(\mathbf{p}) \end{pmatrix} \\ &-i\,\mathrm{Tr}\ln\left(1 - \frac{|\bar{\Delta}|^2}{p_0^2 - \xi^2(\mathbf{p})}\right),\end{aligned} \tag{1.82}$$

up to an irrelevant constant. The integral over the loop energy k_0 gives for the corresponding effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \int_{\mathbf{k}} \left[E(\mathbf{k}) - \xi(\mathbf{k})\right]. \tag{1.83}$$

To this one-loop result we have to add the tree term $|\bar{\Delta}|^2/\lambda_0$. Expanding $E(\mathbf{k})$ in a Taylor series, we see that the effective Lagrangian also contains a term quadratic in $\bar{\Delta}$. This term amounts to a renormalization of the coupling constant; we find to this order for the renormalized coupling constant λ:

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{1}{2}\int_{\mathbf{k}} \frac{1}{|\xi(\mathbf{k})|}. \tag{1.84}$$

The integral at the right-hand side diverges, to regularize it we introduce a momentum cutoff Λ. In this way we obtain

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{m}{2\pi^2}\Lambda, \tag{1.85}$$

where we omitted the (irrelevant) finite part of the integral. It should be remembered that the bare coupling constant λ_0 is negative, so that there is an attractive interaction between the fermions. We can distinguish two limits. One, if the bare coupling constant is taken to zero, $\lambda_0 \to 0^-$, which is the famous weak-coupling BCS limit. Second, the limit where the bare coupling is taken to minus infinity $\lambda_0 \to -\infty$. This is the strong-coupling limit, where the attractive interaction is such that the fermions form tightly bound pairs [27]. These composite bosons have a weak repulsive interaction and can undergo a Bose-Einstein condensation (see succeeding section).

To this order there is no renormalization of the chemical potential, so that we can write $\mu = \mu_0$.

Since there are two unknowns contained in the theory, $\bar{\Delta}$ and μ, we need a second equation to determine these variables in the mean-field approximation. To find the second equation we note that the average fermion number N, which is obtained by differentiating the effective action (1.68) with respect to μ

$$N = \frac{\partial S_{\rm eff}}{\partial \mu}, \tag{1.86}$$

is fixed. If the system is spacetime independent, this reduces to

$$\bar{n} = -i \,\mathrm{tr} \int_k G_0(k)\tau_3, \tag{1.87}$$

where $\bar{n} = N/V$, with V the volume of the system, is the constant fermion number density, τ_3 is the diagonal Pauli matrix in Nambu space,

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.88}$$

and $G_0(k)$ is the Feynman propagator,

$$\begin{aligned} G_0(k) &= \begin{pmatrix} k_0 - \xi(\mathbf{k}) & -\bar{\Delta} \\ -\bar{\Delta}^* & k_0 + \xi(\mathbf{k}) \end{pmatrix}^{-1} \\ &= \frac{1}{k_0^2 - E^2(\mathbf{k}) + i\eta} \begin{pmatrix} k_0\, \mathrm{e}^{ik_0\eta} + \xi(\mathbf{k}) & \bar{\Delta} \\ \bar{\Delta}^* & k_0\, \mathrm{e}^{-ik_0\eta} - \xi(\mathbf{k}) \end{pmatrix}. \end{aligned} \tag{1.89}$$

Here, η is an infinitesimal positive constant that is to be set to zero at the end of the calculation. The exponential functions in the diagonal elements of the propagator are an additional convergence factor needed in nonrelativistic theories [28]. If the integral over the loop energy k_0 in the particle number equation (1.87) is carried out, it takes the familiar form

$$\bar{n} = \int_{\mathbf{k}} \left(1 - \frac{\xi(\mathbf{k})}{E(\mathbf{k})} \right) \tag{1.90}$$

The two equations (1.78) and (1.87) determine $\bar{\Delta}$ and μ. They are usually evaluated in the weak-coupling BCS limit. However, as was first pointed out by Leggett [27], they can also be easily solved in the strong-coupling limit (see succeeding section), where the fermions are tightly bound in pairs. More recently, also the crossover between

the weak-coupling BCS limit and the strong-coupling composite boson limit has been studied in detail [29, 30, 31, 32].

We continue by writing the order parameter $\Delta_{\rm mf}$ as

$$\Delta_{\rm mf}(x) = \bar{\Delta}\, {\rm e}^{2i\varphi(x)}, \tag{1.91}$$

where $\bar{\Delta}$ is a spacetime-independent solution of the mean-field equation (1.77). This approximation, where the phase of the order parameter is allowed to vary in spacetime while the modulus is kept fixed, is called the London limit. The phase field $\varphi(x)$, which is a background field, physically represents the Goldstone mode of the spontaneously broken U(1) symmetry. Our objective is to derive the effective theory describing this Goldstone mode.

To this end we decompose the Grassmann field as, cf. [33]

$$\psi_\sigma(x) = {\rm e}^{i\varphi(x)}\chi_\sigma(x) \tag{1.92}$$

and substitute the specific form (1.91) of the order parameter in the partition function (1.66). Instead of the effective action (1.68) we now obtain

$$S_{\rm eff} = -i{\rm Tr}\ln\begin{pmatrix} p_0 - \partial_0\varphi - \xi(\mathbf{p}+\nabla\varphi) & -\bar{\Delta} \\ -\bar{\Delta}^* & p_0 + \partial_0\varphi + \xi(\mathbf{p}-\nabla\varphi) \end{pmatrix}, \tag{1.93}$$

where the derivative $\tilde{\partial}_\mu\varphi$ of the Goldstone field plays the role of an Abelian background gauge field. We next write this effective action in the equivalent form

$$S_{\rm eff} = -i{\rm Tr}\ln\left[G_0^{-1}\left(1 - G_0\Lambda\right)\right] = i{\rm Tr}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(G_0\Lambda\right)^\ell, \tag{1.94}$$

where in the last step we ignored an irrelevant constant and the matrix Λ is given by

$$\Lambda(\mathbf{p}) = U\tau_3 + \frac{1}{m}\mathbf{p}\cdot\nabla\varphi + \frac{i}{2m}\nabla^2\varphi, \tag{1.95}$$

with U the Galilei-invariant combination

$$U = \partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2. \tag{1.96}$$

We shall consider only the first two terms in the series at the right-hand side of (1.94), and ignore for the moment higher than first derivatives on the Goldstone field φ. The first term yields

$$S_{\rm eff}^{(1)} = i\,{\rm Tr}\,G_0(p)\tau_3\left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2\right]. \tag{1.97}$$

On account of (1.87), this can be written as

$$S_{\rm eff}^{(1)} = -\int_x \bar{n}\left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2\right], \tag{1.98}$$

where $\bar{n} = k_{\rm F}^3/3\pi^2$ is the fermion number density. We see that the phase of the order parameter appears only in the Galilei-invariant combination (1.96).

We continue with the second term in the series (1.94). We shall first restrict ourselves to the weak-coupling BCS limit, where the chemical potential is well approximated by the Fermi energy, $\mu \approx k_{\rm F}^2/2m$. In this limit, we can make the approximation

$$\int_{\mathbf{k}} \approx \nu(0) \int \frac{{\rm d}^2\hat{k}}{4\pi} \int_0^\infty {\rm d}\epsilon \approx \nu(0) \int \frac{{\rm d}^2\hat{k}}{4\pi} \int_{-\infty}^\infty {\rm d}\xi, \tag{1.99}$$

where $\nu(0) = mk_{\rm F}/2\pi^2$ is the density of states per spin degree of freedom at the Fermi level and $\int {\rm d}^2\hat{k}$ denotes the integral over the solid angle. In the last step, we extended the range of the $\xi = \epsilon - \mu$ integration from $[-\mu, \infty)$ to $(-\infty, \infty)$ which is justified by the rapid convergence of the integrals involved. After some algebra, using the integrals

$$\int_{k_0} \frac{1}{(k_0^2 - E^2 + i\eta)^l} = i\,(-1)^l \frac{\Gamma(l-\frac{1}{2})}{\Gamma(l)} \frac{1}{E^{2l-1}}, \tag{1.100}$$

$$\int_\xi \frac{1}{E^{2l+1}} = \sqrt{\pi} \frac{\Gamma(l)}{\Gamma(l+\frac{1}{2})} \frac{1}{|\bar{\Delta}|^{2l}}, \tag{1.101}$$

with $l = 1, 2, \ldots$, and $\Gamma(l)$ the gamma function, we obtain

$$S_{\rm eff}^{(2)} = \nu(0) \int_x \left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2 \right]^2, \tag{1.102}$$

which is again invariant under Galilei transformations.

Equations (1.98) and (1.102) give the effective theory of the Goldstone mode associated with the spontaneous breakdown of the global U(1) symmetry [34],

$$\mathcal{L}_{\rm eff} = -\bar{n} \left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2 \right] + \nu(0) \left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2 \right]^2. \tag{1.103}$$

As expected, it is of the same form as the effective theory we obtained in Eq. (1.21) of the previous section describing a hydrodynamic sound wave. After rescaling $\varphi \to m\varphi$, the two forms become identical. The only difference being that the present Goldstone field φ is a compact field. For the velocity c of the Anderson-Bogoliubov mode we find

$$c^2 = \tfrac{1}{3} v_{\rm F}^2, \tag{1.104}$$

where $v_{\rm F} = k_{\rm F}/m$ is the Fermi velocity. The effective theory (1.103) has been rederived by various other authors [35, 36, 37].

Graphically, the effective theory is represented by the diagrams depicted in Fig. 1.1, where a line with a shaded bubble inserted stands for i times the *full* Green function G and the black bubble denotes i times the *full* interaction Γ of the χ_σ-fields, introduced in (1.92), with the background field U which is denoted by a wiggly line. Both G and Γ are 2×2 matrices. The full interaction is obtained from the inverse Green function by differentiation with respect to the chemical potential,

$$\Gamma = -\frac{\partial G^{-1}}{\partial \mu}. \tag{1.105}$$

Figure 1.1: Graphical representation of the effective theory (1.103). The symbols are explained in the text.

This follows because U, as defined in (1.96), appears in the theory only in the combination $\mu - U$. To lowest order, the interaction is given by $-\tau_3$. Employing Eq. (1.87), with G_0 replaced by the full Green function, we conclude that the first diagram indeed corresponds to the first part of the effective theory (1.103). The second diagram without the wiggly lines denotes i times the (0 0)-component of the *full* polarization tensor, Π_{00}, at zero energy transfer and low momentum $\mathbf{q}$,

$$i \lim_{\mathbf{q}\to 0} \Pi_{00}(0,\mathbf{q}) = \lim_{\mathbf{q}\to 0} \mathrm{tr} \int_k \tau_3 G\,\Gamma\, G\,(k_0, \mathbf{k}+\mathbf{q}), \tag{1.106}$$

where the minus sign associated with the fermion loop is included. To see that this represents the second part of the effective theory we invoke an argument due to Gavoret and Nozières [38]. By virtue of relation (1.105) between the full Green function G and the full interaction Γ, the (0 0)-component of the polarization tensor can be cast in the form

$$\begin{aligned} \lim_{\mathbf{q}\to 0} \Pi_{00}(0,\mathbf{q}) &= i \lim_{\mathbf{q}\to 0} \mathrm{tr} \int_k \tau_3 G \frac{\partial G^{-1}}{\partial \mu} G(k_0, \mathbf{k}+\mathbf{q}) \\ &= -i \frac{\partial}{\partial \mu} \lim_{\mathbf{q}\to 0} \mathrm{tr} \int_k \tau_3 G(k_0, \mathbf{k}+\mathbf{q}) \\ &= \frac{\partial \bar{n}}{\partial \mu} = -\frac{1}{V}\frac{\partial^2 \Omega}{\partial \mu^2}, \end{aligned} \tag{1.107}$$

where Ω is the thermodynamic potential and V the volume of the system. The right-hand side of (1.107) is $\bar{n}^2\kappa$, with κ the compressibility. Because it is related to the macroscopic sound velocity c via

$$\kappa = \frac{1}{m\bar{n}c^2}, \tag{1.108}$$

we conclude that the (0 0)-component of the full polarization tensor satisfies the so-called compressibility sum rule of statistical physics [38]

$$\lim_{\mathbf{q}\to 0} \Pi_{00}(0,\mathbf{q}) = -\frac{1}{V}\frac{\partial^2 \Omega}{\partial \mu^2} = \frac{\bar{n}}{mc^2}. \tag{1.109}$$

It now follows immediately that the second diagram in Fig. 1.1 represents the second part of the effective theory (1.103).

Both the particle number density and the sound velocity can thus also be obtained from the thermodynamic potential Ω by differentiating it with respect to the chemical potential:

$$\bar{n} = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu}; \qquad \frac{1}{c^2} = -\frac{1}{V}\frac{m}{\bar{n}}\frac{\partial^2\Omega}{\partial\mu^2}. \tag{1.110}$$

In the weak-coupling BCS limit, the spectrum of the elementary fermionic excitations does not differ significantly from the free-particle spectrum. The thermodynamic potential may therefore be approximated by that of a free fermion gas. The partition function (1.63) with $\mathcal{L}$ the free Lagrangian (1.62) reads

$$Z_{\rm free} = \exp\left(-i\int_x \mathcal{V}_{\rm free}\right), \tag{1.111}$$

where the spacetime integral over the potential $\mathcal{V}_{\rm free}$ is given by minus the effective action (1.68) with Δ set to zero. We used a potential rather than an action in (1.111) because the free fermion system is homogeneous in spacetime. Carrying out the integrals over the loop energy and momentum, we obtain for the potential

$$\mathcal{V}_{\rm free} = -\frac{4\sqrt{2}}{15\pi^2}m^{3/2}\mu^{5/2}. \tag{1.112}$$

This physically represents the ground-state energy per unit volume. The thermodynamic potential (at the absolute zero of temperature) is obtained by integrating $\mathcal{V}$ over space,

$$\Omega = \int_{\mathbf{x}} \mathcal{V}. \tag{1.113}$$

With the help of (1.110), the velocity of the Anderson-Bogoliubov mode can now be calculated; this yields again (1.104).

One can of course continue and calculate higher-order terms of the effective theory. Since these involve always higher-order derivatives, they are irrelevant at low energy and small momentum. To identify the expansion parameter, let us compute the quadratic terms in the Goldstone fields, involving fourth-order derivatives. A somewhat tedious but straightforward calculation yields in the weak-coupling BCS limit:

$$\mathcal{L}''_{\rm eff} = \frac{\nu(0)}{6|\bar{\Delta}|^2}\left[(\partial_0^2\varphi)^2 - \tfrac{2}{3}v_{\rm F}^2(\partial_0^2\varphi)(\nabla^2\varphi) + \tfrac{1}{5}v_{\rm F}^4(\nabla^2\varphi)^2\right]. \tag{1.114}$$

This equation shows us that the higher-order terms have either an extra factor $\partial_0^2/|\bar{\Delta}|^2$ or $(v_{\rm F}\nabla)^2/|\bar{\Delta}|^2$, where we recall that the BCS correlation length ξ_0 at zero temperature is given by

$$\xi_0 = \frac{v_{\rm F}}{\pi|\bar{\Delta}|}. \tag{1.115}$$

It sets the scale over which the modulus of the order parameter varies. At low energy and small momentum these additional terms can indeed be ignored. In deriving (1.114) we integrated various times by parts and neglected the ensuing total derivatives. With the additional terms included, the energy spectrum of the Anderson-Bogoliubov mode becomes

$$E^2(\mathbf{k}) = \tfrac{1}{3}v_{\rm F}^2\mathbf{k}^2\left(1 - \tfrac{2}{45}\pi^2\xi_0^2\mathbf{k}^2\right), \tag{1.116}$$

in accordance with Refs. [39, 40, 1]. The minus sign in (1.79) shows the stability of this Goldstone mode against decaying.

When one considers even higher energies and momenta, one also has to account for variations in the modulus of the order parameter so that the London limit, where the modulus is taken to be constant, is no longer applicable.

1.4 Composite Boson Limit

In this section we shall investigate the strong-coupling limit of the pairing theory. In this limit, the attractive interaction between the fermions is such that they form tightly bound pairs of mass $2m$. To explicate this limit in arbitrary space dimension d, we swap the bare coupling constant for a more convenient parameter—the binding energy ϵ_a of a fermion pair in vacuum [41]. Both parameters characterize the strength of the contact interaction. To see the connection between the two, let us consider the Schrödinger equation for the problem at hand. In reduced coordinates, it reads

$$\left[-\frac{\nabla^2}{m}+\lambda_0\,\delta(\mathbf{x})\right]\psi(\mathbf{x})=-\epsilon_a, \tag{1.117}$$

where the reduced mass is $m/2$ and the delta-function potential, with $\lambda_0<0$, represents the attractive contact interaction $\mathcal{L}_{\rm i}$ in (1.59). We stress that this is a two-particle problem in vacuum; it is not the famous Cooper problem of two interacting fermions on top of a filled Fermi sea. The equation is most easily solved by Fourier transforming it. This yields the bound-state equation

$$\psi(\mathbf{k})=-\frac{\lambda_0}{\mathbf{k}^2/m+\epsilon_a}\psi(0), \tag{1.118}$$

or

$$-\frac{1}{\lambda_0}=\int_{\mathbf{k}}\frac{1}{\mathbf{k}^2/m+\epsilon_a}. \tag{1.119}$$

This equation allows us to swap the coupling constant for the binding energy ϵ_a. When substituted in the gap equation (1.78), the latter becomes

$$\int_{\mathbf{k}}\frac{1}{\mathbf{k}^2/m+\epsilon_a}=\frac{1}{2}\int_{\mathbf{k}}\frac{1}{E(\mathbf{k})}. \tag{1.120}$$

By inspection, it is easily seen that this equation has a solution given by [27]

$$\bar{\Delta}\to 0,\qquad \mu_0\to-\tfrac{1}{2}\epsilon_a, \tag{1.121}$$

where it should be noted that the chemical potential is negative here. This is the strong-coupling limit. To appreciate the physical significance of the specific value found for the chemical potential in this limit, we note that the spectrum $E_{\rm b}(\mathbf{q})$ of the two-fermion bound state measured relative to the pair chemical potential $2\mu_0$ reads

$$E_{\rm b}(\mathbf{q})=-\epsilon_a+\frac{\mathbf{q}^2}{4m}-2\mu_0. \tag{1.122}$$

The negative value for μ_0 found in (1.121) is precisely the condition for a Bose-Einstein condensation of the composite bosons in the $\mathbf{q} = 0$ state.

To investigate this limit further, we consider the effective action (1.68) and expand $\Delta(x)$ around a constant value $\bar{\Delta}$ satisfying the gap equation (1.78),

$$\Delta(x) = \bar{\Delta} + \tilde{\Delta}(x). \tag{1.123}$$

We obtain in this way,

$$S_{\rm eff} = i\,{\rm Tr}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left[G_0(p)\begin{pmatrix} 0 & \tilde{\Delta} \\ \tilde{\Delta}^* & 0 \end{pmatrix}\right]^{\ell}, \tag{1.124}$$

where G_0 is given in (1.89). We are interested in terms quadratic in $\tilde{\Delta}$. Employing the derivative expansion outlined on the pages 13-14, we find

$$\begin{aligned} S^{(2)}_{\rm eff}(q) = \tfrac{1}{2} i\,{\rm Tr}\,&\frac{1}{p_0^2 - E^2(\mathbf{p})}\frac{1}{(p_0+q_0)^2 - E^2(\mathbf{p}-\mathbf{q})} \\ &\times\Big\{\bar{\Delta}^2\,\tilde{\Delta}^*\tilde{\Delta}^* + [p_0+\xi(\mathbf{p})][p_0+q_0-\xi(\mathbf{p}-\mathbf{q})]\tilde{\Delta}\tilde{\Delta}^* \\ &\quad + \bar{\Delta}^{*^2}\tilde{\Delta}\tilde{\Delta} + [p_0-\xi(\mathbf{p})][p_0+q_0+\xi(\mathbf{p}-\mathbf{q})]\tilde{\Delta}^*\tilde{\Delta}\Big\}, \end{aligned} \tag{1.125}$$

where $q_\mu = i\tilde{\partial}_\mu$. It is to be recalled here that the derivative p_μ operates on everything to its right, while $\tilde{\partial}_\mu$ operates only on the first object to its right. Let us for a moment ignore the derivatives in this expression. After carrying out the integral over the loop energy k_0 and using the gap equation (1.78), we then obtain

$$\mathcal{L}^{(2)}(0) = -\frac{1}{8}\int_{\mathbf{k}}\frac{1}{E^3(\mathbf{k})}\left(\bar{\Delta}^2\,\tilde{\Delta}^{*2} + \bar{\Delta}^{*^2}\tilde{\Delta}^2 + 2|\bar{\Delta}|^2|\tilde{\Delta}|^2\right). \tag{1.126}$$

In the composite boson limit $\bar{\Delta} \to 0$, so that the spectrum (1.79) of the elementary fermionic excitations can be approximated by

$$E(\mathbf{k}) \approx \epsilon(\mathbf{k}) + \tfrac{1}{2}\epsilon_a. \tag{1.127}$$

The remaining integrals in (1.126) become elementary in this limit,

$$\int_{\mathbf{k}}\frac{1}{E^3(\mathbf{k})} = \frac{4\Gamma(3-d/2)}{(4\pi)^{d/2}}m^{d/2}\epsilon_a^{d/2-3}. \tag{1.128}$$

We next consider the terms involving derivatives in (1.125). Following Ref. [29] we set $\bar{\Delta}$ to zero here. The integral over the loop energy is easily carried out, with the result

$$\begin{aligned} \mathcal{L}^{(2)}(q) \;=\; & -\frac{1}{2}\int_{\mathbf{k}}\frac{1}{q_0 - \mathbf{k}^2/m + 2\mu_0 - \mathbf{q}^2/4m}\tilde{\Delta}\tilde{\Delta}^* \\ & -\frac{1}{2}\int_{\mathbf{k}}\frac{1}{-q_0 - \mathbf{k}^2/m + 2\mu_0 - \mathbf{q}^2/4m}\tilde{\Delta}^*\tilde{\Delta}. \end{aligned} \tag{1.129}$$

The integral over the loop momentum $\mathbf{k}$ will be carried out using the dimensional-regularized integral

$$\int_{\mathbf{k}} \frac{1}{(\mathbf{k}^2 + M^2)} = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \frac{1}{(M^2)^{1-d/2}} \tag{1.130}$$

to suppress irrelevant ultraviolet divergences. To illustrate the power of dimensional regularization, let us consider the case $d = 3$ in detail. Introducing a momentum cutoff, we find in the large-Λ limit

$$\int_{\mathbf{k}} \frac{1}{(\mathbf{k}^2 + M^2)} = \frac{\Lambda}{2\pi^2} - \frac{1}{4\pi} M + \mathcal{O}\left(\frac{1}{\Lambda}\right). \tag{1.131}$$

From (1.130) however only the finite part emerges. This exemplifies that terms diverging with a strictly positive power of the momentum cutoff are suppressed in dimensional regularization. These contributions, which come from the ultraviolet region, cannot physically be very relevant because the simple BCS model (1.59) stops being valid here and new theories are required. It is a virtue of dimensional regularization that these irrelevant divergences are suppressed.

We thus obtain in the strong-coupling limit

$$\begin{aligned} \int_{\mathbf{k}} \frac{1}{q_0 - \mathbf{k}^2/m - \epsilon_a - \mathbf{q}^2/4m} = \\ -\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} m^{d/2} (-q_0 + \epsilon_a + \mathbf{q}^2/4m)^{d/2-1}, \end{aligned} \tag{1.132}$$

or expanded in derivatives

$$\begin{aligned} \int_{\mathbf{k}} \frac{1}{q_0 - \mathbf{k}^2/m - \epsilon_a - \mathbf{q}^2/4m} = \\ -\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-1} - \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-2} \left(q_0 - \frac{\mathbf{q}^2}{4m}\right). \end{aligned} \tag{1.133}$$

The first term at the right-hand side yields as contribution to the effective theory

$$\mathcal{L}_\lambda^{(2)} = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-1} |\tilde{\Delta}|^2. \tag{1.134}$$

To this we have to add the contribution $|\tilde{\Delta}|^2/\lambda_0$ coming from the tree potential, i.e., the last term in the partition function (1.67). But this combination is no other than the one needed to define the renormalized coupling constant via (1.84), which in the strong-coupling limit reads using dimensional regularization

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} m^{d/2} \epsilon_a^{d/2-1}. \tag{1.135}$$

In other words, the contribution (1.134) can be combined with the tree contribution to yield the term $|\tilde{\Delta}|^2/\lambda$. Expanding the square root in (1.132) in powers of the derivative

q_μ using the value (1.121) for the chemical potential, and pasting the pieces together, we obtain for the terms quadratic in $\tilde{\Delta}$ [29],

$$\mathcal{L}^{(2)} = \frac{1}{2}\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} m^{d/2}\epsilon_a^{d/2-2}\,\tilde{\Psi}^\dagger\, M_0(q)\,\tilde{\Psi}, \qquad \tilde{\Psi} = \begin{pmatrix} \tilde{\Delta} \\ \tilde{\Delta}^* \end{pmatrix}, \tag{1.136}$$

where $M_0(q)$ is the 2×2 matrix,

$$M_0(q) = \tag{1.137}$$
$$\begin{pmatrix} q_0 - \mathbf{q}^2/4m - (2-d/2)|\bar{\Delta}|^2/\epsilon_a & -(2-d/2)\bar{\Delta}^2/\epsilon_a \\ -(2-d/2)\bar{\Delta}^{*2}/\epsilon_a & -q_0 - \mathbf{q}^2/4m - (2-d/2)|\bar{\Delta}|^2/\epsilon_a \end{pmatrix}.$$

As we will see shortly in Sec. 1.6, this describes the superfluid state of a weakly interacting composite boson system. On comparing with Eq. (1.170) below, we conclude that the composite bosons have—as expected—a mass $m_{\rm b} = 2m$ twice the fermion mass m, and a small chemical potential

$$\mu_{0,\rm b} = (2-d/2)\frac{|\bar{\Delta}|^2}{\epsilon_a}. \tag{1.138}$$

From (1.137) one easily extracts the so-called Bogoliubov spectrum and the velocity c_0 of the sound mode it describes (see Sec. 1.6),

$$c_0^2 = \frac{\mu_{0,\rm b}}{m_{\rm b}} = (1-d/4)\frac{|\bar{\Delta}|^2}{m\epsilon_a}. \tag{1.139}$$

Also the number density $\bar{n}_{0,\rm b}$ of condensed composite bosons,

$$\bar{n}_{0,\rm b} = \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} m^{d/2}\epsilon_a^{d/2-2}|\bar{\Delta}|^2 \tag{1.140}$$

as well as the weak repulsive interaction $\lambda_{0,\rm b}$ between the composite bosons,

$$\lambda_{0,\rm b} = (4\pi)^{d/2}\frac{1-d/4}{\Gamma(2-d/2)}\frac{\epsilon_a^{1-d/2}}{m^{d/2}} \tag{1.141}$$

follow immediately. We in this way have explicitly demonstrated that the BCS theory in the composite boson limit maps onto the Bogoliubov theory.

In concluding this section, we remark that in $d = 2$ various integrals we encountered become elementary for arbitrary values of $\bar{\Delta}$. For example, the gap equation (1.120) reads explicitly in $d = 2$

$$\epsilon_a = \sqrt{\mu^2 + |\bar{\Delta}|^2} - \mu, \tag{1.142}$$

while the particle number equation (1.90) becomes

$$\bar{n} = \frac{m}{2\pi}\left(\sqrt{\mu^2 + |\bar{\Delta}|^2} + \mu\right). \tag{1.143}$$

Since in two dimensions,

$$\bar{n} = \frac{k_{\rm F}^2}{2\pi} = \frac{m}{\pi}\epsilon_{\rm F}, \tag{1.144}$$

with $k_{\rm F}$ and $\epsilon_{\rm F} = k_{\rm F}^2/2m$ the Fermi momentum and energy, the two equations can be combined to yield [41]

$$\frac{\epsilon_a}{\epsilon_{\rm F}} = 2\frac{\sqrt{\mu^2+|\Delta|^2}-\mu}{\sqrt{\mu^2+|\bar{\Delta}|^2}+\mu}. \tag{1.145}$$

The composite boson limit we have been discussing in this section is easily retrieved from these more general equations. Also note that in this limit, $\bar{n} = 2\bar{n}_{0,\rm b}$.

1.5 Superfluid ^{3}He-a

In this section we extend the previous analysis to a more complicated system, namely that of superfluid ^{3}He [43]. Whereas the Cooper-pairing in a BCS superconductor is in a spin-singlet state, the pairing in ^{3}He is in a triplet state. This means that besides the global U(1)^N group generated by the particle number N also the SO(3)^S spin rotation group has to be considered. Moreover, since on account of the exclusion principle the pairs must have an odd angular momentum, also the space rotation group must be included. In ^{3}He, it is generally excepted that the pairing is in the $L = 1$ state. The relevant symmetry group is now much larger than that of a BCS system which leads to a much richer structure of possible superfluid phases [44, 45]. Since in most phases the three symmetries become intertwined, superfluid ^{3}He displays surprising physical properties [46].

We will consider a ^{3}He film. The specific superfluid state we will study is the two-dimensional analog of the three-dimensional ^{3}He-A phase, the so-called ^{3}He-a phase, first considered by Stein and Cross [47] (see also Ref. [40]). This state is characterized by the magnetic quantum numbers $m_S = 0, m_L = -1$, where the orbital magnetic quantum number m_L may be considered as the projection of the orbital angular momentum on the nonexisting z-axis. The corresponding symmetry breakdown is

$$\mathrm{SO}(3)^S \times \mathrm{SO}(2)^L \times \mathrm{U}(1)^N \supset \mathrm{SO}(2)^S \times \mathrm{U}(1)^{L+N}, \tag{1.146}$$

where the generator of the residual symmetry group U(1)$^{L+N}$ is the sum of the generators of the group SO(2)^L of space rotations and the group U(1)^N of phase transformations. We see that in this state, the spin rotation group SO(3)^S is spontaneously broken to SO(2)^S, as in ferro- and antiferromagnets. Since $m_S = 0$, the superfluid ^{3}He-a phase is, in addition, an antiferromagnet. We choose to represent the order parameter by the matrix A_i^α with spin index $\alpha = 1, 2, 3$ referring to the x, y or z component in spin space, and with orbital index $i = 1, 2$ referring to the x or y component in real space. (We will always denote vector spin indices—such as α—as superscript to distinguish them from ordinary space indices.) In this basis, the ^{3}He-a order parameter is given by [48]

$$A_i^\alpha = \bar{\Delta}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -i \end{pmatrix}, \tag{1.147}$$

where $\bar{\Delta}$ is a constant whose physical significance will become clear when we proceed. Without loss of generality, we have chosen $\bar{\Delta}$ to be real.

It follows from (1.146) that the coset space is three dimensional, implying three Goldstone modes. To make these degrees of freedom explicit, we write the order parameter in the more general form

$$A_i^\alpha = \bar{\Delta} d^\alpha \, (\mathbf{e}^1 - i\mathbf{e}^2)_i \, \mathrm{e}^{2i\varphi}, \tag{1.148}$$

where $\mathbf{e}^1$ and $\mathbf{e}^2$ are two orthonormal vectors in real space, d^α is a unit vector in spin space, parameterizing the coset SO(3)^S/SO(2)^S $\simeq$ S^2, and 2φ is the phase of the order parameter. The residual U(1)$^{L+N}$ symmetry is reflected in the invariance of the order parameter (1.148) under the combined action of a U(1) transformation $\varphi \to \varphi + \frac{1}{2}\alpha$ and a rotation of the $\mathbf{e}$-frame through α about the z axis: $(\mathbf{e}^1 - i\mathbf{e}^2) \to (\mathbf{e}^1 - i\mathbf{e}^2)\exp(-i\alpha)$. The transformation parameter α may be taken to be spacetime dependent. The specific form (1.147) is recovered by choosing d^α in the z direction of spin space, and $\mathbf{e}^1$ and $\mathbf{e}^2$ in the x and y direction of real space, respectively, and by taking $\varphi = 0$.

Superfluid ^{3}He can be modeled by the Lagrangian [39]

$$\mathcal{L} = \psi_\sigma^*[i\partial_0 - \xi(-i\nabla)]\psi_\sigma - \tfrac{1}{2}\lambda_0[\psi_\sigma^*(-i\hat{\partial}_i)\psi_\tau^*]\,[\psi_\tau(-i\hat{\partial}_i)\psi_\sigma], \tag{1.149}$$

where the last term, with λ_0 a positive coupling constant, is an attractive interaction term appropriate for P-wave pairing. In (1.149), ψ_σ are Grassmann fields describing the ^{3}He atoms with spin $\sigma = \uparrow, \downarrow$, and $\xi(-i\nabla)$ is as defined below (1.59). Finally, $\hat{\partial}_i = (\partial_i - \overleftarrow{\partial}_i)/2k_\mathrm{F}$, with $i = 1, 2$. We again linearize the theory by introducing auxiliary fields Δ and Δ^* using the functional identity

$$\begin{aligned}\exp\left\{-i\frac{\lambda_0}{2}\int_x [\psi_\sigma^*(-i\hat{\partial}_i)\psi_\tau^*]\,[\psi_\tau(-i\hat{\partial}_i)\psi_\sigma]\right\} = \\ \int \mathrm{D}\Delta^*\mathrm{D}\Delta \exp\left(-\frac{i}{2}\int_x \Big\{\Delta_{\sigma\tau}^{*\,i}\,[\psi_\tau(-i\hat{\partial}_i)\psi_\sigma] + [\psi_\sigma^*(-i\hat{\partial}_i)\psi_\tau^*]\,\Delta_{\tau\sigma}^i \\ -\frac{1}{\lambda_0}\Delta_{\sigma\tau}^{*\,i}\Delta_{\tau\sigma}^i\Big\}\right).\end{aligned} \tag{1.150}$$

The Euler-Lagrange equation for Δ shows that classically it merely stands for a product of two Grassmann fields:

$$\Delta_{\sigma\tau}^i = \lambda_0 \psi_\sigma(-i\hat{\partial}_i)\psi_\tau. \tag{1.151}$$

As may be inferred from this equation, Δ is symmetric in its spin indices, i.e., it is a spin triplet. Moreover, the index i indicates that the field is a vector under spatial rotations. The partition function Z of the theory can now be represented by the functional integral:

$$\begin{aligned}Z = \int \mathrm{D}\Psi^\dagger\mathrm{D}\Psi \int \mathrm{D}\Delta^*\mathrm{D}\Delta \, \exp\left(\frac{i}{2\lambda_0}\int_x \Delta_{\sigma\tau}^{*\,i}\Delta_{\tau\sigma}^i\right) \\ \times \exp\left[\frac{i}{2}\int_x \Psi^\dagger \begin{pmatrix} p_0 - \xi(\mathbf{p}) & -\frac{1}{2}\{p_i/k_\mathrm{F}, \Delta^i\}g^\dagger \\ -\frac{1}{2}g\{p_i/k_\mathrm{F}, \Delta^{*\,i}\} & p_0 + \xi(\mathbf{p}) \end{pmatrix}\Psi\right],\end{aligned} \tag{1.152}$$

where $\{\ ,\ \}$ denotes the anticommutator. We recall that the operator $p_\mu = i\tilde{\partial}_\mu$ acts on everything to its right. The derivatives $\partial_i \Delta^i$ and $\partial_i \Delta^{*\,i}$ contained in (1.152) arise after a partial integration in the corresponding terms in (1.150). The tensor g in (1.152) is the metric spinor

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.153}$$

and Ψ stands for the Nambu multiplet

$$\Psi = \begin{pmatrix} \psi \\ g\psi^* \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_\uparrow \\ g\psi_\downarrow \end{pmatrix}. \tag{1.154}$$

The reason for introducing the spinor g in this multiplet is that the two symmetric matrices Δ^i $(i = 1, 2)$ may be expressed as a linear combination of the symmetric matrices $\sigma^\alpha g$, with σ^α $(\alpha = 1, 2, 3)$ the Pauli matrices:

$$\Delta^i_{\sigma\tau} = A^\alpha_i (\sigma^\alpha g)_{\sigma\tau}, \tag{1.155}$$

where A^α_i are the expansion coefficients, so that

$$\Delta^i g^\dagger = A^\alpha_i \sigma^\alpha. \tag{1.156}$$

In this way, the metric spinor g disappears from the argument of the exponential function in (1.152). This will simplify the calculation. With the value (1.148) for the matrix A^α_i, the Lagrangian appearing in the partition function (1.152) can be cast in the concise form

$$\mathcal{L} = \tfrac{1}{2}\Psi^\dagger \left(p_0 - \xi(\mathbf{p})\tau_3 - \Lambda(\mathbf{p}) - \frac{\bar{\Delta}}{2k_{\rm F}} \tau_i \{p_i, d^\alpha\} \sigma^\alpha \right) \Psi, \tag{1.157}$$

where τ_i $(i = 1, 2)$ are Pauli matrices in Nambu space which should not be confused with the Pauli matrices σ^α in spin space. The vertex $\Lambda(\mathbf{p})$ is the same as the one we encountered before in the context of the BCS theory [see Eq. (1.95)]. In deriving (1.157) we have given the $\mathbf{e}$-frame the standard orientation.

We proceed in the same way as in the previous section, and decompose the Grassmann fields as in Eq. (1.92). The partition function (1.152) then becomes after carrying out the Gaussian integration over the fermionic degrees of freedom

$$Z = \int \mathrm{D}\Delta^* \mathrm{D}\Delta \, \exp\left(i S_{\rm eff}[\Delta^*, \Delta] + \frac{i}{2\lambda_0} \Delta^{*\,i}_{\sigma\tau} \Delta^i_{\tau\sigma} \right), \tag{1.158}$$

with $S_{\rm eff}$ the effective action,

$$S_{\rm eff} = -\tfrac{1}{2} i \, \mathrm{Tr} \log \left(p_0 - \xi(\mathbf{p})\tau_3 - \Lambda(\mathbf{p}) - \frac{\bar{\Delta}}{2k_{\rm F}} \tau_i \{p_i, d^\alpha\} \sigma^\alpha \right). \tag{1.159}$$

A general orientation of the $\mathbf{e}$-frame, which may vary in spacetime, is accounted for by requiring the action to be invariant under the residual $\mathrm{U}(1)^{L+N}$ symmetry. This is achieved by the replacement

$$\tilde{\partial}_\mu \varphi \to \tilde{\partial}_\mu \varphi - \mathbf{e}^1 \cdot \tilde{\partial}_\mu \mathbf{e}^2. \tag{1.160}$$

Since in this section we are not interested in the Goldstone modes of the spontaneously broken spin rotation group, we keep the spin vector d^α fixed in the z direction.

We continue as in the previous section and expand the effective action in a Taylor series:

$$S_{\rm eff} = \tfrac{1}{2} i {\rm Tr} \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left[G_0(p) \Lambda(\mathbf{p}) \right]^\ell , \tag{1.161}$$

where the Feynman propagator now reads

$$\begin{aligned} G_0(k) &= \left[k_0 - \xi(\mathbf{k}) \tau^3 - \frac{\bar{\Delta}}{k_{\rm F}} \mathbf{k} \cdot \boldsymbol{\tau} \sigma^3 \right]^{-1} \\ &= \frac{1}{k_0^2 - E^2(\mathbf{k}) + i\eta} \left[k_0 {\rm e}^{i k_0 \eta \tau_3} + \xi(\mathbf{k}) \tau_3 + \frac{\bar{\Delta}}{k_{\rm F}} \mathbf{k} \cdot \boldsymbol{\tau} \sigma^3 \right] . \end{aligned} \tag{1.162}$$

Here, $E(\mathbf{k})$ is the spectrum of the elementary fermionic excitations

$$E^2(\mathbf{k}) = \xi^2(\mathbf{k}) + \left(\frac{\bar{\Delta}}{k_{\rm F}} \right)^2 . \tag{1.163}$$

The constant $\bar{\Delta}$ introduced in (1.148) is—apart from a factor $k_{\rm F}$—seen to be the energy gap of these excitations at the Fermi circle.

The expression (1.161) can be evaluated in the weak-coupling limit along the lines of Sec. 1.3 to obtain the effective theory of the Abelian Goldstone mode [49]. We arrive in this way again at the theory (1.103) we calculated from the BCS theory with $\nu(0) = m/2\pi$ now the two-dimensional density of states per spin degree of freedom and $\bar{n} = k_{\rm F}^2/2\pi$ the two-dimensional fermion number density. It describes a sound wave traveling with the speed $v_{\rm F}/\sqrt{2}$. It is remarkable that to this order the additional contributions arising from the derivatives in the interaction term of the Lagrangian (1.149) cancel so that the same effective theory is obtained as in the previous section where these derivatives were absent.

In the next chapter we shall consider the effective theory describing the antiferromagnetic spin waves associated with the spontaneous breakdown of the spin rotation group SO(3)^S $\supset$SO(2)^S in this particular superfluid ^{3}He phase. Since this symmetry is an internal symmetry unrelated to Galilei invariance, there is no reason to expect a theory that is invariant under Galilei transformations.

1.6 Weakly Interacting Bose Gas

The model commonly used to describe a weakly interacting Bose gas (for a general introduction, see the textbooks [50, 42, 51]), is defined by the Lagrangian [52]

$$\mathcal{L} = \phi^* \big[i\partial_0 - \epsilon(-i\nabla) + \mu_0 \big] \phi - \lambda_0 |\phi|^4, \tag{1.164}$$

where $\epsilon(-i\nabla) = -\nabla^2/2m$ and μ_0 is the chemical potential. The self-coupling is taken to be positive, $\lambda_0 > 0$, so that the contact interaction is repulsive. We will treat this model perturbatively in a loop expansion. For this to be applicable the (renormalized)

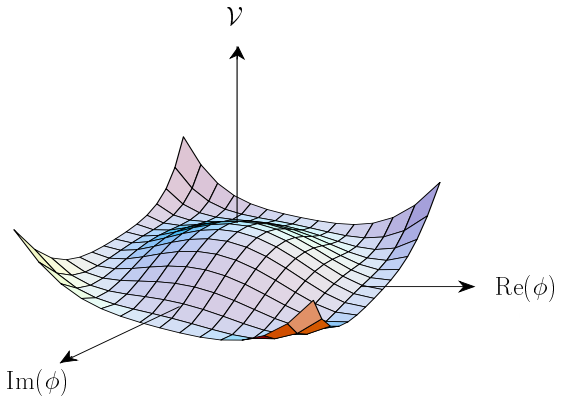


Figure 1.2: Graphical representation of the potential energy (1.165).

coupling constant must be small. Superfluid ^{4}He is a strongly interacting system where this premise does not hold. Fortunately, after almost two decades of experimental efforts by various groups in the US as well as in Europe, Bose-Einstein condensation (BEC) has recently been found in weakly interacting Bose gases. A group in Boulder, Colorado were the first to produce such a BEC in a cloud of ^{87}Rb atoms trapped in a magnetic field [53]. The condensate formed at a temperature around 170 nK and was comprised of up to 2000 atoms; it was stable for about 15 s. A second unequivocal observation of BEC was subsequently established in a system of Na atoms, with a condensate containing two orders of magnitude more particles than in the Colorado experiments [54]. Various other systems of bosonic alkali atoms are presently under study.

At zero temperature, the global U(1) symmetry is spontaneously broken by a non-trivial ground state. This can be easily seen by considering the shape of the potential energy

$$\mathcal{V} = -\mu_0|\phi|^2 + \lambda_0|\phi|^4, \tag{1.165}$$

depicted in Fig. 1.2 which is seen to have a minimum away from the origin $\phi = 0$. To account for this, we shift ϕ by a (complex) constant $\bar{\phi}$ and write

$$\phi(x) = \mathrm{e}^{i\varphi(x)}\,[\bar{\phi} + \tilde{\phi}(x)]. \tag{1.166}$$

The scalar field $\varphi(x)$ is a background field representing the Goldstone mode of the spontaneously broken global U(1) symmetry. At zero temperature, the constant value

$$|\bar{\phi}|^2 = \frac{1}{2}\frac{\mu_0}{\lambda_0} \tag{1.167}$$

minimizes the potential energy. It physically represents the number density of particles contained in the condensate for the total particle number density is given by

$$n(x) = |\phi(x)|^2. \tag{1.168}$$

Because $\bar{\phi}$ is a constant, the condensate is a uniform, zero-momentum state. That is, the particles residing in the ground state are in the $\mathbf{k} = 0$ mode. In terms of the new variables, the quadratic terms of the Lagrangian (1.164) may be cast in the matrix form

$$\mathcal{L}_0 = \tfrac{1}{2}\tilde{\Phi}^\dagger M_0(p)\tilde{\Phi}, \qquad \tilde{\Phi} = \begin{pmatrix} \tilde{\phi} \\ \tilde{\phi}^* \end{pmatrix}, \tag{1.169}$$

with

$$M_0(p) = \begin{pmatrix} p_0 - \epsilon(\mathbf{p}) + \mu_0 - U - 4\lambda_0|\bar{\phi}|^2 & -2\lambda_0\bar{\phi}^2 \\ -2\lambda_0\bar{\phi}^{*2} & -p_0 - \epsilon(\mathbf{p}) + \mu_0 - U - 4\lambda_0|\bar{\phi}|^2 \end{pmatrix}, \tag{1.170}$$

where U is the Galilei-invariant combination (1.96). In writing this we have omitted a term $\nabla^2\varphi$ containing two derivatives which is irrelevant at low momentum and also a term of the form $\nabla\varphi \cdot \mathbf{j}$, where $\mathbf{j}$ is the Noether current associated with the global U(1) symmetry,

$$\mathbf{j} = \frac{1}{2im}\phi^* \overset{\leftrightarrow}{\nabla} \phi. \tag{1.171}$$

This term, which after a partial integration becomes $-\varphi\nabla \cdot \mathbf{j}$, is also irrelevant at low energy and small momentum because in a first approximation the particle number density is constant, so that the classical current satisfies the condition

$$\nabla \cdot \mathbf{j} = 0. \tag{1.172}$$

The Feynman propagator of the theory is easily extracted by inverting the matrix M_0 with the background field U set to zero. This yields upon using the mean-field value (1.167) for $\bar{\phi}$

$$G_0(k) = \frac{1}{k_0^2 - E^2(\mathbf{k}) + i\eta} \times \begin{pmatrix} k_0\, \mathrm{e}^{ik_0\eta} + \epsilon(\mathbf{k}) + 2\lambda_0|\bar{\phi}|^2 & -2\lambda_0\bar{\phi}^2 \\ -2\lambda_0\bar{\phi}^{*2} & -k_0\, \mathrm{e}^{-ik_0\eta} + \epsilon(\mathbf{k}) + 2\lambda_0|\bar{\phi}|^2 \end{pmatrix}, \tag{1.173}$$

with η a small positive constant that is to be set to zero after the k_0 integration has been performed, and $E(\mathbf{k})$ the single-particle Bogoliubov spectrum [55],

$$\begin{aligned} E(\mathbf{k}) &= \sqrt{\epsilon^2(\mathbf{k}) + 2\mu_0\epsilon(\mathbf{k})} \\ &= \sqrt{\epsilon^2(\mathbf{k}) + 4\lambda_0|\bar{\phi}|^2\epsilon(\mathbf{k})}. \end{aligned} \tag{1.174}$$

The most notable feature of this spectrum is that it is gapless, behaving for small momentum as

$$E(\mathbf{k}) \sim u_0\,|\mathbf{k}|, \tag{1.175}$$

with $u_0 = \sqrt{\mu_0/m}$ a velocity which is sometimes referred to as the microscopic sound velocity. It was first shown by Beliaev [56] that the gaplessness of the single-particle spectrum persists at the one-loop order. This was subsequently proven to hold to all orders in perturbation theory by Hugenholtz and Pines [57]. For large momentum, the Bogoliubov spectrum takes a form

$$E(\mathbf{k}) \sim \epsilon(\mathbf{k}) + 2\lambda_0|\bar{\phi}|^2 \tag{1.176}$$

typical for a nonrelativistic particle with mass m moving in a medium. To highlight the condensate we have chosen here the second form in (1.174) where μ_0 is replaced with $2\lambda_0|\bar{\phi}|^2$.

Since gapless modes in general require a justification for there existence, we expect the gaplessness of the single-particle spectrum to be a result of Goldstone's theorem. This is corroborated by the relativistic version of the theory. There, one finds two spectra, one corresponding to a massive Higgs particle which in the nonrelativistic limit becomes too heavy and decouples from the theory, and one corresponding to the Goldstone mode of the spontaneously broken global U(1) symmetry [58]. The latter reduces in the nonrelativistic limit to the Bogoliubov spectrum. Also, when the theory is coupled to an electromagnetic field, one finds that the single-particle spectrum acquires an energy gap. This is what one expects to happen with the spectrum of a Goldstone mode when the Higgs mechanism is operating. The equivalence of the single-particle excitation and the collective density fluctuation has been proven to all orders in perturbation by Gavoret and Nozières [38].

Given this observation we immediately infer that at low energy and small momentum the superfluid phase is described by the theory we encountered before

$$\mathcal{L}_{\text{eff}} = -\bar{n}\left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2\right] + \frac{\bar{n}}{2mc^2}\left[\partial_0\varphi + \frac{1}{2m}(\nabla\varphi)^2\right]^2, \tag{1.177}$$

of a nonrelativistic sound wave, with the real scalar field φ representing the Goldstone mode of the spontaneously broken global U(1) symmetry. The alert reader might be worrying about an apparent mismatch in the number of degrees of freedom in the normal and the superfluid phase. Whereas the normal phase is described by a complex ϕ-field, the superfluid phase is described by a real scalar field φ. The resolution of this paradox lies in the spectrum of the modes [59]. In the normal phase, the spectrum $E(\mathbf{k}) = \mathbf{k}^2/2m$ is linear in the energy, so that only positive energies appear in the Fourier decomposition, and one needs—as is well know from standard quantum mechanics—a complex field to describe a single particle. In the superfluid phase, where the spectrum is $E^2(\mathbf{k}) = c^2\mathbf{k}^2$, the counting goes differently. The Fourier decomposition now contains positive as well as negative energies and a single real field suffice to describe this mode. In other words, although the number of fields is different, the number of degrees of freedom is the same in both phases.

The effective theory is graphically again represented by Fig. 1.1. To lowest order, the inverse propagator is given by the matrix M_0 in (1.170), so that the vertex describing the interaction between the U and the $\tilde{\Phi}$-fields is minus the unit matrix. In terms of the full Green function G, the particle number density now reads

$$\bar{n} = \frac{i}{2}\,\mathrm{tr}\int_k G(k). \tag{1.178}$$

The (0 0)-component of the *full* polarization tensor, Π_{00}, at zero energy transfer and low momentum $\mathbf{q}$,

$$i \lim_{\mathbf{q}\to 0} \Pi_{00}(0,\mathbf{q}) = -\frac{1}{2} \lim_{\mathbf{q}\to 0} \mathrm{tr} \int_k G\,\Gamma\,G\,(k_0, \mathbf{k}+\mathbf{q}) \tag{1.179}$$

contains a symmetry factor $\frac{1}{2}$ which is absent in fermionic theories. Following the same steps as in the case of superconductors, we again arrive at the compressibility sum rule (1.109).

The diagrams of Fig. 1.1 can also be evaluated in a loop expansion (thereby integrating out the complex scalar field $\tilde{\phi}$) to obtain explicit expressions for the particle number density $\bar{n}$ and the sound velocity c to any given order [60]. In doing so, one encounters—apart from ultraviolet divergences which can be renormalized away—also infrared divergences because the Bogoliubov spectrum is gapless. When however all on-loop contributions are added together, these divergences are seen to cancel [60]. We will in this report not proceed in this way, but instead compute the thermodynamic potential Ω from which both $\bar{n}$ and c are obtained by differentiating with respect to the chemical potential [see (1.110)].

In the approximation (1.169) of ignoring higher than second order in the fields, the integration over the $\tilde{\Phi}$ field is Gaussian. Carrying out this integral, we obtain for the partition function

$$Z = \mathrm{e}^{-i\int_x \mathcal{V}_0} \int \mathrm{D}\tilde{\Phi} \exp\left(i\int_x \mathcal{L}_0\right) = \mathrm{e}^{-i\int_x \mathcal{V}_0}\, \mathrm{Det}^{-1/2}[M_0(p)], \tag{1.180}$$

where $\mathcal{V}_0$ is the potential (1.165) with ϕ replaced by $\bar{\phi}$,

$$\mathcal{V}_0 = -\frac{1}{4}\frac{\mu_0^2}{\lambda_0}. \tag{1.181}$$

Writing

$$Z = \exp\left[-i\int_x (\mathcal{V}_0 + \mathcal{V}_{\mathrm{eff}})\right], \tag{1.182}$$

we find from (1.180) that the effective potential is given to this order by

$$\mathcal{V}_{\mathrm{eff}} = -\frac{i}{2}\mathrm{tr}\int_k \ln[M_0(k)]. \tag{1.183}$$

The use of a potential rather than an action here is to indicate that we are working with a spacetime-independent condensate, so that also $U = 0$. The easiest way to evaluate the loop integral over k_μ is to assume—without loss of generality—that $\bar{\phi}$ is real and to first differentiate the expression with respect to μ_0

$$\frac{\partial}{\partial\mu_0}\mathrm{tr}\int_k \ln\begin{pmatrix} k_0-\epsilon(\mathbf{k})-\mu_0 & -\mu_0 \\ -\mu_0 & -k_0-\epsilon(\mathbf{k})-\mu_0 \end{pmatrix} = -2\int_k \frac{\epsilon(\mathbf{k})}{k_0^2 - E^2(\mathbf{k}) + i\eta}, \tag{1.184}$$

where $E(\mathbf{k})$ is the Bogoliubov spectrum (1.174). The integral over k_0 can be carried out with the help of a contour integration, yielding

$$\int_k \frac{\epsilon(\mathbf{k})}{k_0^2 - E^2(\mathbf{k}) + i\eta} = -\frac{i}{2} \int_{\mathbf{k}} \frac{\epsilon(\mathbf{k})}{E(\mathbf{k})}. \tag{1.185}$$

This in turn is easily integrated with respect to μ_0. Putting the pieces together, we obtain for the effective potential

$$\mathcal{V}_{\rm eff} = \frac{1}{2} \int_{\mathbf{k}} E(\mathbf{k}). \tag{1.186}$$

The remaining integral over the loop momentum $\mathbf{k}$ is diverging in the ultraviolet. We regularize it by introducing a momentum cutoff Λ. The model (1.164) is only valid far below the cutoff. Above it, the model breaks down and new physics starts. In the large-Λ limit, we arrive at

$$\mathcal{V}_{\rm eff} = \frac{1}{12\pi^2}\Lambda^3\mu_0 - \frac{1}{4\pi^2}\Lambda m\mu_0^2 + \frac{8}{15\pi^2}m^{3/2}\mu_0^{5/2}, \tag{1.187}$$

ignoring an irrelevant term ($\propto \Lambda^5$) independent of the chemical potential. Equation (1.187) contains two ultraviolet-diverging terms. It should be realized that they arise from a region where the model (1.164) is not applicable. They cannot therefore be of deep significance. As a consequence of the uncertainty principle, stating that large momenta correspond to small distances, terms arising from the ultraviolet region are always local and can be absorbed by redefining the parameters appearing in the Lagrangian [61]. Since $\mu_0 = 2\lambda_0|\bar{\phi}|^2$, the two ultraviolet-diverging terms in (1.187) can be absorbed by introducing the renormalized parameters

$$\mu = \mu_0 - \frac{1}{6\pi^2}\lambda_0\Lambda^3 \tag{1.188}$$

$$\lambda = \lambda_0 - \frac{1}{\pi^2}m\lambda_0^2\Lambda. \tag{1.189}$$

Because the diverging terms are—at least to this order—of a form already present in the original Lagrangian, the theory is called "renormalizable". The renormalized parameters are the physical ones that are to be identified with those measured in experiment. In this way, we see that the contributions to the loop integral stemming from the ultraviolet region are of no importance. What remains is the finite part

$$\mathcal{V}_{\rm eff} = \frac{8}{15\pi^2}m^{3/2}\mu^{5/2}. \tag{1.190}$$

This result could have been obtained directly without renormalization if we, instead of introducing a momentum cutoff to regularize the integrals, had employed analytic regularization. In such a regularization scheme, where for example the integrals are analytically continued to arbitrary real values of the space dimension d, divergences proportional to powers of the cutoff never show up as was demonstrated in Sec. 1.4. Only logarithmic divergences appear as $1/\epsilon$ poles, where ϵ tends to zero when the

parameter in which the analytic continuation is carried out is given its physical value. These logarithmic divergences $\ln(\Lambda/E)$, with E an energy scale, are relevant also in the infrared because for fixed cutoff $\ln(\Lambda/E) \to -\infty$ when E is taken to zero.

In so-called "nonrenormalizable" theories, the terms which are diverging in the ultraviolet are still local but not of a form present in the original Lagrangian. Whereas in former days such theories were rejected because there supposed lack of predictive power, the modern view is that there are no fundamental theories and that there is no basic difference between renormalizable and nonrenormalizable theories [62]. Even a renormalizable theory like (1.164) should be extended to include all higher-order terms such as a $|\phi|^6$-term which are allowed by symmetry. These additional terms render the theory "nonrenormalizable". This does not however change the predictive power of the theory. The point is that when describing the physics at an energy scale E far below the cutoff, the higher-order terms are suppressed by powers of E/Λ, as follows from dimensional analysis. Therefore, far below the cutoff, the nonrenormalizable terms are negligible.

The thermodynamic potential Ω becomes to the order in which we are working

$$\Omega = \int_{\mathbf{x}} (\mathcal{V}_0 + \mathcal{V}_{\text{eff}}) \,. \tag{1.191}$$

We are now in a position to determine the particle number density and the sound velocity using (1.110). We find

$$\bar{n} = \frac{1}{2}\frac{\mu}{\lambda}\left(1 - \frac{8\lambda}{3\pi^2} m^{3/2}\mu^{1/2}\right) \tag{1.192}$$

and

$$c^2 = \frac{\mu}{m}\left(1 + \frac{4\lambda}{3\pi^2} m^{3/2}\mu^{1/2}\right), \tag{1.193}$$

where in the last equation we made an expansion in the (renormalized) coupling constant λ which was assumed to be small. These equations reveal that the expansion is more precisely one in terms of the dimensionless parameter $\lambda m^{3/2}\mu^{1/2}$, or reintroducing Planck's constant $\lambda m^{3/2}\mu^{1/2}/\hbar^3$.

Up to this point, we have considered the chemical potential to be the independent parameter, thereby assuming the presence of a reservoir that can freely exchange particles with the system under study. It can thus contain any number of particles, only the average number is fixed by external conditions. From the experimental point of view it is, however, often more realistic to consider the particle number fixed. If this is the case, the particle number density $\bar{n}$ should be considered as independent variable and the chemical potential should be expressed in terms of it. This can be achieved by inverting relation (1.192):

$$\begin{aligned} \mu &= 2\bar{n}\lambda\left[1 + \frac{8}{3\pi^2}(2\bar{n}m^3\lambda^3)^{1/2}\right] \\ &= \frac{4\pi\bar{n}a}{m}\left[1 + \frac{32}{3}\left(\frac{\bar{n}a^3}{\pi}\right)^{1/2}\right], \end{aligned} \tag{1.194}$$

where in the last step we employed the relation between the (renormalized) coupling constant λ and the S-wave scattering length a [63, 42],

$$\lambda = \frac{2\pi a}{m}. \tag{1.195}$$

This relation follows from comparing the differential cross sections for scattering of a slowly moving boson from a hard-core sphere of radius a and from a delta function potential with strength λ in the Born approximation. For the sound velocity (1.193) expressed in terms of the particle number density we find

$$\begin{aligned} c^2 &= \frac{2\bar{n}\lambda}{m}\left[1 + \frac{4}{\pi^2}\left(2\bar{n}m^3\lambda^3\right)^{1/2}\right] \\ &= \frac{4\pi\bar{n}a}{m^2}\left[1 + 16\left(\frac{\bar{n}a^3}{\pi}\right)^{1/2}\right]. \end{aligned} \tag{1.196}$$

It is important to note that c^2 is linear in the coupling constant. Without the interparticle interaction characterized by λ, the sound velocity would be zero. Moreover, the interaction must be repulsive for the system to support sound waves. Let us in this connection mention that a third experimental group has reported evidence of BEC in a system of ^{7}Li atoms [64]. This would be a surprising result as ^{7}Li atoms have—in contrast to ^{87}Rb and Na atoms—an attractive interaction and thus a negative scattering length.

The effective theory (1.177) can also be put in a equivalent form

$$\mathcal{L}_{\text{eff}} = -\bar{n}U(x) + \frac{1}{4}U(x)\frac{1}{\lambda_0}U(x), \tag{1.197}$$

which can be easily generalized to systems with long-ranged interactions. A case of particular interest to us is the Coulomb potential

$$V(\mathbf{x}) = \frac{e_0^2}{|\mathbf{x}|}, \tag{1.198}$$

whose Fourier transform in d space dimensions reads

$$V(\mathbf{k}) = 2^{d-1}\pi^{(d-1)/2}\Gamma\left[\tfrac{1}{2}(d-1)\right]\frac{e_0^2}{|\mathbf{k}|^{d-1}}. \tag{1.199}$$

The simple contact interaction $L_{\text{i}} = -\lambda_0\int_{\mathbf{x}}|\phi(x)|^4$ in (1.164) gets now replaced by

$$L_{\text{i}} = -\frac{1}{2}\int_{\mathbf{x},\mathbf{y}}|\phi(t,\mathbf{x})|^2 V(\mathbf{x}-\mathbf{y})|\phi(t,\mathbf{y})|^2. \tag{1.200}$$

The rationale for using the three-dimensional Coulomb potential even when considering charges confined to move in a lower dimensional space is that the electromagnetic interaction remains three-dimensional. The effective theory (1.197) now becomes in the Fourier representation

$$\mathcal{L}_{\text{eff}} = -\bar{n}U(k) + \frac{1}{2}U(k_0,\mathbf{k})\frac{1}{V(\mathbf{k})}U(k_0,-\mathbf{k}) \tag{1.201}$$

and leads to the dispersion relation

$$E^2(\mathbf{k}) = 2^d \pi^{(d-1)/2} \Gamma\left[\tfrac{1}{2}(d-1)\right] \frac{\bar{n} e_0^2}{m} |\mathbf{k}|^{3-d}. \tag{1.202}$$

For $d = 3$, this yields the famous plasma mode with an energy gap given by the plasma frequency $\omega_{\rm p}^2 = 4\pi\bar{n}e_0^2/m$.

To appreciate under which circumstances the Coulomb interaction becomes important, we note that for electronic systems $1/|\mathbf{x}| \sim k_{\rm F}$ for dimensional reasons and the fermion number density $\bar{n} \sim k_{\rm F}^d$, where $k_{\rm F}$ is the Fermi momentum. The ratio of the Coulomb interaction energy $\epsilon_{\rm C}$ to the Fermi energy $\epsilon_{\rm F} = k_{\rm F}^2/2m$ is therefore proportional to $\bar{n}^{-1/d}$. This means that the lower the electron number density, the more important the Coulomb interaction becomes.

For later reference, we close this section by calculating the fraction of particles residing in the condensate. In deriving the Bogoliubov spectrum (1.174), we set $|\bar{\phi}|^2 = \mu_0/2\lambda_0$. For our present consideration we have to keep $\bar{\phi}$ as independent variable. The spectrum of the elementary excitation expressed in terms of $\bar{\phi}$ is

$$E(\mathbf{k}) = \sqrt{\left[\epsilon(\mathbf{k}) - \mu_0 + 4\lambda_0|\bar{\phi}|^2\right]^2 - 4\lambda_0^2|\bar{\phi}|^4}\,. \tag{1.203}$$

It reduces to the Bogoliubov spectrum when the mean-field value (1.167) for $\bar{\phi}$ is inserted. Equation (1.186) for the effective potential is still valid, and so is (1.191). We thus obtain for the particle number density

$$\bar{n} = \frac{\mu_0}{2\lambda_0} - \frac{1}{2}\frac{\partial}{\partial\mu_0}\int_{\mathbf{k}} E(\mathbf{k})\Big|_{|\bar{\phi}|^2=\mu_0/2\lambda_0}, \tag{1.204}$$

where the mean-field value for $\bar{\phi}$ is to be substituted after the differentiation with respect to the chemical potential has been carried out. The integral in (1.204) gives rise to an ultraviolet-diverging term which can can be cancelled by going over to the renormalized parameters (1.189) and (1.188) in the first term of (1.204). In the second term, being a one-loop result, we may to this order simply replace the bare by the (one-loop) renormalized parameters. We find in this way [65]

$$\bar{n} = \frac{\mu}{2\lambda} + \frac{1}{3\pi^2}(2m\lambda|\bar{\phi}|^2)^{3/2}, \tag{1.205}$$

or for the so-called depletion of the condensate [66, 67]

$$\frac{\bar{n}}{\bar{n}_0} - 1 \approx \frac{8}{3}\left(\frac{\bar{n}a^3}{\pi}\right)^{1/2}, \tag{1.206}$$

where

$$\bar{n}_0 = \frac{\mu}{2\lambda} \tag{1.207}$$

is the number density of particles in the condensate. Equation (1.206) shows that even at zero temperature not all the particles reside in the condensate. Due to the interparticle repulsion, particles are removed from the ground state and put in states of

finite momentum. It has been estimated that in superfluid ^{4}He—a strongly interacting system—only about 8% of the particles condense in the zero-temperature state [68]. However, it is well known (see next section) that at zero temperature all the particles nevertheless participate in the superfluid motion [69]. Apparently, the condensate drags the normal fluid along with it.

1.7 BEC at Finite Temperature

We continue to consider BEC in a weakly interacting Bose gas at finite temperature. Some finite-temperature considerations can be found in a textbook by Popov [70]. At the absolute zero of temperature, we saw that the Bogoliubov spectrum (1.174) vanishes linearly as the momentum goes to zero. Since this property is a direct consequence of the spontaneously broken global U(1) symmetry, we expect it to hold at any temperature below the critical temperature $T_{\rm c}$. We shall compute the sound velocity in the vicinity of $T_{\rm c}$.

To calculate the thermodynamic potential at finite temperature, we have to generalize the zero-temperature quantum field theory used up to this point to finite temperature. For our purposes, this is achieved simply by going over to imaginary time $t \to -i\tau$, with $0 \le \tau \le 1/T$, and by replacing integrals over energy k_0 with summations over frequencies [71],

$$\int_{k_0} g(k_0) \to i\,T \sum_n g(i\,\omega_n), \tag{1.208}$$

where g is an arbitrary function, and ω_n are the so-called Matsubara frequencies,

$$\omega_n = \begin{cases} 2n\pi T & \text{for bosons} \\ (2n+1)\pi T & \text{for fermions.} \end{cases} \tag{1.209}$$

These rules allow us to calculate the partition function Z at finite temperature. The thermodynamic potential is then obtained using the relation

$$\Omega = -T\ln(Z). \tag{1.210}$$

As in the previous section, we evaluate the thermodynamic potential for the case where the condensate is spacetime independent. That is, $\bar\phi$ introduced in (1.166) is assumed to depend only on temperature and the scalar field φ representing the Goldstone mode is set to zero. It was shown in (1.191) that in the Bogoliubov approximation of ignoring terms higher than second order in the complex field $\tilde\phi$, the thermodynamic potential is given by the effective potential. Applying the rules outlined above to the zero-temperature expression (1.183), we obtain

$$\mathcal{V}_{\rm eff}(T) = \frac{T}{2} \sum_n \int_{\mathbf{k}} \mathrm{tr}\,\ln[M_0(i\omega_n, \mathbf{k})], \tag{1.211}$$

with the matrix M_0 given by (1.170) with $U = 0$. The mean-field value of the potential energy is given by

$$\mathcal{V}_0(T) = -\mu_0|\bar\phi(T)|^2 + \lambda_0|\bar\phi(T)|^4, \tag{1.212}$$

with $|\bar{\phi}(T)|^2$ denoting the number density of particle residing in the condensate.

The summation over the Matsubara frequencies can be carried out using standard methods. In this way, one obtains for the effective potential

$$\mathcal{V}_{\rm eff}(T) = \frac{1}{2}\int_{\mathbf{k}} E(\mathbf{k}) + T\int_{\mathbf{k}} \ln\left(1-\mathrm{e}^{-E(\mathbf{k})/T}\right). \tag{1.213}$$

The first term at the right-hand side has been analyzed in the previous section and was shown to lead to a renormalization of the parameters λ_0 and μ_0 in addition to the contribution (1.190) which we will ignore here. We shall study the second term in the vicinity of the critical point by expanding it in a high-temperature series. The expansion is justified only if $T_{\rm c}$ is in the high-temperature regime. It will turn out that this is indeed the case for the weak-coupling theory we are discussing. We obtain for the thermodynamic potential

$$\begin{aligned}\frac{\Omega}{V} = \mathcal{V}(T) + \frac{(2m)^{3/2}}{2\pi^2}T^{5/2}\int_0^\infty \mathrm{d}q \\ \times\Bigg\{q^2\,\ln\left(1-\mathrm{e}^{-q^2}\right) + \frac{1}{T}\big(4\lambda|\bar{\phi}|^2-\mu\big)\,\frac{q^2}{\mathrm{e}^{q^2}-1} \\ -\frac{1}{2T^2}\left[4\lambda^2|\bar{\phi}|^4\,\frac{1}{\mathrm{e}^{q^2}-1} + \big(4\lambda|\bar{\phi}|^2-\mu\big)^2\,\frac{q^2\mathrm{e}^{q^2}}{\big(\mathrm{e}^{q^2}-1\big)^2}\right] \\ +\mathcal{O}\left(\frac{1}{T^3}\right)\Bigg\},\end{aligned} \tag{1.214}$$

where the integration variable is $q = |\mathbf{k}|/\sqrt{2mT}$. The potential is denoted by $\mathcal{V}(T)$ to indicate that we included the zero-temperature renormalization of the parameters in (1.212). The first integral appearing here is finite and yields

$$\int_0^\infty \mathrm{d}y y^2\,\ln\left(1-\mathrm{e}^{-y^2}\right) = -\tfrac{1}{2}\Gamma(\tfrac{3}{2})\zeta(\tfrac{5}{2}), \tag{1.215}$$

with Γ the gamma function and ζ the Riemann zeta function. The last two integrals, however, are infrared divergent. We regularize these by analytically continue the following equations to arbitrary values of α

$$\int_0^\infty \frac{\mathrm{d}x}{x}\frac{x^\alpha}{\mathrm{e}^x-1} = \Gamma(\alpha)\zeta(\alpha), \tag{1.216}$$

$$\int_0^\infty \frac{\mathrm{d}x}{x}\frac{x^\alpha}{(\mathrm{e}^x-1)^2} = \Gamma(\alpha)[\zeta(\alpha-1)-\zeta(\alpha)], \tag{1.217}$$

with the result

$$\begin{aligned}\frac{\Omega}{V} = \mathcal{V}(T) + \left(\frac{m}{2\pi}\right)^{3/2}\Big\{-\zeta(\tfrac{5}{2})T^{5/2} + \big(4\lambda|\bar{\phi}|^2-\mu\big)\zeta(\tfrac{3}{2})T^{3/2} \\ -\left[4\lambda^2|\bar{\phi}|^4 + \tfrac{1}{2}\big(4\lambda|\bar{\phi}|^2-\mu\big)^2\right]\zeta(\tfrac{1}{2})T^{1/2}\Big\}.\end{aligned} \tag{1.218}$$

To this order in $1/T$ the thermodynamic potential is of the Landau form, involving terms up to fourth order in the order parameter $\bar{\phi}$:

$$\frac{\Omega}{V} = \alpha_0 - \alpha_1|\bar{\phi}|^2 + \alpha_2|\bar{\phi}|^4, \tag{1.219}$$

with

$$\alpha_0 = \left(\frac{m}{2\pi}\right)^{3/2}\left[-\zeta(\tfrac{5}{2})T^{5/2} - \mu\zeta(\tfrac{3}{2})T^{3/2} - \tfrac{1}{2}\mu^2\zeta(\tfrac{1}{2})T^{1/2}\right] \tag{1.220}$$

a $\bar{\phi}$-independent term, and

$$\alpha_1 = \mu - 4\lambda\left(\frac{m}{2\pi}\right)^{3/2}\left[\zeta(\tfrac{3}{2})T^{3/2} + \mu\zeta(\tfrac{1}{2})T^{1/2}\right] \tag{1.221}$$

$$\alpha_2 = \lambda\left[1 - 12\lambda\left(\frac{m}{2\pi}\right)^{3/2}\zeta(\tfrac{1}{2})T^{1/2}\right]. \tag{1.222}$$

We note, however, that our expansion is not one in terms of $\bar{\phi}$ since the higher-order terms in $1/T$ contain, besides higher-order terms in $\bar{\phi}$, also $|\bar{\phi}|^2$- and $|\bar{\phi}|^4$-terms.

The critical temperature is determined by setting the coefficient α_1 to zero. An approximate solution to this equation is given by

$$T_{\rm c} = \frac{\pi}{\left[\sqrt{2}\,\zeta(\frac{3}{2})\right]^{2/3}}\frac{1}{m}\left(\frac{\mu}{\lambda}\right)^{2/3} - \tfrac{2}{3}\frac{\zeta(\frac{1}{2})}{\zeta(\frac{3}{2})}\mu + \mathcal{O}(\lambda^{2/3}), \tag{1.223}$$

where we expanded in the coupling constant λ. This justifies the high-temperature expansion we have been using because the leading term is of the order $\lambda^{-2/3}$ which is large for a weak-coupling theory.

Equation (1.223) expresses the critical temperature in terms of μ, the chemical potential. As we mentioned in the previous section, from the experimental point of view it is sometimes more realistic to consider the particle number density as independent variable. To express $T_{\rm c}$ in terms of the particle number density, we invert the equation for n obtained from (1.219) with $\bar{\phi} = 0$,

$$\bar{n}(T_{\rm c}) = -\frac{1}{V}\frac{\partial\Omega}{\partial\mu}\bigg|_{T=T_{\rm c}} = \zeta(\tfrac{3}{2})\left(\frac{mT_{\rm c}}{2\pi}\right)^{3/2} + \mu\left(\frac{m}{2\pi}\right)^{3/2}\zeta(\tfrac{1}{2})T_{\rm c}^{1/2}, \tag{1.224}$$

to obtain $\mu = \mu(\bar{n})$. When this is substituted in the condition $\alpha_1 = 0$, the latter can be manipulated in the form

$$\bar{n}(T_{\rm c}) - 4\bar{n}(T_{\rm c})\lambda\left(\frac{m}{2\pi}\right)^{3/2}\zeta(\tfrac{1}{2})T_{\rm c}^{1/2} - \left(\frac{m}{2\pi}\right)^{3/2}\zeta(\tfrac{3}{2})T_{\rm c}^{3/2} = 0, \tag{1.225}$$

from which the critical temperature as function of the particle number density can be determined. Setting λ equal to zero, we obtain the critical temperature

$$T_0 = \frac{2\pi}{m}\left(\frac{\bar{n}}{\zeta(\frac{3}{2})}\right)^{2/3} \tag{1.226}$$

of a free Bose gas. To first nontrivial order in an expansion in λ, we find

$$\begin{aligned} T_{\rm c} &= T_0 \left[1 - \frac{4}{3\pi} \frac{\zeta(\frac{1}{2})}{\zeta^{1/3}(\frac{3}{2})} m\lambda \bar{n}^{1/3}(T_{\rm c}) \right] \\ &= T_0 \left[1 - \frac{8}{3} \zeta(\tfrac{1}{2}) \left(\frac{\bar{n}(T_{\rm c}) a^3}{\zeta(\frac{3}{2})} \right)^{1/3} \right], \end{aligned} \tag{1.227}$$

where in the last equation we replaced the coupling constant λ with the scattering length a using relation (1.195). This equation implies a (slight) increase of the critical temperature due to the weak interaction since $\zeta(\frac{1}{2}) < 0$. This is qualitatively different from the strongly interacting ^{4}He system. A free gas with ^{4}He parameters at vapor pressure would have a critical temperature of about 3 K, whereas liquid ^{4}He becomes superfluid at the *lower* temperature of 2.17 K.

The value of the order parameter $\bar{\phi}(T)$ obtained from Eq. (1.219) reads

$$|\bar{\phi}(T)|^2 = \frac{1}{2} \frac{\alpha_1}{\alpha_2}. \tag{1.228}$$

Near the critical temperature this can be cast in the form

$$\begin{aligned} |\bar{\phi}(T)|^2 &\approx 3\zeta(\tfrac{3}{2}) \left(\frac{mT_{\rm c}}{2\pi} \right)^{3/2} \left(1 - \frac{T}{T_{\rm c}} \right) \\ &\approx 3\bar{n}(T_{\rm c}) \left(1 - \frac{T}{T_{\rm c}} \right). \end{aligned} \tag{1.229}$$

Recalling that $|\bar{\phi}(T)|^2$ represents the particle number density $\bar{n}_0(T)$ of the condensate, we see that when the critical temperature is approached from below, the Bose-Einstein condensate is drained of particles, and that at $T_{\rm c}$ it vanishes altogether.

We are now in a position to investigate the spectrum of the elementary excitations at finite temperature. If one simply substitutes the value (1.228) for $\bar{\phi}(T)$ in the Bogoliubov spectrum (1.203) one finds, contrary to what is expected, that it has an energy gap. The solution to this paradox, which also exists in the relativistic $|\phi|^4$-theory [71], lays in the observation that, as at zero temperature [63], perturbation theory should be carried out consistently. This has not been done up to this point. Whereas Eq. (1.228) included thermal fluctuations, the spectrum (1.203) did not. To fix this, we go back to Eq. (1.221) and note that it reflects a change in the chemical potential due to thermal fluctuations. With this modification the finite-temperature spectrum becomes gapless—in accordance with a theorem due to Hohenberg and Martin [72].

Let us work this out in some detail near the critical temperature, where $|\bar{\phi}(T)|^2$ is given by (1.229). Retaining only the leading term in the $1/T$ expansion, we infer from (1.221) the following change in the chemical potential

$$\mu \to \tilde{\mu} = \mu - 4\zeta(\tfrac{3}{2})\lambda \left(\frac{mT}{2\pi} \right)^{3/2}, \tag{1.230}$$

and $2\lambda|\bar{\phi}|^2 = \tilde{\mu}$, where μ is given a tilde to indicate that it is dressed by thermal fluctuations. In this way, the finite-temperature spectrum becomes

$$E(\mathbf{k}) = \sqrt{\epsilon^2(\mathbf{k}) + 4\lambda|\bar{\phi}(T)|^2 \epsilon(\mathbf{k})}, \tag{1.231}$$

which is indeed gapless. Since we included the zero-temperature renormalization of the parameters stemming from the first term at the right-hand side of (1.213), the renormalized parameters λ and μ feature in these equations. The gaplessness of the finite-temperature spectrum was shown to be true in all orders of perturbation theory by Hohenberg and Martin [72]. The result we just obtained is completely analogous to the zero-temperature Bogoliubov spectrum (1.174). The expression for the sound velocity at finite temperature we extract from the spectrum reads

$$\begin{aligned} c^2(T) &= \frac{2\lambda}{m}|\bar{\phi}(T)|^2 \\ &= 6\frac{\lambda}{m}\bar{n}(T_{\rm c})\left(1-\frac{T}{T_{\rm c}}\right). \end{aligned} \tag{1.232}$$

It vanishes when the temperature approaches $T_{\rm c}$. This is in accord with the observation that the gapless Goldstone mode of the spontaneously broken global U(1) symmetry vanishes at the critical temperature. We note that also at finite temperature, the sound velocity squared is linear in the coupling constant—without the interparticle repulsion the sound velocity would be zero.

At the end of the preceding section we mentioned the well-known fact that despite only a fraction of the particles at zero temperature reside in the zero-momentum state, all particles participate in the superfluid motion. To show this, let us assume the entire system moves with a velocity $\mathbf{u}$ relative to the laboratory system. As is known from standard hydrodynamics the time derivate in the frame following the motion of the fluid is $\partial_0 + \mathbf{u}\cdot\nabla$ [see Eq. (1.8)]. If we insert this in the Lagrangian (1.164) of the near-ideal Bose gas, it becomes

$$\mathcal{L} = \phi^*[i\partial_0 - \epsilon(-i\nabla) + \mu_0 - \mathbf{u}\cdot(-i\nabla)]\phi - \lambda_0|\phi|^4, \tag{1.233}$$

where the extra term features the total momentum $\int_{\mathbf{x}} \phi^*(-i\nabla)\phi$ of the system. The velocity $\mathbf{u}$ multiplying this is on the same footing as the chemical potential which multiplies the particle number $\int_{\mathbf{x}} |\phi|^2$. Whereas μ_0 is associated with particle number conservation, $\mathbf{u}$ is related to the conservation of momentum.

In the two-fluid picture, the condensate can move with a different velocity $\mathbf{v}_{\rm s}$ as the rest of the system. To bring this out we introduce new fields, cf. (1.166)

$$\phi(x) \to \phi'(x) = {\rm e}^{im\mathbf{v}_{\rm s}\cdot\mathbf{x}}\phi(x) \tag{1.234}$$

in terms of which the Lagrangian becomes [73]

$$\mathcal{L} = \phi^*\big[i\partial_0 - \epsilon(-i\nabla) + \mu_0 - \tfrac{1}{2}m\mathbf{v}_{\rm s}\cdot(\mathbf{v}_{\rm s}-2\mathbf{u}) - (\mathbf{u}-\mathbf{v}_{\rm s})\cdot(-i\nabla)\big]\phi - \lambda_0|\phi|^4, \tag{1.235}$$

where we dropped the primes on ϕ again. Both velocities appear in this expression. Apart from the change $\mathbf{u} \to \mathbf{u} - \mathbf{v}_{\rm s}$ in the second last term, the field transformation resulted in a change of the chemical potential

$$\mu_0 \to \mu_{\rm eff} := \mu_0 - \tfrac{1}{2}m\mathbf{v}_{\rm s}\cdot(\mathbf{v}_{\rm s} - 2\mathbf{u}) \tag{1.236}$$

where $\mu_{\rm eff}$ may be considered as an effective chemical potential.

The equations for the Bogoliubov spectrum and the thermodynamic potential are readily written down for the present case when we keep these two changes in mind. In particular, the effective potential (1.213) reads

$$\mathcal{V}_{\rm eff}(T) = \frac{1}{2}\int_{\mathbf{k}} E(\mathbf{k}) + T\int_{\mathbf{k}} \ln\left(1 - \mathrm{e}^{-[E(\mathbf{k})+(\mathbf{u}-\mathbf{v}_{\rm s})\cdot\mathbf{k}]/T}\right), \tag{1.237}$$

where $E(\mathbf{k})$ is the Bogoliubov spectrum (1.203) with the replacement Eq. (1.236). The mean-field potential $\mathcal{V}_0(T)$ is given by (1.212) with the same replacement. The momentum density, or equivalently, the mass current $\mathbf{g}$ of the system is obtained in this approximation by differentiating the potential $\mathcal{V}_0(T)+\mathcal{V}_{\rm eff}(T)$ with respect to $-\mathbf{u}$. We find, using the equation

$$\frac{\partial \mu_{\rm eff}}{\partial \mathbf{u}} = m\mathbf{v}_{\rm s} \tag{1.238}$$

that it is given by

$$\mathbf{g} = mn\mathbf{v}_{\rm s} - \int_{\mathbf{k}} \frac{\mathbf{k}}{\exp\{[E(\mathbf{k}) + (\mathbf{u}-\mathbf{v}_{\rm s})\cdot\mathbf{k}]/T\} - 1}. \tag{1.239}$$

The last term is the contribution stemming from the elementary excitations. In the zero-temperature limit, this term vanishes and $\mathbf{g} = mn\mathbf{v}_{\rm s}$. This equation, comprising the total particle number density n, shows that at zero temperature indeed all the particles are involved in the superflow, despite the fact that only a fraction of them resides in the condensate [69]. When the condensate moves with the same velocity as the rest ($\mathbf{v}_{\rm s} = \mathbf{u}$), the last term in (1.239) vanishes again, now by symmetry.

Assuming that the difference between the normal and superfluid velocity is small, we may expand the last term in (1.239) to linear order in this difference to find

$$\mathbf{g} = \rho\mathbf{v}_{\rm s} + \frac{1}{3T}\int_{\mathbf{k}} \mathbf{k}^2 \frac{\mathrm{e}^{E(\mathbf{k})/T}}{(\mathrm{e}^{E(\mathbf{k})/T}-1)^2}(\mathbf{u}-\mathbf{v}_{\rm s}), \tag{1.240}$$

where $\rho = mn$ is the total mass density of the fluid, and we used the general result

$$\int_{\mathbf{k}} k^i k^j f(|\mathbf{k}|) = \frac{1}{3}\delta^{ij}\int_{\mathbf{k}} \mathbf{k}^2 f(|\mathbf{k}|), \tag{1.241}$$

with $f(|\mathbf{k}|)$ an arbitrary function depending only on the length of $\mathbf{k}$. The last term stemming from the elementary excitations may be used to define the normal mass density $\rho_{\rm n}$,

$$\rho_{\rm n} = \frac{1}{3T}\int_{\mathbf{k}} \mathbf{k}^2 \frac{\mathrm{e}^{E(\mathbf{k})/T}}{(\mathrm{e}^{E(\mathbf{k})/T}-1)^2}. \tag{1.242}$$

Writing

$$\rho = \rho_{\rm s} + \rho_{\rm n} \tag{1.243}$$

for the total mass density, we may cast (1.240) in the form

$$\mathbf{g} = \rho_{\rm s}\mathbf{v}_{\rm s} + \rho_{\rm n}\mathbf{u}_{\rm n}. \tag{1.244}$$

These last two equations are the basic equations of the two-fluid model [69]. The model was introduced by Tisza [74] using idea's of London to give a phenomenological description of superfluid ^{4}He. It not only successfully explained various startling experimental properties of the strongly interacting system, but also predicted new phenomena which were later confirmed by experiment

1.8 BCS at Finite Temperature

We in this section apply the high-temperature expansion to the BCS theory to show that it yields the usual Ginzburg-Landau theory [75].

The one-loop effective potential for a uniform system at finite temperature is readily seen to take the form

$$\mathcal{V}_{\rm eff} = -\int_{\mathbf{k}} E(\mathbf{k}) - 2\,T\int_{\mathbf{k}} \ln\left(1+\mathrm{e}^{-E(\mathbf{k})/T}\right), \tag{1.245}$$

with $E(\mathbf{k})$ the BCS spectrum (1.79) of the elementary fermionic excitations. The factor -2 in (1.245) arises because there are two fermion species, with spin $\sigma =\uparrow$ and $\downarrow$, respectively. To the one-loop potential we have to add the tree contribution

$$\mathcal{V}_0 = -\frac{|\bar{\Delta}|^2}{\lambda_0}. \tag{1.246}$$

The form (1.245) for the effective potential is usually not taken as starting point to derive the Ginzburg-Landau theory. Normally, one first carries out the integration over the loop energy ξ and then performs the sum over the Matsubara frequencies [76]. But here we follow—in analogy with the previous calculation—the opposite route and have carried out the summation first and will now perform the integration over ξ (in the weak-coupling limit),

$$\mathcal{V}_{\rm eff} = -2\,T\nu(0)\int_{\xi} \ln\left(1+\mathrm{e}^{-E/T}\right), \tag{1.247}$$

where we ignored the first term in (1.245) corresponding to the one-loop quantum contribution to the effective potential which we studied in Sec. 1.3, save for the renormalization of λ_0 it leads to. Our problem is seen to reduce to one in a single dimension. To cope with divergences that will appear we dimensional regularize the integral and consider the problem in $1-\epsilon$ dimensions. Equation (1.247) then becomes

$$\mathcal{V}_{\rm eff} = -4\,T^2\nu(0)\int_0^\infty \mathrm{d}y\, y^{-\epsilon} \ln\left(1+\mathrm{e}^{-\sqrt{y^2+|\bar{\Delta}|^2/T^2}}\right), \tag{1.248}$$

with y the dimensionless variable $y=\xi/T$. We kept the one-dimensional integration measure K_1, where

$$K_d = \frac{2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}, \tag{1.249}$$

is the area of a unit sphere in d spatial dimensions divided by $(2\pi)^d$. Because $\bar{\Delta}$ appears only in the combination $|\bar{\Delta}|^2/T^2$, it follows that an expansion in this parameter is tantamount to one in $1/T$. Carrying out this expansion, we arrive at

$$\mathcal{V}_{\rm eff} = 2\nu(0)\int_0^\infty \mathrm{d}y\, y^{-\epsilon}\Bigg\{-2\,T^2\ln\left(1+\mathrm{e}^{-y}\right) + |\bar{\Delta}|^2\frac{1}{y}\frac{1}{\mathrm{e}^y+1} \\ -\frac{1}{4}\frac{|\bar{\Delta}|^4}{T^2}\left[\frac{1}{y^3}\frac{1}{\mathrm{e}^y+1}+\frac{1}{y^2}\frac{\mathrm{e}^y}{(\mathrm{e}^y+1)^2}\right] \\ +\mathcal{O}\left(\frac{1}{T^4}\right)\Bigg\}. \tag{1.250}$$

In analogy with what we did in the case of a weakly interacting Bose gas, we regularize the infrared-divergent integrals by analytically continue the following equations to arbitrary values of α

$$\int_0^\infty \mathrm{d}x\,\ln\left(1+\mathrm{e}^{-x}\right) = \frac{\pi^2}{12}, \tag{1.251}$$

$$\int_0^\infty \frac{\mathrm{d}x}{x}\frac{x^\alpha}{\mathrm{e}^x+1} = \Gamma(\alpha)(1-2^{1-\alpha})\zeta(\alpha), \tag{1.252}$$

$$\int_0^\infty \frac{\mathrm{d}x}{x}\frac{x^\alpha}{(\mathrm{e}^x+1)^2} = \Gamma(\alpha)\left[(1-2^{1-\alpha})\zeta(\alpha)-(1-2^{2-\alpha})\zeta(\alpha-1)\right]. \tag{1.253}$$

In this way we obtain in the limit $\epsilon\to 0$

$$\mathcal{V}_{\rm eff} = -\frac{\pi^2}{3}\nu(0)T^2 - \nu(0)|\bar{\Delta}|^2\left[\frac{1}{\epsilon}+\gamma+2\ln(2)+2\zeta'(0)\right] \\ -\frac{7}{4}\zeta'(-2)\nu(0)\frac{|\bar{\Delta}|^4}{T^2}. \tag{1.254}$$

To make contact with the standard approach we use the identities

$$\zeta'(0) = -\tfrac{1}{2}\ln(2\pi), \qquad \zeta'(-2) = -\frac{1}{4\pi^2}\zeta(3) \tag{1.255}$$

and substitute

$$\frac{1}{\epsilon}\to\ln\left(\frac{\omega_{\rm D}}{T}\right), \tag{1.256}$$

where the Debeye energy $\omega_{\rm D}$, being a measure of the inverse lattice spacing, is the physical ultraviolet cutoff and the temperature T is the relevant infrared scale. [This correspondence between the pole $1/\epsilon$ of dimensional regularization and the logarithm $\ln(\Lambda)$ appearing in the regularization with a cutoff is commonly used in the context of quantum field theory.] The critical temperature is again determined by the condition that the coefficient of the term quadratic in the order parameter be zero, i.e.,

$$\nu(0)\left[\ln\left(\frac{\omega_{\rm D}}{T}\right)+\gamma-\ln\left(\tfrac{1}{2}\pi\right)\right]+\frac{1}{\lambda}=0, \tag{1.257}$$

where we included the tree contribution (1.246) with the bare coupling λ_0 replaced with the renormalized one (1.84). This yields the standard result for the critical temperature $T_{\rm c}$

$$T_{\rm c} = \frac{2}{\pi}{\rm e}^{\gamma}\omega_{\rm D}{\rm e}^{1/\nu(0)\lambda}, \tag{1.258}$$

where it should be kept in mind that the coupling constant λ is negative in the weak-coupling limit. Employing this expression for the critical temperature, we can manipulate the effective potential (1.254) with the tree contribution (1.246) added in the canonical form

$$\mathcal{V} = -\frac{\pi^2}{3}\nu(0)T^2 + \nu(0)\ln\left(\frac{T}{T_{\rm c}}\right)|\bar{\Delta}|^2 + \frac{7\zeta(3)}{16\pi^2}\nu(0)\frac{|\bar{\Delta}|^4}{T^2}, \tag{1.259}$$

valid close to the critical temperature. The minimization of this potential with respect to $\bar{\Delta}$ yields the well-known temperature dependence of the order parameter near the critical temperature

$$|\bar{\Delta}(T)|^2 \approx \frac{8\pi^2}{7\zeta(3)}T_{\rm c}^2\left(1 - \frac{T}{T_{\rm c}}\right), \tag{1.260}$$

which should be compared with Eq. (1.229) we obtained for a superfluid.

Chapter 2

Induced Quantum Numbers

Solitons play an important role in condensed matter physics (for a general introduction see Ref. [77, 78]). One of the most famous solitons is the magnetic vortex in a superconductor first discussed by Abrikosov [79]. In various models, solitons are found to have peculiar quantum numbers associated with them. We, in this chapter, exclusively study solitons arising in fermionic systems. The unusual quantum numbers induced by the solitons can then be computed by integrating out the fermionic degrees of freedom. The effective theory thus obtained contains the same information as the original fermionic model at low energy and small momentum. In particular, approximate expressions for the fermionic currents—so-called Goldstone-Wilczek currents [80]—can be extracted from it. To see how this is connected to the quantum numbers of solitons, it should be realized that the effective theory and the ensuing Goldstone-Wilczek currents are built from background fields, which in our definition include possible Goldstone fields. But solitons are precisely specified by these fields. Hence, when a specific field configuration describing a soliton is substituted in the Goldstone-Wilczek currents, the induced quantum numbers localized on the soliton can be evaluated. These quantum numbers arising in purely bosonic effective theories can be peculiar in that they can have fermionic, or even anyonic characteristics. We will encounter various examples of this along the way.

We will also consider phase transitions where the induced quantum numbers localized on the solitons change from fermionic to bosonic. Such a transition, first considered by Wen and Zee [81], is called a statistics-changing phase transition.

2.1 Skyrmion Lattice

In this section we investigate the effective theory describing the spin degrees of freedom of superfluid ^{3}He-a. In this superfluid state, as we remarked in Sec. 1.5, the $\mathrm{SO}^S(3)$ spin rotation group is spontaneously broken to the group $\mathrm{SO}^S(2)$ of spin rotations about the preferred spin axis. Since the state is characterized by the magnetic quantum number $m_S = 0$, we expect it to display besides superfluid also antiferromagnetic behavior. The spin waves of an ordinary uniaxial antiferromagnet, which are the

Goldstone modes of the spontaneously broken SO^S(3) symmetry, are known to be described by the O(3) nonlinear sigma model [82]. In 2+1 dimensions, the model allows for a topological term in the action—the so-called Hopf term [81].

It has been suggested by Dzyaloshinski, Polyakov, and Wiegmann [83] that the Hopf term with $\theta = \pi$ should be included in the effective theory describing a spin-$\frac{1}{2}$ Heisenberg antiferromagnet in 2+1 dimensions. But subsequent microscopic calculations showed that the topological term cannot be derived from the Heisenberg model [84]. It will turn out that in ^{3}He-a the situation is different in that here the quantum-induced effective action does accommodate such a topological term.

To derive the effective theory, we use the well-known fact that the unit spin vector $d^\alpha(x)$ in (1.148) can always be rotated in the third direction by introducing a spacetime-dependent 2×2 SU(2)-matrix $s(x)$:

$$d^\alpha(x)\sigma^\alpha = s(x)\sigma^3 s^\dagger(x). \tag{2.1}$$

We next introduce the decomposition

$$\Psi = \begin{pmatrix} \psi \\ g\psi^* \end{pmatrix} = \begin{pmatrix} s\chi \\ sg\chi^* \end{pmatrix} \tag{2.2}$$

in the Lagrangian (1.157). The theory then takes the form of an SU(2) gauge theory:

$$\mathcal{L} = \frac{1}{2} X^\dagger \left(p_0 - B_0 - \xi(p_i - B_i)\tau_3 - \frac{\bar{\Delta}}{2k_{\rm F}} \{(p_i - B_i)\tau_i, \sigma^3\} \right) X, \tag{2.3}$$

where $X = (\chi, g\chi^*)^{\rm T}$, with the superscript T indicating the transpose, and the 2×2 matrix-valued field

$$B_\mu = -is^\dagger \tilde{\partial}_\mu s = B_\mu^\alpha \sigma^\alpha \tag{2.4}$$

plays the role of a gauge field. Remember that $\tilde{\partial}_\mu = (\partial_0, -\nabla)$. Also recall that the Pauli matrices τ_3 and τ_i $(i = 1, 2)$ operate in Nambu space, whereas the Pauli matrices σ^α $(\alpha = 1, 2, 3)$ operate in spin space. After integrating out the fermionic degrees of freedom, we now obtain instead of (1.159) the effective one-loop action

$$S_{\rm eff} = -\frac{i}{2} \,{\rm Tr}\log \left(p_0 - B_0 - \xi(\mathbf{p} - \mathbf{B})\tau_3 - \frac{\bar{\Delta}}{2k_{\rm F}} \{p_i - B_i, \sigma^3\}\tau_i \right). \tag{2.5}$$

Expanded in a Taylor series this becomes, apart from an irrelevant constant,

$$S_{\rm eff} = \frac{i}{2} \,{\rm Tr} \sum_{\ell=1}^{\infty} \frac{1}{\ell} [G_0(p)\Lambda(\mathbf{p})]^\ell, \tag{2.6}$$

where the vertex $\Lambda(\mathbf{p})$ is now given by

$$\Lambda(\mathbf{p}) = B_0 + \frac{1}{2m}[(\mathbf{B})^2 - 2\mathbf{p}\cdot\mathbf{B} - i\nabla\cdot\mathbf{B}]\tau_3 - \frac{\bar{\Delta}}{k_{\rm F}} B_i^3 \tau_i, \tag{2.7}$$

and $G_0(p)$ is the propagator (1.162). We have written $\mathbf{B}\cdot\mathbf{B}$ as $(\mathbf{B})^2$ so as not to confuse it with the second spin component ($\alpha = 2$) of the spatial vector $\mathbf{B}^\alpha$.

The first term $S_{\rm eff}^{(1)}$ in the expansion (2.6) is readily shown to give

$$S_{\rm eff}^{(1)} = -\frac{\bar{n}}{4m}\,{\rm tr}\int_x (\mathbf{B})^2. \tag{2.8}$$

The trace tr stands for the trace over the remaining sigma matrix indices, where one should bear in mind that the B_μ's defined by (2.4) are elements of the SU(2) algebra. For the second term $S_{\rm eff}^{(2)}$ in the expansion (2.6) we obtain to lowest order in derivatives:

$$S_{\rm eff}^{(2)} = \frac{1}{4}\nu(0)\,{\rm tr}\int_x \left\{(B_0)^2 - (\sigma^3 B_0)^2 + \frac{v_{\rm F}^2}{2}\left[(\mathbf{B})^2 + \gamma(\sigma^3\mathbf{B})^2\right]\right\}, \tag{2.9}$$

where γ stands for

$$\gamma = 1 + \left(\frac{\bar{\Delta}}{\mu}\right)^2 \ln\left(\frac{c\Lambda}{\bar{\Delta}}\right), \tag{2.10}$$

with Λ an ultraviolet energy cutoff and c an irrelevant numerical constant. The term proportional to $(\bar{\Delta}/\mu)^2$ stems from the last term in the vertex (2.7). Since $\bar{\Delta} << \mu \approx \epsilon_{\rm F}$ in the weak-coupling limit we are considering, the factor $(\bar{\Delta}/\mu)^2 \ln(c\Lambda/\bar{\Delta})$ is negligible compared to 1, and may be set to zero. Adding the contributions (2.8) and (2.9), and using the identity

$${\rm tr}[(B_\mu)^2 - (\sigma^3 B_\mu)^2] = (\partial_\mu d^\alpha)^2, \tag{2.11}$$

which is easily derived from the definitions (2.1) and (2.4), we conclude that, apart from a possible topological term, the effective theory describing the antiferromagnetic spin waves is the O(3) nonlinear sigma model [49]

$$\mathcal{L}_{\rm eff} = \tfrac{1}{4}\nu(0)\left[(\partial_0 d^\alpha)^2 - \tfrac{1}{2}v_{\rm F}^2(\partial_i d^\alpha)^2\right]. \tag{2.12}$$

It is gratifying to see that the $\mathbf{B}$-terms obtained from the first and second term in the expansion of the effective action conspire to precisely generate the right combination ${\rm tr}[(\mathbf{B})^2 - (\sigma^3\mathbf{B})^2]$.

The O(3) nonlinear sigma model can be equivalently represented by the CP^1 model

$$\mathcal{L}_{\rm eff} = \nu(0)(\partial_\mu z^\dagger \partial_\mu z - B_\mu^3 B_\mu^3), \tag{2.13}$$

adopting units such that the spin-wave velocity is unity $v_{\rm F}/\sqrt{2} = 1$. Here, the complex scalar $z = (z_1, z_2)^{\rm T}$, which is subject to the constraint $z^\dagger z = 1$, is defined by the equation

$$d^\alpha = z^\dagger \sigma^\alpha z, \tag{2.14}$$

and is related to the 2×2 matrix s in the following way

$$s = \begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}. \tag{2.15}$$

In terms of z, the field B_μ^3 reads

$$B_\mu^3 = -iz^\dagger \tilde{\partial}_\mu z. \tag{2.16}$$

Since d^α defines z only up to a phase, the Lagrangian (2.13) has a *local* gauge symmetry:

$$z(x) \to \mathrm{e}^{i\alpha(x)} z(x). \tag{2.17}$$

In the original variables, such a transformation corresponds to a spin rotation about the preferred spin axis. That is, the unbroken component of the global SO^S(3) spin rotation group becomes a *local* gauge symmetry in the effective theory. We will refer to this symmetry as local spin gauge symmetry. The reader is referred to Ref. [85] for a general discussion on this phenomenon.

We next turn to the topological term. When written in terms of the spin gauge field B^3_μ, the Hopf term takes the form of a Chern-Simons term:

$$\theta S_{\mathrm{CS}} = \frac{\theta}{4\pi^2} \int_x \epsilon_{\mu\nu\lambda} B^3_\mu \tilde{\partial}_\nu B^3_\lambda, \tag{2.18}$$

where θ is a real parameter.

The question whether a Chern-Simons term is induced at the quantum level can be addressed in great generality. To the author's knowledge, So [86]; Ishikawa and Matsuyama [87] were the first to reveal the underlying structure, showing that θ is determined by the Feynman propagator $G(k)$ of the underlying fermionic theory

$$\theta = \frac{1}{24\pi} \epsilon_{\mu\nu\lambda} \int_k \mathrm{tr} \left(G \frac{\partial G^{-1}}{\partial k_\mu} G \frac{\partial G^{-1}}{\partial k_\nu} G \frac{\partial G^{-1}}{\partial k_\lambda} \right). \tag{2.19}$$

They also showed that the right-hand side of (2.19) can acquire a topological meaning. This work appears to have been largely ignored in the literature, and the results have later been rederived by others [88, 89].

For ^{3}He-a, where according to Eq. (1.162) the inverse propagator reads

$$G_0^{-1}(k) = k_0 - \xi(\mathbf{k})\tau_3 - \frac{\bar{\Delta}}{k_{\mathrm{F}}} \mathbf{k} \cdot \boldsymbol{\tau} \sigma^3, \tag{2.20}$$

the θ-parameter takes the value [90]

$$\theta = \mathrm{sgn}(\mathbf{e}^1 \times \mathbf{e}^2)\pi. \tag{2.21}$$

Here, sgn denotes the signum, and it should be realized that the cross product of two vectors is a scalar in two space dimensions. The dependence of θ on the orientation of the $\mathbf{e}$-frame is crucial because it changes sign under both a parity transformation and time inversion. Under a parity transformation in 2+1 dimensions, one spatial coordinate is reflected:

$$(t, x_1, x_2) \to (t, -x_1, x_2); \tag{2.22}$$

and an arbitrary vector V_μ transforms as:

$$V_{0,2}(t, x_1, x_2) \to V_{0,2}(t, -x_1, x_2),\ V_1(t, x_1, x_2) \to -V_1(t, -x_1, x_2). \tag{2.23}$$

The sign change in θ offsets the one in the Chern-Simons term $\mathcal{L}_{\mathrm{CS}}$ under these transformations, so that the combination $\theta\mathcal{L}_{\mathrm{CS}}$ is invariant [88, 90].

The appearance of a Chern-Simons term in the effective theory of a nonrelativistic model might seem surprising. However, the two possible terms $\epsilon_{ij}B_0^3\partial_i B_j^3$ and $\epsilon_{ij}B_i^3\partial_0 B_j^3$ one can write down are related by the requirement of invariance under the spin gauge transformation $B_\mu^3 \to B_\mu^3 + \tilde{\partial}_\mu\alpha$, in such a way that the two can be combined into a single Chern-Simons term. The presence of both terms can be checked explicitly by invoking the derivative expansion we used before. The specific form of the interaction, reflected in the last term of the vertex (2.7), as well as of the propagator (1.162), are crucial in obtaining them.

It was pointed out by Wilczek and Zee [91] that the Hopf term imparts spin to the solitons of the (2+1)-dimensional O(3) nonlinear sigma model. These solitons, first discussed by Belavin and Polyakov [92], belong to the class of topological defects that are regular throughout coordinate space. That is, they do not possess a singular core as is often the case. They are characterized by the topological charge [77]

$$Q = \frac{1}{8\pi}\int_{\mathbf{x}} \epsilon_{ij}\epsilon^{\alpha\beta\gamma} d^\alpha \partial_i d^\beta \partial_j d^\gamma, \tag{2.24}$$

where the spin indices α, β, γ run over $1, 2, 3$, and the antisymmetric Levi-Civita symbol is defined such that $\epsilon^{123} = 1$. We repeat that vector spin indices are always represented by superscripts.

The natural language for the description and classification of defects in ordered systems is provided by homotopy theory. (For general accounts on homotopy theory see, for example, [93], [94] or [95].) The way to characterize a general defect of dimension $\epsilon_{\rm d}$ is to surround it by a spherical surface of dimension r such that [96]

$$r = d - \epsilon_{\rm d} - 1, \tag{2.25}$$

with d the dimension of the medium under consideration. The minus one at the right-hand side represents the distance from the defect to the surrounding hypersphere. In each point of this surface, the Goldstone fields define a map of the r-sphere S^r onto the coset space G/H. Topologically stable defects arise when the contour in the coset space is not contractable. They are therefore classified by the homotopy groups $\pi_r(G/H)$.

For the case at hand, the topological charge (2.24) is the winding number of the map

$$d^\alpha(\mathbf{x}) : {\rm S}^2_{\mathbf{x}} \to {\rm S}^2 \tag{2.26}$$

of compactified space ${\rm S}^2_{\mathbf{x}}$ into the internal two-sphere ${\rm S}^2$ parameterized by d^α, where it is to be noted that the second homotopy group $\pi_2({\rm S}^2) = Z$. The corresponding topological current, which is conserved independently of the field equations, is

$$J_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}\epsilon^{\alpha\beta\gamma} d^\alpha \tilde{\partial}_\nu d^\beta \tilde{\partial}_\lambda d^\gamma. \tag{2.27}$$

This current may alternatively be written in terms of the field B_μ^3 as:

$$J_\mu = \frac{1}{2\pi}\tilde{G}_\mu, \tag{2.28}$$

where the dual field $\tilde{G}_\mu$ is defined by

$$\tilde{G}_\mu = \epsilon_{\mu\nu\lambda}\tilde{\partial}_\nu B^3_\lambda. \tag{2.29}$$

The simplest soliton has $Q = 1$ and is called a skyrmion. Due to the Chern-Simons term, skyrmions acquire fractional spin $\theta/2\pi$ and statistics θ [91, 97]. The values $\theta = \pm\pi$ found in superfluid ^{3}He-a imply that here the skyrmions have spin $\frac{1}{2}$ and are fermions. The fractional statistics (for an introductory review, see Ref. [98]) imparted by the topological term to the skyrmion can be easily understood by rewriting the Chern-Simons term (2.18) as

$$\theta\mathcal{L}_{\rm CS} = \frac{\theta}{2\pi} J_\mu B^3_\mu, \tag{2.30}$$

where J_μ is the topological current (2.27). This shows that the effect of the Chern-Simons term is to couple the topological current to the spin gauge field B^3_μ, with a charge $\theta/2\pi$. In other words, besides carrying flux, skyrmions also carry charge. Now, if a skyrmion winds around another skyrmion, its wavefunction will acquire a phase $\exp(i\theta)$ through the Aharonov-Bohm effect, thus turning skyrmions into anyons with statistics parameter θ.

A skyrmion configuration is given by

$$d^\alpha(\mathbf{x}) = \begin{pmatrix} \cos\phi\, \sin u(r) \\ \sin\phi\, \sin u(r) \\ \cos u(r) \end{pmatrix}, \tag{2.31}$$

where (r, ϕ) are circle coordinates in the spatial plane, while $u(r)$ is a function with $u(r) = 0$ at the origin of the skyrmion ($r = 0$) and $u(r) \to \pi$ for $r \to \infty$. The corresponding spin gauge-field configuration B^3_μ, which can be found by solving (2.1) for s, reads

$$B^3_\mu = \tilde{\partial}_\mu\alpha + \tfrac{1}{2}(1 - \cos u)\tilde{\partial}_\mu\phi, \tag{2.32}$$

where α is an arbitrary gauge parameter originating from the circumstance that s is determined by d^α only up to a phase factor $\exp(i\alpha\sigma^3)$. We fix the gauge by demanding that the "magnetic" flux

$$\Phi = \int_{\mathbf{x}} \tilde{G}_0 \tag{2.33}$$

is regular everywhere. Since $\epsilon_{ij}\partial_i\partial_j\phi$ is singular at the origin, we choose $\alpha = 0$. This leads to

$$\tilde{G}_0 = \frac{1}{2r}\frac{\mathrm{d}u}{\mathrm{d}r}\sin u. \tag{2.34}$$

The corresponding magnetic flux $\Phi(R)$ through a disk of radius R is

$$\Phi(R) = \pi[1 - \cos u(R)]. \tag{2.35}$$

With $u(\infty) = \pi$, it follows that the flux piercing the plane is 2π. That is, each skyrmion carries one unit of flux. As pointed out by Huang, Koike, and Polonyi [99], these conclusions may be nicely visualized by placing a Dirac monopole with unit magnetic

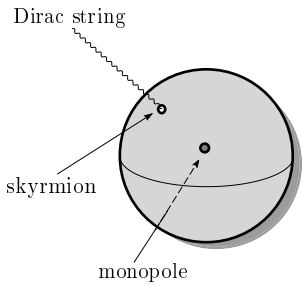


Figure 2.1: Compactified space represented by a two-sphere with a monopole at its origin. The Dirac string pierces the plane at the center of the skyrmion.

charge outside of the system at the origin of a two-sphere representing the compactified spatial plane, and letting the Dirac string pierce the plane at infinity. Alternatively, one may extract a unit flux tube so that the Dirac string pierces the plane at the origin where the center of the skyrmion resides (see Fig. 2.1).

Let us mention that we are working in the ordered phase of the O(3) nonlinear sigma model. The (2+1)-dimensional model $\mathcal{L} = (\partial_\mu d^\alpha)^2/2g^2$, with g the coupling constant, has a phase transition into a disordered, strong-coupling phase at [100]

$$g_{\rm cr}^{-2} = \frac{3}{2\pi^2}\Lambda, \tag{2.36}$$

where Λ is an ultraviolet energy cutoff. Since in our case, according to (2.12), $g^{-2} = \nu(0)/2 = m/4\pi$, with $m \sim \mathrm{GeV} >> \Lambda$, we are indeed in the ordered, weak-coupling phase. The two phases differ in the way the so-called flux symmetry generated by the topological current (2.28) is realized [101]. In the weak-coupling phase the flux symmetry is unbroken, while in the strong-coupling phase it is realized in the so-called Kosterlitz-Thouless mode. In the latter phase, there is—at least in perturbation theory—algebraic long-range order due to the presence of a gapless Kosterlitz-Thouless boson. It has been pointed out by Huang, Koike, and Polonyi [99], that a condensation of the Dirac monopoles would invalidate this picture. If this happens, they argued, the flux symmetry becomes anomalous and the Kosterlitz-Thouless boson acquires an energy gap so that the algebraic long-range order of the ordered, strong-coupling phase is lost. These authors found numerical evidence supporting this scenario, but Bitar and Manousakis [102] criticized their numerical analysis and concluded that the monopoles do not spoil the perturbative picture.

We next study the connection between the microscopic fermionic model and the effective theory, i.e., the nonlinear sigma model with a Chern-Simons term added. In particular, we will be interested in how the spin currents j_μ^α of the original model

$$\begin{aligned} j_0^\alpha &= \psi^\dagger \tfrac{1}{2}\sigma^\alpha \psi \\ j_i^\alpha &= \frac{1}{2m}\psi^\dagger \tfrac{1}{2}\sigma^\alpha (-i \overleftrightarrow{\partial_i})\psi, \end{aligned} \tag{2.37}$$

with j_0^α the spin density, are represented in the effective model. In (2.37), $\overleftrightarrow{\partial}_i = \partial_i - \overleftarrow{\partial}_i$, is the gradient operator acting to the right and left. Since the global SO^S(3) spin rotation group is spontaneously broken to SO^S(2) in ^{3}He-a, the two components j_μ^a, with $a = 1, 2$ are supercurrents, meaning that the corresponding charges are transported by the spin waves without dissipation. We shall demonstrate that the induced spin currents are represented by the isospin currents I_μ^α of the nonlinear sigma model,

$$I_\mu^\alpha = \epsilon^{\alpha\beta\gamma} d^\beta \frac{\partial \mathcal{L}}{\partial(\partial_\mu d^\gamma)} = \frac{1}{2}\nu(0)\epsilon^{\alpha\beta\gamma} d^\beta \tilde{\partial}_\mu d^\gamma; \tag{2.38}$$

and that due to the presence of the Chern-Simons term, the spin currents acquire an additional contribution.

Let us rewrite the spin currents (2.37) in terms of χ and the fields $B_\mu = -is^\dagger \tilde{\partial}_\mu s$, where ψ and χ are related via (2.2). We find

$$\begin{aligned} j_0^\alpha &= R^{\alpha\beta}\chi^\dagger \tfrac{1}{2}\sigma^\beta \chi \\ j_i^\alpha &= R^{\alpha\beta}\frac{1}{2m}\left[\chi^\dagger \tfrac{1}{2}\sigma^\beta(-i\overleftrightarrow{\partial}_i)\chi - \chi^\dagger B_i^\beta \chi\right], \end{aligned} \tag{2.39}$$

where $R^{\alpha\beta}$ is an orthogonal rotation matrix defined via

$$s^\dagger \sigma^\alpha s = R^{\alpha\beta}\sigma^\beta. \tag{2.40}$$

If we differentiate the Lagrangian (2.3) with respect to the gauge fields, we obtain

$$\begin{aligned} -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial B_0^\alpha} &= \chi^\dagger \tfrac{1}{2}\sigma^\alpha \chi \\ \frac{1}{2}\frac{\partial \mathcal{L}}{\partial B_i^\alpha} &= \frac{1}{2m}\left[\chi^\dagger \tfrac{1}{2}\sigma^\alpha(-i\overleftrightarrow{\partial}_i)\chi - \chi^\dagger B_i^\alpha \chi\right], \end{aligned} \tag{2.41}$$

where in the last equation with $\alpha = 3$ we neglected the contribution arising from the interaction term. This is justified because $\bar{\Delta}/k_{\rm F} << 1$. It follows from (2.39) and (2.41) that

$$j_0^\alpha = -\frac{1}{2}R^{\alpha\beta}\frac{\partial \mathcal{L}}{\partial B_0^\beta}, \quad j_i^\alpha = \frac{1}{2}R^{\alpha\beta}\frac{\partial \mathcal{L}}{\partial B_i^\beta} \tag{2.42}$$

If we now wish to calculate j_μ^α in the effective theory, we have to replace $\mathcal{L}$ in this equation with the effective Lagrangian. Omitting the Chern-Simons term for the moment, we obtain in this way

$$j_\mu^\alpha = -\nu(0)R^{\alpha b}B_\mu^b, \quad (b = 1, 2). \tag{2.43}$$

We proceed to show that the right-hand side of this equation is the isospin current (2.38). To this end we write:

$$\begin{aligned} I_\mu^\alpha &= \tfrac{1}{4}i\nu(0)\tilde{\partial}_\mu d^\beta d^\gamma \,\mathrm{tr}\,\sigma^\alpha\sigma^\beta\sigma^\gamma \\ &= \tfrac{1}{4}i\nu(0)\mathrm{tr}\, s^\dagger\sigma^\alpha\tilde{\partial}_\mu s + \tfrac{1}{4}\nu(0)\mathrm{tr}\, s^\dagger\sigma^\alpha s\sigma^3 B_\mu\sigma^3, \end{aligned} \tag{2.44}$$

where we used (2.1) and the definition of B_μ. Then, employing (2.40) and performing the trace, we see that the induced spin currents are indeed given by the isospin currents of the nonlinear sigma model [103]

$$j^\alpha_\mu = I^\alpha_\mu. \tag{2.45}$$

We next turn our attention to the anomalous contribution to the spin currents arising from the Chern-Simons term:

$$j^\alpha_{\theta,\mu} = -\frac{\theta}{4\pi^2} R^{\alpha 3}\, \tilde{G}_\mu. \tag{2.46}$$

In particular, we are interested in the spin carried by a skyrmion. The zero-component of the dual field describing a skyrmion configuration is given by Eq. (2.34). To find the spin, we calculate the projection of the spin density $j^\alpha_{\theta,0}$ onto the spin-quantization axis. Usually this axis is fixed in spacetime, but for a skyrmion configuration the spin-quantization axis $d^\alpha(\mathbf{x})$ varies in space. The spin σ of a skyrmion is therefore given by

$$\sigma = \int_{\mathbf{x}} d^\alpha j^\alpha_{\theta,0}. \tag{2.47}$$

This quantity is most easily evaluated when use is made of the identity

$$d^\alpha R^{\alpha\beta} = \delta^{3\beta}, \tag{2.48}$$

which is readily derived by multiplying the definition of the R matrix, Eq. (2.40), with d^α and using (2.1). We find in this way that the spin of a skyrmion is related to its topological charge (2.28) via

$$\sigma = \frac{\theta}{2\pi} Q. \tag{2.49}$$

More precisely, a skyrmion with topological charge $Q = 1$ has a spin $\theta/2\pi$ in accord with its statistics.

This result can also be derived by directly employing (2.42) with (2.48):

$$d^\alpha j^\alpha_0 = -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial B^3_0}, \tag{2.50}$$

and replacing $\mathcal{L}$ with the Chern-Simons term.

Equation (2.49) is the anomalous contribution to the total spin of the system. Hence, the difference in the number $N_{\uparrow,\downarrow}$ of spin-$\uparrow$ and spin-$\downarrow$ ^{3}He atoms is given by the skyrmion number [103]

$$N_\uparrow - N_\downarrow = \pm Q. \tag{2.51}$$

It follows that a finite number of skyrmions can be created by taking $N_\uparrow \neq N_\downarrow$. If the difference $|N_\uparrow - N_\downarrow|$ is large enough the skyrmions, which are fermions due to the Chern-Simons term, may form a hexagonal lattice (see Fig. 2.2) analogous to the one formed by so-called Anderson-Toulouse-Chechetkin vortices [104] in rotating superfluid ^{3}He-A [105]. Experimentally, however, such a skyrmion lattice may be difficult to realize because an external magnetic field necessary to spin polarize the system locks the spin-quantization axis d^α to a plane normal to the field.

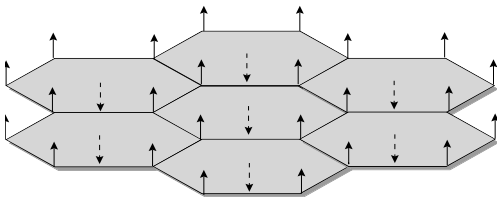


Figure 2.2: Hexagonal spin-$\frac{1}{2}$ skyrmion lattice. The arrows denote the direction of the spin vector $d^\alpha(\mathbf{x})$.

2.2 Peierls Instability

In this section we shall discuss a model describing quasi one-dimensional metals. The effective low-energy, small-momentum continuum theory is shown to have two phases both of which possess a rich soliton structure. The two phases differ in a topological term which is induced in one phase, but not in the other. This so-called θ-term is shown to change the statistics of certain solitons.

One-dimensional metals are frequently discussed in terms of the Hubbard model defined by the lattice Hamiltonian [82]

$$H_{\rm H} = -t\sum_j (c^\dagger_{j\sigma}c_{j+1,\sigma} + \text{h.c.}) + U\sum_j n_{j\uparrow}n_{j\downarrow}. \tag{2.52}$$

It contains two parameters, U and t, representing the Coulomb interaction between a pair of valence electrons on the same atom, and the matrix element for the hopping of an electron to a neighboring atom, respectively [106]. When U is chosen positive, the Coulomb interaction is repulsive. In (2.52), the operator $c^\dagger_{j\sigma}$ creates an electron of spin $\sigma(=\uparrow,\downarrow)$ at the location of the jth atom, and $n_{j\sigma} = c^\dagger_{j\sigma}c_{j\sigma}$ is the electron-number operator at site j. The sum $\sum_j$ is over all lattice sites. It is assumed that the Coulomb interaction is highly screened, with a screening length of the order of a lattice spacing a, so that we can restrict ourselves to the on-site Coulomb repulsion. In addition, it is assumed that the electrons are well localized, so that the tight-binding approximation is valid.

The Hubbard Hamiltonian (2.52) may be rewritten in the equivalent form

$$H_{\rm H} = -t\sum_j (c^\dagger_{j\sigma}c_{j+1,\sigma} + \text{h.c.}) - \frac{2}{3}U\sum_j \left(S^\alpha_j\right)^2, \tag{2.53}$$

where we dropped an irrelevant constant. Here, $S^\alpha_j = \frac{1}{2}c^\dagger_{j\sigma}\sigma^\alpha_{\sigma\tau}c_{j\tau}$, with σ^α $(\alpha = 1,2,3)$ the Pauli matrices, is the electron spin operator at site j. The representation (2.53) which explicitly exhibits the spin operator is most convenient to derive the effective theory.

In the large-U limit, the Hubbard model at half-filling, where the number of electrons equals the number of sites, can be mapped onto the spin-$\frac{1}{2}$ Heisenberg model

[82]

$$H = J \sum_j S_j^\alpha S_{j+1}^\alpha, \tag{2.54}$$

with coupling constant $J = 4t^2/U > 0$. Because J is positive, spins on neighboring sites favor antiferromagnetic coupling. This model, solved by Bethe, is known to be gapless [107]. In the opposite limit of weak coupling $U << t$, the Coulomb repulsion is a small perturbation and we expect the model to behave qualitatively the same as a free-electron gas. These two arguments taken together make it plausible that the one-dimensional Hubbard model at half-filling is gapless for all values of the coupling constant, implying that it exhibits metallic behavior.

Although the Hubbard chain is meant to describe a one-dimensional metal, it completely ignores the electron-phonon interaction. This coupling, particularly in one dimension, can however be very important. A famous argument due to Peierls [108] shows that a one-dimensional metal with a partially filled conduction band is unstable towards a periodic distortion of the linear lattice which opens an energy gap at the Fermi points. Because of this, Peierls concluded that such a one-dimensional system at the absolute zero of temperature would probably never have metallic properties and would instead be an insulator. Fröhlich [109] showed that a periodic lattice distortion is accompanied by a modulation of the electron number density with the same periodicity—a so-called charge density wave [110].

Being a fundamental property of one-dimensional metals, we wish to extend the Hubbard model so as to account for the Peierls instability. At half-filling, the periodic distortion has a period of twice the lattice spacing. The modulation of the electron-number density resulting from this instability is given by

$$H_{\rm P} = \Delta \sum_{j\sigma} (-1)^j n_{j\sigma}, \tag{2.55}$$

where Δ is the so-called Peierls energy gap, it is typical of the order of 100 K. By adding this term to the Hamiltonian (2.53) we account for a crucial aspect of the one-dimensional electron-phonon interaction in the Hubbard model.

We examine the model in the limit of low energy and small momentum by going to the continuum. This is facilitated by introducing the operators $a_{\rm e},\ a_{\rm o}$ with engineering dimension $\frac{1}{2}$ via

$$\begin{aligned} c_{2j} &\rightarrow i^{2j}\sqrt{2a}\, a_{\rm e}(2j) \\ c_{2j+1} &\rightarrow i^{2j+1}\sqrt{2a}\, a_{\rm o}(2j+1), \end{aligned} \tag{2.56}$$

for even and odd sites, respectively. Here, as well as in the following, we suppress spin indices. Summation over these hidden indices is always implied. The powers of i in (2.56) represent the factor $\exp(ik_{\rm F}x_1)$, with $x_1 = 2ja$ for even sites and $x_1 = (2j+1)a$ for odd sites, respectively, where $k_{\rm F} = \pi/2a$ is the Fermi momentum at half-filling. In terms of the new operators, the hopping term of the Hubbard model becomes

$$-t\sum_j (c_j^\dagger c_{j+1} + \text{h.c.}) \rightarrow -2ita \sum_j \Big\{ \left[a_{\rm e}^\dagger(2j) - a_{\rm e}^\dagger(2j+2)\right] a_{\rm o}(2j+1)$$

$$+a_{\rm o}^\dagger(2j+1)\left[a_{\rm e}(2j+2)-a_{\rm e}(2j)\right]\Big\}. \tag{2.57}$$

With the derivative of the operator $a_{\rm o}$ defined by

$$\partial_1 a_{\rm o}=\lim_{a\to 0}\frac{a_{\rm o}(2j+1)-a_{\rm o}(2j-1)}{2a}, \tag{2.58}$$

and a similar definition for $\partial_1 a_{\rm e}$, we obtain for this term in the continuum limit

$$-t\sum_j(c_j^\dagger c_{j+1}+{\rm h.c.})\to 2ta\int_{x_1}\psi^\dagger\alpha(-i\partial_1)\psi. \tag{2.59}$$

Here, we introduced the multiplet ψ containing the even-site and odd-site species of fermions,

$$\psi=\begin{pmatrix} a_{\rm e}\\ a_{\rm o}\end{pmatrix}, \tag{2.60}$$

with $\alpha=\tau_1$ the first Pauli matrix which should not be confused with the spin matrices σ^α. In deriving (2.59) we replaced the summation over the lattice sites with an integral: $2a\sum_j\to\int {\rm d}x_1$. The right-hand side of (2.59) is of the form of a massless Dirac Hamiltonian. It describes Bloch electrons with a gapless dispersion relation that is linear in the crystal momentum k_1

$$E(k_1)=v_{\rm F}|k_1|, \tag{2.61}$$

where $v_{\rm F}=2ta$ is the Fermi velocity. A one-dimensional model described by this dispersion relation would exhibit metallic properties. As expected, the term (2.55) representing the Peierls instability drastically changes this behavior. In the continuum limit, this term becomes a mass term in the Dirac theory

$$\begin{aligned}\Delta\sum_j(-1)^j n_j\ &\to 2a\Delta\sum_j\left[a_{\rm e}^\dagger(2j)a_{\rm e}(2j)-a_{\rm o}^\dagger(2j+1)a_{\rm o}(2j+1)\right]\\ &\to\Delta\int_{x_1}\psi^\dagger\beta\psi,\end{aligned} \tag{2.62}$$

with $\beta=\tau_3$ being the diagonal Pauli matrix. As a result, the gapless spectrum (2.61) changes into a spectrum with an energy gap

$$E^2(k_1)=v_{\rm F}^2k_1^2+\Delta^2, \tag{2.63}$$

and the metallic properties of the free model are lost.

We treat the Coulomb interaction of the Hubbard model (2.53) by replacing it with

$$-\frac{2}{3}U\sum_j\left(S_j^\alpha\right)^2\to\frac{4}{3}U\sum_j M_j\left[\frac{M_j}{2}\left(d_j^\alpha\right)^2-(-1)^j S_j^\alpha d_j^\alpha\right], \tag{2.64}$$

where the operator field d_j^α satisfies the constraint equation

$$M_j d_j^\alpha=(-1)^j S_j^\alpha \tag{2.65}$$

and $\left(d_j^\alpha\right)^2 = 1$ at every site j. In the low-energy, small-momentum limit, the modulus M_j can be taken as a constant $\bar{M}$. The factor $(-1)^j$, alternating sign from site to site, is included in (2.64) because we are interested in the antiferromagnetic properties of the model. In the mean-field approximation, the operator d_j^α is considered to be a classical field. It then describes the staggered magnetization and thus becomes the order parameter of the Néel state. In this approximation, the continuum limit of the relevant term of the Coulomb interaction becomes

$$-\frac{4}{3}\bar{M}U\sum_j(-1)^j S_j^\alpha d_j^\alpha \to \Sigma\int_{x_1}\psi^\dagger\beta d^\alpha\sigma^\alpha\psi, \tag{2.66}$$

where we introduced the effective coupling constant $\Sigma = -\frac{2}{3}\bar{M}U$. The term (2.66) together with the two other terms (2.59) and (2.62) describe the Néel state of the extended Hubbard model in the mean-field approximation. The corresponding Lagrangian reads

$$\mathcal{L} = \bar{\psi}(i\tilde{\partial\!\!\!/} - \Delta - \Sigma\, d^\alpha\sigma^\alpha)\psi, \qquad \bar{\psi} = \psi^\dagger\beta, \tag{2.67}$$

where the Fermi velocity $v_{\rm F}$ is set to unity, and

$$\gamma_0 = \beta = \tau_3, \quad \gamma_1 = \beta\alpha = i\tau_2 \tag{2.68}$$

are the Dirac matrices. Moreover, $\tilde{\partial\!\!\!/} = \tilde{\partial}_\mu\gamma_\mu$. Although mean-field theory should in general not be trusted in lower dimensions, it in this case seems to capture the essential physics.

The effective theory we are seeking is obtained by integrating out the electron fields. (We tacitly switched here from the operator formalism to the functional-integral approach.) In our discussion, Δ is considered to be a constant. If one is interested in the dynamics of the charge density wave, which we are not, Δ is assumed to have a spacetime-dependent phase, $\Delta(x) = \bar{\Delta}\exp[i\varphi(x)]$.

Since the Lagrangian (2.67) is quadratic in the electron fields, these are readily integrated out. The ensuing effective action $S_{\rm eff}$ reads

$$S_{\rm eff} = -i{\rm Tr}\,\ln(\partial\!\!\!/ - \Delta - \Sigma\, d^\alpha\sigma^\alpha), \tag{2.69}$$

which we again evaluate in a derivative expansion [26]. On our way, we encounter integrals which diverge in the ultraviolet. To handle these, we use dimensional regularization and generalize the integrals to arbitrary spacetime dimensions D.

The lowest-order term is of the form $r(d^\alpha)^2$, where r is a constant that tends to infinity when the dimensional-regularization parameter $\epsilon = 1 - \frac{1}{2}D$ is taken to zero. This term merely renormalizes the first term at the right-hand side of (2.64). Neglecting for the moment a possible topological term, we obtain as next order in derivatives the contribution [111]

$$\mathcal{L}_{\rm eff} = \frac{1}{2g^2}\left(\partial_\mu d^\alpha\right)^2, \tag{2.70}$$

where $1/g^2 = \Sigma^2/6\pi\Delta^2$. This is a kinetic term for the staggered magnetization induced by quantum effects. Note that $1/g^2 > 0$, which is required for stability. The O(3) nonlinear sigma model (2.70) in 1+1 spacetime dimensions is known to have only

one phase, viz. the so-called quantum disordered phase, with a finite correlation length [112]. Due to strong infrared interactions of the "Goldstone modes", they acquire an energy gap for all values of the coupling constant g. For this reason one frequently refers to the (1+1)-dimensional state described by the effective theory (2.70), with the constraint $(d^\alpha)^2 = 1$, as a *short-ranged* Néel state. In higher spacetime dimensions the model has a phase transition between an ordered, weak-coupling phase $(g < g_{\rm cr})$ and a disordered, strong-coupling $(g > g_{\rm cr})$ phase, where $g_{\rm c}$ is the critical coupling.

The form (2.69) of the action is not convenient to derive the topological term. To obtain this term we instead follow Jaroszewics [113] and introduce, as we have done in the previous section, the decomposition $\psi = s\chi$, with $s(x)$ the spacetime-dependent SU(2) spin rotation matrix defined in (2.1). The effect of this unitary transformation is to rotate the staggered magnetization in every point of spacetime into a fixed direction. With the help of this decomposition, the Lagrangian (2.67) can be cast in the equivalent form

$$\mathcal{L} = \bar{\chi}(i\tilde{\not\partial} - \Delta - \Sigma\sigma^3 - \not{B})\chi, \tag{2.71}$$

where $B_\mu = -is^\dagger\tilde{\partial}_\mu s = B_\mu^\alpha\sigma^\alpha$ is the 2×2 matrix-valued field we encountered before in the context of superfluid ^{3}He-a. As was the case there, this Lagrangian is invariant under the *local* spin gauge transformation

$$\chi(x) \to \mathrm{e}^{-i\alpha(x)\sigma^3}\chi(x), \quad B_\mu^3(x) \to B_\mu^3(x) + \tilde{\partial}_\mu\alpha(x), \tag{2.72}$$

with spin gauge field B_μ^3. This local gauge freedom derives from the circumstance that the transformed rotation matrix $s' = s\exp(i\alpha\sigma^3)$ also satisfies (2.1), i.e., $s'^\dagger\sigma^\alpha d^\alpha s' = \sigma^3$. That is to say, d^α determines the spin rotation matrix s only up to a multiplicative U(1) factor $\exp(i\alpha\sigma^3)$. In terms of B_μ, the kinetic term $(\partial_\mu d^\alpha)^2$ is given by

$$(\partial_\mu d^\alpha)^2 = 4\left(B_\mu^a\right)^2, \tag{2.73}$$

where the summation over the spin index only involves the components $a = 1, 2$. The kinetic term is thus seen to be independent of the spin gauge field B_μ^3. The term $\left(B_\mu^3\right)^2$, which would give the gauge field a mass, is not generated as it would violate local spin gauge symmetry. On the other hand, the topological term that in principle can arise in this context depends only on B_μ^3

$$\mathcal{L}_\theta = \frac{\theta}{2\pi}\epsilon_{\mu\nu}\tilde{\partial}_\mu B_\nu^3. \tag{2.74}$$

It contains, as is frequently the case with topological terms, the antisymmetric Levi-Civita symbol and it is linear in derivatives. It can alternatively be written in terms of the O(3) nonlinear sigma field as [82]:

$$\mathcal{L}_\theta = \frac{\theta}{8\pi}\epsilon_{\mu\nu}\epsilon^{\alpha\beta\gamma}d^\alpha\tilde{\partial}_\mu d^\beta\tilde{\partial}_\nu d^\gamma, \tag{2.75}$$

where we recognize the winding number of the map

$$d^\alpha(x) : \mathrm{S}_x^2 \to \mathrm{S}^2 \tag{2.76}$$

of compactified spacetime S^2_x into the internal two-sphere S^2 parameterized by d^α, cf. (2.26).

Since it is independent of B^1_μ and B^2_μ, these components may, as far as deriving the topological term is concerned, be neglected in the Lagrangian (2.71). This leads us to consider the following Lagrangian

$$\begin{aligned}\mathcal{L} &= \bar{\chi}_{\mathrm{e}}\left[i\tilde{\partial}\!\!\!/ - (\Delta+\Sigma) - B\!\!\!/^3\right]\chi_{\mathrm{e}} \\ &\quad + \bar{\chi}_{\mathrm{o}}\left[i\tilde{\partial}\!\!\!/ - (\Delta-\Sigma) + B\!\!\!/^3\right]\chi_{\mathrm{o}},\end{aligned} \tag{2.77}$$

where we made explicit the two fermion species χ_{e} and χ_{o}, which are seen to have different masses and to couple to the local spin gauge field in an opposite way. This is because the Néel state is characterized by a staggered magnetization, flipping sign when we go from an even to an odd site.

To investigate whether the topological term (2.74) is induced by the fermions, we could proceed as before and apply the derivative expansion. But in 1+1 dimensions we have another tool available, namely that of bosonization, which we shall use instead. Since (2.77) contains only a U(1) symmetry, we can employ Abelian bosonization. One could also bosonize the original theory (2.67), as was first done by Wen and Zee [81], but this involves the more intricate non-Abelian bosonization. The Abelian bosonization rules—on which we shall comment in the next chapter—read

$$\begin{aligned}\bar{\chi}_{\mathrm{e}} i\tilde{\partial}\!\!\!/\chi_{\mathrm{e}} &\rightarrow \tfrac{1}{2}(\partial_\mu\phi_{\mathrm{e}})^2 \\ \bar{\chi}_{\mathrm{e}}\chi_{\mathrm{e}} &\rightarrow K\cos(\sqrt{4\pi}\phi_{\mathrm{e}}) \\ \bar{\chi}_{\mathrm{e}}\gamma_\mu\chi_{\mathrm{e}} &\rightarrow \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\tilde{\partial}_\nu\phi_{\mathrm{e}},\end{aligned} \tag{2.78}$$

with K a for our purposes irrelevant positive constant and ϕ_{e} a real Bose field. With these and similar rules for χ_{o}, the Lagrangian (2.77) can be represented in bosonized form as

$$\mathcal{L}_{\mathrm{bos}} = \frac{1}{2}(\partial_\mu\phi_+)^2 + \frac{1}{2}(\partial_\mu\phi_-)^2 - \sqrt{\frac{2}{\pi}}\epsilon_{\mu\nu}\phi_-\,\tilde{\partial}_\mu B^3_\nu - \mathcal{V}, \tag{2.79}$$

where we introduced the fields

$$\phi_\pm = (\phi_{\mathrm{e}} \pm \phi_{\mathrm{o}})/\sqrt{2}, \tag{2.80}$$

and where the potential $\mathcal{V}$ is given by

$$\mathcal{V} = 2\Delta K\cos(\sqrt{2\pi}\phi_+)\,\cos(\sqrt{2\pi}\phi_-) - 2\Sigma K\sin(\sqrt{2\pi}\phi_+)\,\sin(\sqrt{2\pi}\phi_-). \tag{2.81}$$

Note that in (2.79) only the field ϕ_- couples to the local spin gauge field. The potential $\mathcal{V}$ has a minimum

$$\mathcal{V}_{\min} = \begin{cases} -2|\Delta|K & \text{for} \quad |\Delta| > |\Sigma| \\ -2|\Sigma|K & \text{for} \quad |\Delta| < |\Sigma|. \end{cases} \tag{2.82}$$

at

$$\phi_\pm = 0,\ |\phi_\mp| = \sqrt{\frac{\pi}{2}} \quad \text{for} \ \ \Delta > \Sigma > 0; \tag{2.83}$$

$$\phi_+ = \phi_- = \pm\frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{for } 0 < \Delta < \Sigma, \tag{2.84}$$

and similar values for Δ and Σ negative. With these constant ϕ-values, the third term in the Lagrangian (2.79) takes the form of the topological term (2.74). We find that the θ-parameter, which is an angle, is given by [111]

$$\theta = \pi[\mathrm{sgn}(\Delta + \Sigma) - \mathrm{sgn}(\Delta - \Sigma)], \tag{2.85}$$

valid for arbitrary sign of Δ and Σ.

More specific, $\theta = 0 \,(\mathrm{mod}\, 2\pi)$ when $|\Delta| > |\Sigma|$, and $\theta = \pi \,(\mathrm{mod}\, 2\pi)$ when $|\Delta| < |\Sigma|$. That is, for $|\Delta| > |\Sigma|$ the effective theory is simply the O(3) nonlinear sigma model (2.70), with coupling constant squared $g^2 > 6\pi$. As we remarked before, the state it describes is the short-ranged Néel state with finite correlation length.

For $|\Delta| < |\Sigma|$, on the other hand, (2.70) is to be augmented by the topological term (2.74). In terms of the field d^α, the effective theory now reads

$$\mathcal{L}_{\mathrm{eff}} = \frac{1}{2g^2}\left(\partial_\mu d^\alpha\right)^2 + \frac{\theta}{8\pi}\epsilon_{\mu\nu}\epsilon^{\alpha\beta\gamma} d^\alpha \tilde{\partial}_\mu d^\beta \tilde{\partial}_\nu d^\gamma, \tag{2.86}$$

with $\theta = \pm\pi$. The θ-term changes the physical content of the O(3) nonlinear sigma model in a dramatic way. This term has been extensively studied in the context of the antiferromagnetic Heisenberg spin chain which can also be mapped onto the nonlinear sigma model [114]. Whereas the half-integer spin chain is known to be gapless, the integer spin chain has a gap in the excitation spectrum—the so-called Haldane gap. The two cases differ in the topological term which is induced with a coefficient $\theta = 2\pi\sigma \,(\mathrm{mod}\, 2\pi)$, where σ is the spin. For integer spins the coefficient is zero and the excitation spectrum has an energy gap. For half-integer spins, on the other hand, the coefficient is nonzero and it can be shown that as a consequence the spectrum becomes gapless [115]. Using these results, we obtain the phase diagram depicted in Fig. 2.3. The standard Hubbard chain is recovered by setting $\Delta = 0$. It is therefore effectively described by (2.70), implying that the model has a gapless spectrum and thus metallic properties for all values of the Coulomb interaction U. The disordered phase of the effective theory, describing the insulating state of the extended Hubbard model, becomes accessible when the Peierls energy gap Δ is taken sufficiently large compared to Σ, measuring the strength of the Coulomb interaction.

The fermions of the original theory appear in the bosonized theory as solitons to which we next turn. An example of a soliton in the regime $0 < \Delta < \Sigma$, where the groundstates are given by (2.84), is the configuration $\phi_+(x_1) = \phi_-(x_1) = f(x_1)$, where the function $f(x_1)$ interpolates between the two different ground states

$$f(-\infty) = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \quad f(\infty) = -\frac{1}{2}\sqrt{\frac{\pi}{2}}. \tag{2.87}$$

This soliton carries one unit of fermion-number charge N. This follows because according to the bosonization rules (2.78), the fermion-number current j_μ is represented as

$$j_\mu = \sqrt{\frac{2}{\pi}}\epsilon_{\mu\nu}\tilde{\partial}_\nu\phi_+, \tag{2.88}$$

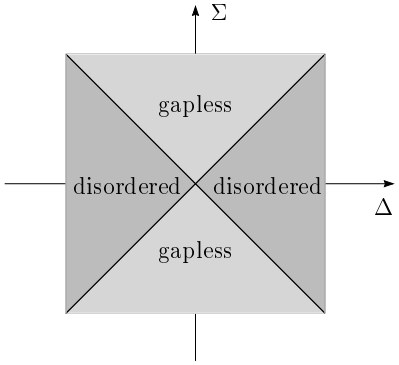


Figure 2.3: Phase diagram of the extended Hubbard model (2.67), where Σ represents the on-site Coulomb interaction and Δ is the Peierls energy gap.

implying that

$$N = \int_{x_1} j_0 = -\sqrt{\frac{2}{\pi}}\left[\phi_+(\infty) - \phi_+(-\infty)\right] = 1. \tag{2.89}$$

Similar solitons of unit fermion-number charge can be constructed in the other sectors of the phase diagram.

It can be easily demonstrated that these solitons carry in addition a spin $\frac{1}{2}$, so that they have the same quantum numbers as the original fermions [81]. Using arguments similar to those of the previous section, one can show that the spin current, given by

$$j_\mu^\alpha = \bar{\psi}\tfrac{1}{2}\sigma^\alpha\gamma_\mu\psi, \tag{2.90}$$

is again represented in the effective theory by the isospin current I_μ^α of the O(3) non-linear sigma model:

$$I_\mu^\alpha = \epsilon^{\alpha\beta\gamma} d^\beta \frac{\partial\mathcal{L}}{\partial(\partial_\mu d^\gamma)}. \tag{2.91}$$

Also Eq. (2.42) is valid, so that

$$d^\alpha j_0^\alpha = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial B_0^3} = -\frac{1}{2}\sqrt{\frac{2}{\pi}}\partial_1\phi_-. \tag{2.92}$$

This gives as spin for the soliton under consideration

$$\sigma = \int_{x_1} d^\alpha j_0^\alpha = \pm\frac{1}{2}. \tag{2.93}$$

It may therefore be thought of as representing the fundamental fermion of the original theory. Similar conclusions hold for the solitons in the other sectors of the phase diagram.

Up to this point we have only considered the part of the bosonized theory involving the scalar fields $\phi_\pm$. We now turn to the nonlinear sigma field d^α and will argue that twists in this field may give certain solitons an anomalous spin. For definiteness we shall assume in the following that the space dimension is compactified into a circle $-L \leq x_1 \leq L$, with $x_1 = -L$ and $x_1 = L$ labeling the same point. The model (2.67) admits two possible boundary conditions for d^α, namely periodic $d^\alpha(-L) = d^\alpha(L)$ and antiperiodic $d^\alpha(-L) = -d^\alpha(L)$ ones [81]. That is, the two phases split up into an even and odd sector.

We are interested in the contribution $j^\alpha_{\theta,\mu}$ to the spin current stemming from the topological term $\mathcal{L}_\theta$:

$$j^\alpha_{\theta,\mu} = \epsilon^{\alpha\beta\gamma} d^\beta \frac{\partial \mathcal{L}_\theta}{\partial(\partial_\mu d^\gamma)} = \frac{\theta}{4\pi} \epsilon_{\mu\nu} \tilde{\partial}_\nu d^\alpha. \tag{2.94}$$

The corresponding charge S^α_θ is zero in the even sector, and

$$S^\alpha_\theta = \int_{x_1} j^\alpha_{\theta,0} = \frac{\theta}{4\pi} \left[d^\alpha(L) - d^\alpha(-L) \right] = \frac{\theta}{2\pi} d^\alpha(-L), \tag{2.95}$$

in the odd sector of the theory. It follows that S^α_θ is nonzero only in the gapless phase where $\theta = \pm\pi$, and with antiperiodic boundary conditions imposed on d^α. A soliton in this part of the theory acquires consequently an extra contribution σ_θ to the spin given by

$$\sigma_\theta = \pm \tfrac{1}{2}. \tag{2.96}$$

Since the solitons have a canonical spin of $\frac{1}{2}$, the topological term in the odd sector of the gapless phase transmutes these fermions into bosons [81]. This ability of $\mathcal{L}_\theta$ to change statistics it shares with the Chern-Simons term in (2+1)-dimensional theories.

We next consider the critical lines $\Delta = \pm\Sigma$. At these lines, either the component $\chi_{\rm e}$ or $\chi_{\rm o}$ in (2.77) is gapless, and the above analysis does not apply. Let us first focus on the critical line $\Delta = \Sigma$. The bosonized form of (2.77) then reads

$$\begin{aligned} \mathcal{L} \;\; = \;\; & \frac{1}{2}(\partial_\mu \phi_{\rm e})^2 + \frac{1}{2}(\partial_\mu \phi_{\rm o})^2 - (\Delta + \Sigma) K \cos(\sqrt{4\pi}\phi_{\rm e}) \\ & - \sqrt{\frac{1}{\pi}} \epsilon_{\mu\nu} (\phi_{\rm e} - \phi_{\rm o}) \tilde{\partial}_\mu B^3_\nu, \end{aligned} \tag{2.97}$$

where the gaplessness of the field $\chi_{\rm o}$ is reflected in the absence of a cosine term for $\phi_{\rm o}$. As a result, the integration over this field is a simple Gaussian which can easily be carried out to give a gauge-invariant mass term for the local spin gauge field

$$\mathcal{L}_{\rm mass} = \frac{1}{2\pi} B^3_\mu \left(g_{\mu\nu} - \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\partial^2} \right) B^3_\nu. \tag{2.98}$$

This mass generation by gapless fermions in two spacetime dimensions is the famous Schwinger mechanism [116]. The contribution of $\chi_{\rm e}$ to the effective theory may be approximated by substituting the value $\phi_{\rm e} = 0$, or $\sqrt{\pi}/2$ in (2.97), corresponding to the minimum of the potential $(\Delta + \Sigma) K \cos(\sqrt{4\pi}\phi_{\rm e})$ for $\Delta = \Sigma < 0$, or $\Delta = \Sigma > 0$.

It follows that only in the latter case a topological term with $\theta = \pi$ is generated. The Schwinger mechanism operates similarly at the critical line $\Delta = -\Sigma$. Here, the topological term (with $\theta = \pi$) persists only at the lower half of the line, defined by $\Delta = -\Sigma > 0$.

2.3 Statistics-Changing Phase Transition

The (1+1)-dimensional model (2.71) we discussed in the previous section has similar characteristics in 2+1 dimensions [117, 118, 119, 120]. While the Dirac Hamiltonian naturally appears in (1+1)-dimensional systems where it describes the fermionic excitations around the two Fermi points [82], this is not the case in 2+1 dimensions. However, there exist two-dimensional systems—so-called semimetals—that have point-like Fermi surfaces [121]. In these materials, the valence and conduction band intersect in discrete points. Near such a degeneracy of two energy levels, the Hamiltonian describing the full theory may be approximated by a 2×2 Dirac Hamiltonian describing just the two-level subsystem in the vicinity of the diabolic points [122, 123, 124]. In this section, we briefly discuss some of the salient features of the (2+1)-dimensional model.

For our purposes, it suffice to consider the simplified, Abelian version of the model given by (2.77). We have chosen a two-dimensional representation of the Dirac algebra

$$\gamma_0 = \beta = \tau_3, \quad \gamma_1 = i\tau_2, \quad \gamma_2 = i\tau_1, \tag{2.99}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad g_{\mu\nu} = \mathrm{diag}(1, -1, -1), \tag{2.100}$$

with τ the Pauli matrices. With the help of the derivative expansion [26], one can easily compute the induced fermion-number current j_μ. It is given by [119]

$$j_\mu = \frac{\vartheta}{2\pi}\tilde{G}_\mu, \tag{2.101}$$

where $\tilde{G}_\mu$ is the dual field $\tilde{G}_\mu = \epsilon_{\mu\nu\lambda}\tilde{\partial}_\nu B^3_\lambda$ and

$$\vartheta = \tfrac{1}{2}[\mathrm{sgn}(\Delta + \Sigma) - \mathrm{sgn}(\Delta - \Sigma)]. \tag{2.102}$$

The first term here is the contribution from the even-site spinor χ_e, while the second term is the contribution from the odd-site spinor χ_o. The relative minus sign reflects the opposite charge of χ_e and χ_o with respect to the local spin gauge symmetry. We encountered the dual field $\tilde{G}_\mu$ previously in the context of superfluid ^{3}He-a [see (2.29)], where it was shown that it is proportional to the topological current J_μ of the O(3) nonlinear sigma model. The corresponding charge Q is the winding number of the map (2.26).

We thus arrive at the conclusion that the induced fermion-number current j_μ is proportional to the topological current J_μ of the effective theory,

$$j_\mu = \vartheta J_\mu. \tag{2.103}$$

The proportionality constant ϑ is nonzero for $|\Sigma| > |\Delta|$, where $\mathrm{sgn}(\Delta+\Sigma) - \mathrm{sgn}(\Delta - \Sigma) = 2\mathrm{sgn}(\Sigma)$, and zero for $|\Sigma| < |\Delta|$.

Let us first discuss the case $|\Sigma| > |\Delta|$. The situation of two different conserved currents which become proportional to each other is typical for ordered states where the residual symmetry links up different groups of the original symmetry. For the model under consideration, (2.103) implies that the fermion-number symmetry $\mathrm{U}(1)^N$ generated by the charge $N = \int_{\mathbf{x}} j_0$, and the flux symmetry $\mathrm{U}(1)^\Phi$ generated by the flux

$$\Phi = \int_{\mathbf{x}} \tilde{G}_0, \tag{2.104}$$

are spontaneously broken in the following manner:

$$\mathrm{U}(1)^N \times \mathrm{U}(1)^\Phi \supset \mathrm{U}(1)^{N-\vartheta\Phi}. \tag{2.105}$$

To identify the Abelian Goldstone mode associated with this spontaneous symmetry breakdown, we also calculate the induced "Maxwell" term,

$$\mathcal{L} = -\tfrac{1}{4}\Pi G_{\mu\nu}^2, \tag{2.106}$$

with $G_{\mu\nu} = \tilde{\partial}_\mu B_\nu^3 - \tilde{\partial}_\nu B_\mu^3$. To the one-loop order we find for the coefficient Π appearing here [119]

$$\Pi = \frac{1}{8\pi}\left(\frac{1}{|\Delta+\Sigma|} + \frac{1}{|\Delta-\Sigma|}\right). \tag{2.107}$$

If we imagine integrating over the local spin gauge field, the partition function can be written as

$$Z = \int \mathrm{D}B_\mu^3 \exp\left[i\int_x \left(-\tfrac{1}{4}\Pi G_{\mu\nu}^2 - \vartheta\epsilon_{\mu\nu\lambda}A_\mu\tilde{\partial}_\nu B_\lambda^3\right)\right], \tag{2.108}$$

where we omitted a gauge-fixing factor for the spin gauge field and coupled the current j_μ to a background field A_μ. This coupling allows us, by differentiating with respect to A_μ, to compute the induced fermion-number current (2.103).

To make the Goldstone mode explicit we employ a procedure frequently used to derive dual theories and introduce the change of variables $B_\mu^3 \to \tilde{G}_\mu$ in the functional integral. Since the dual field $\tilde{G}_\mu$ fulfills the Bianchi identity $\tilde{\partial}_\mu\tilde{G}_\mu = 0$, we introduce a Lagrange multiplier φ in the functional integral to impose this constraint:

$$Z = \int \mathrm{D}\tilde{G}_\mu \mathrm{D}\varphi \exp\left[i\int_x \left(-\tfrac{1}{2}\Pi\tilde{G}_\mu^2 - \vartheta A_\mu\tilde{G}_\mu + \varphi\tilde{\partial}_\mu\tilde{G}_\mu\right)\right]. \tag{2.109}$$

Performing the Gaussian integral over the dual field $\tilde{G}_\mu$, we obtain an expression for the partition function in terms of a gapless scalar field:

$$Z = \int \mathrm{D}\varphi\, \exp\left[\frac{i}{2\Pi}\int_x (\tilde{\partial}_\mu\varphi + \vartheta A_\mu)^2\right]. \tag{2.110}$$

This is the Goldstone field we were seeking. It should be noted that in 2+1 dimensions, both a massless vector field and a real scalar field represent one degree of freedom.

This can be easily understood by noting that a photon has only one transverse degree of freedom in two space dimensions. In terms of the Goldstone field φ, the fermion-number current becomes

$$j_\mu = -\frac{\vartheta}{\Pi}(\tilde{\partial}_\mu \varphi + \vartheta A_\mu) \tag{2.111}$$

as in BCS theory. From this we conclude that the (2+1)-dimensional model (2.71) exhibits superconductivity. It should be noted that the mechanism leading to this, with the spin gauge field playing a decisive role, is entirely different from that in classic superconductors.

On comparing the expression (2.111) for the fermion-number current with the one obtained directly from (2.108), we conclude that the dual field $\tilde{G}_\mu$ is related to the Goldstone field via

$$\tilde{G}_\mu = -\frac{1}{\Pi}\tilde{\partial}_\mu \phi. \tag{2.112}$$

We next turn to the case $|\Sigma| < |\Delta|$. When one of the two spinors $\chi_{\rm e}$ or $\chi_{\rm o}$ become gapless, i.e., $\Delta + \Sigma = 0$ or $\Delta - \Sigma = 0$, the induced fermion-number current discontinuously drops to zero, as was first noticed by Chen and Wilczek [125]. So, for $|\Sigma| < |\Delta|$ the argument that the fermion-number current and the topological current are proportional, which led to the conclusion that we have the spontaneous symmetry breaking (2.105), is invalid and the symmetries U(1)^N and U(1)$^\Phi$ remain unbroken here. The discontinuities in the induced charges at the lines $|\Sigma| = |\Delta|$ merely reflect that these are critical lines [119].

The restoration of symmetry when one passes from the broken phase to the symmetric phase should be accompanied by a loss of the gapless mode. To see that this is indeed the case we note that besides the Maxwell term, also a Chern-Simons terms is generated here,

$$\mathcal{L}_\theta = \frac{\theta}{4\pi^2}\epsilon_{\mu\nu\lambda} B^3_\mu \tilde{\partial}_\nu B^3_\lambda, \tag{2.113}$$

with $\theta = -\frac{1}{2}\pi[\mathrm{sgn}(\Delta+\Sigma) + \mathrm{sgn}(\Delta-\Sigma)]$. The first and second term here arise from the even-site and odd-site spinors, $\chi_{\rm e}$ and $\chi_{\rm o}$, respectively. Observe that in the ordered, gapless phase where $|\Sigma| > |\Delta|$, the Chern-Simons term is zero. The Euler-Lagrange equation for B^3_μ in the presence of a Chern-Simons term and with the background field A_μ set to zero yields $\tilde{\partial}_\mu G_{\mu\nu} \propto \tilde{G}_\nu$, or in dual form $\tilde{\partial}_\mu \tilde{G}_\nu - \tilde{\partial}_\nu \tilde{G}_\mu \propto G_{\mu\nu}$. Upon taking the divergence of this last equation and using the previous one as well as the Bianchi identity $\tilde{\partial}_\mu \tilde{G}_\mu = 0$, one finds [126]

$$\partial^2 \tilde{G}_\mu \propto \tilde{G}_\mu. \tag{2.114}$$

This equation together with (2.112) shows that φ has become massive. From this we conclude that due to the presence of the Chern-Simons term in the symmetric phase, the gapless mode of the ordered phase vanishes, as it should be.

A last point we like to mention is that the phase transition we have been discussing here is a statistics-changing phase transition. This is easily understood using the result [91] that a Chern-Simons term imparts spin to a skyrmion. The presence of this term (2.113) in the unbroken phase turns a skyrmion, which is an ordinary boson in the broken phase where no Chern-Simons term is generated, into a fermion. Whence, on

crossing the phase boundary $|\Sigma| = |\Delta|$, skyrmions undergo a spin transmutation and change their statistics from fermionic to bosonic or *vice versa*.

2.4 Fluxons

In this section, we examine a gas of electrons confined to a plane. We present a simple method to calculate the quantum numbers induced by a uniform magnetic background field [123, 127]. We will in particular focus on the quantum numbers carried by a fluxon—a point-like object carrying one unit of magnetic flux that can be pictured as being obtained by piercing the spatial plane with a magnetic vortex. It is found that it carries both the charge and spin of a fermion. Unlike what one expects, the spin of the fluxon does not originate from a Chern-Simons term, but from a so-called mixed Chern-Simons term involving two different gauge fields, viz. the electromagnetic gauge field A_μ and spin gauge field B^3_μ which describes the spin degrees of freedom.

Let us consider the Lagrangian

$$\mathcal{L} = \psi^\dagger(i\partial_0 + \mu - H_{\rm P})\psi + b\psi^\dagger \tfrac{1}{2}\sigma^3\psi, \qquad \psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \tag{2.115}$$

governing the dynamics of the Pauli spinor field ψ, with Grassmann components $\psi_\uparrow$ and $\psi_\downarrow$ describing the electrons of spin-$\uparrow$ and $\downarrow$ and chemical potential $\mu \approx \epsilon_{\rm F} = k_{\rm F}^2/2m$ which is well approximated by the Fermi energy. We introduced an external source b coupled to the spin density, to compute the induced spin. The Pauli Hamiltonian

$$H_{\rm P} = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 - g_0\mu_{\rm B}\tfrac{1}{2}\sigma^3 H + eA_0, \tag{2.116}$$

with $\mu_{\rm B} = e/2m$ the Bohr magneton and $g_0 = 2$ the free electron g-factor, contains a Zeeman term which couples the electron spins to the background magnetic field H. Usually this term is omitted. The reason is that in realistic systems the g-factor is much larger two—the value for a free electron. In strong magnetic fields relevant to, say, the QHE, the energy levels of spin-$\downarrow$ electrons are too high to be occupied so that the system is spin polarized, and the electron's spin is irrelevant to the problem. Since we consider a free electron gas, we include the Zeeman term. We describe the uniform magnetic field H by the vector potential $A_0 = A_1 = 0; A_2 = Hx_1$. The energy eigenvalues of the Pauli Hamiltonian are the famous Landau levels

$$E_l^\pm = \frac{|eH|}{m}\left(l + \tfrac{1}{2}\right) - \frac{eH}{m}\sigma_\pm, \tag{2.117}$$

with $\sigma_\pm = \pm\frac{1}{2}$ for spin-$\uparrow$ and spin-$\downarrow$ electrons, respectively.

Integrating out the fermionic degrees of freedom, one finds the one-loop effective action:

$$S_{\rm eff} = \int_x \mathcal{L}_{\rm eff} = -i{\rm Tr}\ln\left(p_0 - H_{\rm P} + \mu + \tfrac{1}{2}b\sigma^3\right). \tag{2.118}$$

The key observation to evaluate this expression is that with our gauge choice, the theory is translational invariant. Although a translation by a distance Δ_1 in the x_1-direction

changes the vector potential, $A_2 \rightarrow A_2 + H\Delta_1$, this change can be canceled by a gauge transformation $A_\mu \rightarrow A_\mu + \tilde{\partial}_\mu \alpha$, with gauge parameter $\alpha = H\Delta_1 x_2$, so that the theory is invariant under the combined symmetry. As a result, the theory effectively reduces to a (0+1)-dimensional theory, and each Landau level is infinitely degenerate. The number of degenerate states per unit area is given by $|eH|/2\pi$ for each level. With these observations, we can write $\mathcal{L}_{\rm eff}$ as

$$\mathcal{L}_{\rm eff} = -i\frac{|eH|}{2\pi}\sum_{l=0}^{\infty}\int_{k_0}\big[\ln\big(k_0 - E_{l,+} + \mu + \tfrac{1}{2}b\big) + \ln\big(k_0 - E_{l,-} + \mu - \tfrac{1}{2}b\big)\big]. \tag{2.119}$$

The presence of only an integral over the loop energy and no momentum integrals reflects that this is effectively a theory in zero space dimensions.

The induced fermion number is obtained by differentiating $\mathcal{L}_{\rm eff}$ with respect to the chemical potential, as can be inferred from the original Lagrangian (2.115), and setting the external source b to zero,

$$j_0 = \frac{\partial \mathcal{L}_{\rm eff}}{\partial \mu}\Big|_{b=0}. \tag{2.120}$$

To evaluate the resulting energy integral, we employ the integral

$$\int \frac{{\rm d}k_0}{2\pi i}\frac{{\rm e}^{ik_0\eta}}{k_0 + \xi - i\xi\eta} = \theta(\xi), \tag{2.121}$$

containing, as usual in nonrelativistic calculations [28], an additional convergence factor $\exp(ik_0\eta)$. The function $\theta(\xi)$ at the right-hand side is the Heaviside unit step function. The value for the induced fermion-number density thus obtained is

$$j_0 = \frac{|eH|}{2\pi}(l_+ + l_-), \tag{2.122}$$

where $l_\pm$ is the number of filled Landau levels for spin-$\uparrow$ and spin-$\downarrow$ electrons,

$$l_\pm = \Xi\left(\frac{m\mu_\pm}{|eH|} + \frac{1}{2}\right), \tag{2.123}$$

with $\Xi(x)$ the integer-part function denoting the largest integer less than x, and

$$\mu_\pm = \mu + \frac{eH}{m}\sigma_\pm \tag{2.124}$$

their effective chemical potentials. Implicit in this framework is the assumption that the Fermi energies of the spin-$\uparrow$ and the spin-$\downarrow$ electrons lie between two Landau levels, so that the integer-part function is well defined. We see from (2.122) that the so-called filling factor ν_H, defined as

$$\nu_H = \frac{j_0}{|eH|/2\pi} = l_+ + l_-, \tag{2.125}$$

takes on integer values only. This was to be expected for an ideal electron gas at zero temperature; given a value of the Fermi energy $\epsilon_{\rm F}$, a Landau level below the Fermi

surface is filled, while a level above it is empty. (When the Fermi energy and that of a Landau level coincide, the value of the integer-part function Ξ is ambiguous.)

If, in addition to a magnetic field, there is also an electric field **E** in the plane, a Hall current is induced perpendicular to the two fields. The current carried by the filled Landau levels is obtained by multiplying the induced density (2.122) with the drift velocity E/H. In this way one finds:

$$j_2 = \frac{\mathrm{sgn}(eH)}{2\pi} e(l_+ + l_-)E, \tag{2.126}$$

where, without loss of generality, the electric field is chosen in the x_1 direction: $A_0 = -Ex_1$. The induced fermion-number density (2.122) and Hall current (2.126) correspond to a Chern-Simons term,

$$\mathcal{L}_\theta = \tfrac{1}{2}\theta e^2 \epsilon_{\mu\nu\lambda} A_\mu \tilde{\partial}_\nu A_\lambda \tag{2.127}$$

in the effective theory, with

$$\theta = \frac{\mathrm{sgn}(eH)}{2\pi}(l_+ + l_-). \tag{2.128}$$

Because of the factor $\mathrm{sgn}(eH)$, which changes sign under a parity transformation, this Chern-Simons term is invariant under such transformations.

We next turn to the magnetic properties of the two-dimensional electron gas in a constant magnetic field. The induced spin density calculated from the effective Lagrangian via

$$s = \frac{\partial \mathcal{L}_{\mathrm{eff}}}{\partial b}\Big|_{b=0}, \tag{2.129}$$

turns out to be independent of the filled Landau levels $l_\pm$, viz.

$$s = \frac{eH}{4\pi}. \tag{2.130}$$

This follows from the symmetry in the spectrum $E^+_{l+1} = E^-_l$ if $eH > 0$, and $E^+_l = E^-_{l+1}$ if $eH < 0$. The spin magnetic moment, or magnetization M, is obtained from (2.130) by multiplying s with twice the Bohr magneton μ_{B},

$$M = g_0 \mu_{\mathrm{B}} s = \frac{e^2}{4\pi m} H. \tag{2.131}$$

This leads to the text-book result for the magnetic spin susceptibility χ_{P}

$$\chi_{\mathrm{P}} = \frac{\partial M}{\partial H} = \frac{e^2}{4\pi m} = 2\mu_{\mathrm{B}}^2\, \nu(0), \tag{2.132}$$

with $\nu(0) = m/2\pi$ the density of states per spin degree of freedom in two space dimensions. To see how the spin contribution (2.132) to the magnetic susceptibility compares to the orbital contribution, we evaluate the k_0-integral in the Lagrangian (2.119) with $b = 0$ to obtain

$$\mathcal{L}_{\mathrm{eff}} = \frac{|eH|}{2\pi} \sum_{l=0}^{\infty} \sum_{\varsigma=\pm} (\mu - E^\varsigma_l)\theta(\mu - E^\varsigma_l). \tag{2.133}$$

The summation over the Landau levels l is easily carried out with the result for small fields

$$\mathcal{L}_{\rm eff} = \frac{1}{4\pi}\sum_{\varsigma=\pm}\left[\mu_\varsigma^2 m - \frac{(eH)^2}{4m}\right] = \frac{\mu^2 m}{2\pi} + \frac{(eH)^2}{8\pi m}[(2\sigma)^2 - 1], \tag{2.134}$$

where $\sigma = \frac{1}{2}$ and $\mu_\pm$ is given by (2.124). The first term at the right-hand side of (2.134), which is independent of the magnetic field, is minus the energy density of a free electron gas particle contribution,

$$-\frac{\mu^2 m}{2\pi} = 2\int_{\mathbf{k}}\left(\frac{k^2}{2m} - \mu\right)\theta\left(\mu - \frac{k^2}{2m}\right). \tag{2.135}$$

The factor 2 at the right-hand side accounts for the spin-$\uparrow$ and spin-$\downarrow$ fermions, while the step function shows that only energy levels below the Fermi level contribute. The energy density is negative because the levels are measured relative to the Fermi energy. The second term in (2.134) yields the low-field susceptibility

$$\chi = (-1)^{2\sigma+1}2\mu_{\rm B}^2\,\nu(0)\left[(2\sigma)^2 - 1)\right], \qquad (d=2). \tag{2.136}$$

We have cast it in a form valid for $\sigma = 0, \frac{1}{2}, 1$. The term involving the factor $(2\sigma)^2$ is the spin contribution which reduces to (2.132) for $\sigma = \frac{1}{2}$. Let us compare (2.136) with the three-dimensional expression

$$\chi = (-1)^{2\sigma+1}2\mu_{\rm B}^2\,\nu(0)\left[(2\sigma)^2 - \tfrac{1}{3}\right], \qquad (d=3). \tag{2.137}$$

where $\nu(0) = mk_F/2\pi^2$ now denotes the three-dimensional density of states per spin degree of freedom at the Fermi sphere. We see that the ratio of orbital to spin contribution to χ is different in the two cases. In addition, whereas a three-dimensional electron gas is paramagnetic ($\chi > 0$) because of the dominance of the spin contribution, the two-dimensional gas is not magnetic ($\chi = 0$) at small fields since the diamagnetic orbital and paramagnetic spin contributions to χ cancel.

The induced currents we just calculated may be used to compute the induced quantum numbers carried by certain scalar field configurations. The case of interest to us is the fluxon. This is a point-like object carrying one unit of magnetic flux $2\pi/e$ which can be described by a magnetic field

$$H_\otimes = \frac{2\pi}{e}\delta(\mathbf{x}). \tag{2.138}$$

According to (2.130), a single fluxon carries a spin $\sigma_\otimes = \frac{1}{2}$ and, since for small fields

$$j_0 \to \frac{\mu m}{\pi} + \frac{|eH|}{2\pi}, \tag{2.139}$$

where the first term at the right-hand side is the two-dimensional fermion-number density $\mu m/\pi = k_{\rm F}^2/2\pi$, also one unit of fermion-number charge. That is, in a nonrelativistic electron gas, a fluxon carries the quantum numbers of a fermion. However, the close connection between the spin of a fluxon and induced Chern-Simons term for

arbitrary large fields that exists in a relativistic context, is lost. This can be traced back to the fact that in nonrelativistic theories, the electron spin is an independent degree of freedom. Below, we point out that the spin of the fluxon does not derive from the ordinary Chern-Simons term, but from a so-called mixed Chern-Simons term. Such a term is absent in a relativistic context.

It is interesting to note that whereas the spin of an antifluxon, which can be described by the magnetic field

$$H_{\odot} = -\frac{2\pi}{e}\delta(\mathbf{x}), \tag{2.140}$$

is $-\frac{1}{2}$, its fermion-number charge is the same as that for a fluxon because of the absolute values appearing in (2.139). This is what one expects in a nonrelativistic theory.

Because fluxons carry the quantum numbers of fermions, the exclusion principle forbids two fluxons to be in the same state. This is important when calculating, for example, the orbital angular momentum L of a state with $N_{\otimes}$ fluxons since they have to be put in successive orbital angular momentum states. We find for L [128]

$$L = 2\sigma_{\otimes}\sum_{\ell=1}^{N_{\otimes}}(\ell-1) = \sigma_{\otimes}N_{\otimes}(N_{\otimes}-1). \tag{2.141}$$

In this way, the total angular momentum $J = S + L$, with $S = \sigma_{\otimes}N_{\otimes}$ the total spin carried by the fluxons, becomes

$$J = S + L = \sigma_{\otimes}N_{\otimes}^2 = \tfrac{1}{2}N_{\otimes}^2. \tag{2.142}$$

We next investigate the origin of the induced spin density (2.130) we found in the nonrelativistic electron gas. To this end we slightly generalize the theory (2.115) and consider the Lagrangian

$$\mathcal{L} = \psi^{\dagger}\left[p_0 - eA_0 + \mu - \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2\right]\psi + g_0\mu_{\mathrm{B}}H^{\alpha}\psi^{\dagger}\tfrac{1}{2}\sigma^{\alpha}\psi. \tag{2.143}$$

It differs from (2.115) in that the spin source term is omitted and the magnetic field in the Zeeman term is allowed to point in any direction in spin space labeled by the index $\alpha = 1, 2, 3$. It is convenient to consider a magnetic field with fixed magnitude, but a spacetime-dependent orientation

$$H^{\alpha}(x) = Hd^{\alpha}(x), \tag{2.144}$$

where H is constant, and d^{α} is a unit vector in spin space. The gauge field A_{μ} appearing in the extended Lagrangian (2.143) still gives the magnetic field perpendicular to the plane $\nabla \times \mathbf{A} = H$. As we did various times before, we make the decomposition

$$\psi(x) = s(x)\chi(x), \tag{2.145}$$

with $s(x)$ the spacetime-dependent SU(2) matrix first introduced in (2.1). In terms of these new variables the Lagrangian (2.143) becomes

$$\mathcal{L} = \chi^{\dagger}\left[p_0 - eA_0 - B_0 + \mu - \frac{1}{2m}(\mathbf{p} - e\mathbf{A} - \mathbf{B})^2\right]\chi + \frac{eH}{2m}\chi^{\dagger}\sigma^3\chi, \tag{2.146}$$

where $B_\mu = -is^\dagger \tilde{\partial}_\mu s$ is the field (2.4). The theory takes again the form of a gauge theory with gauge field B_μ. The spin-density operator

$$j_0^\alpha = \psi^\dagger \tfrac{1}{2}\sigma^\alpha \psi, \tag{2.147}$$

becomes in these new variables [see (2.42)]

$$j_0^\alpha = R^{\alpha\beta}\chi^\dagger \tfrac{1}{2}\sigma^\beta \chi = -\frac{1}{2}R^{\alpha\beta}\frac{\partial \mathcal{L}}{\partial B_0^\beta}. \tag{2.148}$$

In deriving the first equation we employed the identity (2.40), relating the SU(2) matrices in the $j = \frac{1}{2}$ representation,

$$s = \exp\left(i\tfrac{1}{2}\theta^\alpha \sigma^\alpha\right), \tag{2.149}$$

to those in the adjoint representation ($j = 1$),

$$R = \exp\left(i\theta^\alpha J^\alpha_{\rm adj}\right). \tag{2.150}$$

The matrix elements of the generators in the latter representation are

$$\left(J^\alpha_{\rm adj}\right)^{\beta\gamma} = -i\epsilon^{\alpha\beta\gamma}. \tag{2.151}$$

The projection of the spin density j_0^α onto the spin-quantization axis, i.e., the direction d^α of the applied magnetic field [see Eq. (2.50)],

$$s = d^\alpha j_0^\alpha = -\frac{1}{2}\frac{\partial \mathcal{L}}{\partial B_0^3}, \tag{2.152}$$

only involves the spin gauge field B_μ^3. So, when calculating the induced spin density we may set the fields B_μ^1 and B_μ^2 to zero and consider the simpler theory

$$\mathcal{L} = \sum_{\varsigma=\pm} \chi_\varsigma^\dagger \left[p_0 - eA_0^\varsigma + \mu_\varsigma - \frac{1}{2m}(\mathbf{p} - e\mathbf{A}^\varsigma)^2\right]\chi_\varsigma, \tag{2.153}$$

where the effective Fermi energies for the spin-$\uparrow$ and spin-$\downarrow$ electrons are given in (2.124) and $eA_\mu^\pm = eA_\mu \pm B_\mu^3$. Both components $\chi_\uparrow$ and $\chi_\downarrow$ induce a Chern-Simons term, so that in total we have

$$\begin{aligned}
\mathcal{L}_\theta &= \tfrac{1}{2}e^2\epsilon_{\mu\nu\lambda}(\theta_+ A_\mu^+ \tilde{\partial}_\nu A_\lambda^+ + \theta_- A_\mu^- \tilde{\partial}_\nu A_\lambda^-) \\
&= \tfrac{1}{2}(\theta_+ + \theta_-)\epsilon_{\mu\nu\lambda}(e^2 A_\mu \tilde{\partial}_\nu A_\lambda + B_\mu^3 \tilde{\partial}_\nu B_\lambda^3) \\
&\quad + e(\theta_+ - \theta_-)\epsilon_{\mu\nu\lambda} B_\mu^3 \tilde{\partial}_\nu A_\lambda,
\end{aligned} \tag{2.154}$$

where the last term involving two different vector potentials is a mixed Chern-Simons term. The coefficients are given by, cf. (2.128)

$$\theta_\pm = \frac{\mathrm{sgn}(eH)}{2\pi}\, l_\pm, \tag{2.155}$$

assuming that $|eH| > \frac{1}{2}|\epsilon_{ij}\partial_i B_j^3|$, so that the sign of eH is not changed by spin gauge contributions. The integers $l_\pm$ are the number of filled Landau levels for spin-$\uparrow$ and spin-$\downarrow$ electrons given by (2.123). Since $l_+ - l_- = \mathrm{sgn}(eH)$, we obtain for the induced spin density s precisely the result (2.130) we found in the preceding section,

$$s = -\frac{1}{2}\frac{\partial \mathcal{L}_{\mathrm{eff}}}{\partial B_0^3}\Big|_{B_\mu^3=0} = \frac{eH}{4\pi}. \tag{2.156}$$

The present derivation shows that the induced spin in the nonrelativistic electron gas originates not from the standard Chern-Simons term (2.127), but from a mixed Chern-Simons term involving the electromagnetic and spin gauge potential.

The first term in (2.154) is the standard Chern-Simons term, the combination $\theta_+ + \theta_-$ precisely reproduces the result (2.128) and is related to the induced fermion-number density (2.122).

Chapter 3

Dual Theories

In this chapter we shall discuss dual theories of a superfluid ^{4}He film and of a BCS superconductor in two and three space dimensions. The first use of a duality transformation in contemporary physics is generally attributed to Kramers and Wannier who applied such a transformation to the Ising model on a square lattice [129]. The models we consider in this chapter all possess vortex solutions. The stability of these defects, which appear in the original formulations as singular objects, is guaranteed by a non-trivial topology. The surrounding hyper sphere of a vortex is a circle [$r = 1$ in Eq. (2.25)], so that the relevant homotopy group is the fundamental homotopy group. In $d = 2$, a vortex is a point object ($\epsilon_d = 0$), while in $d = 3$ it is a line defect ($\epsilon_d = 1$). A loop circling any of these vortices cannot be deformed to a point without encountering a singularity.

The duality transformations of the models discussed here are aimed at obtaining a field theoretic description of the vortices they contain. Whereas these excitations are of topological nature in the original formulations, they become the elementary excitations of the dual theory.

3.1 Superfluid ^{4}He Film

In this section, we shall derive the dual theory describing a ^{4}He film. It is well known that this system undergoes a Kosterlitz-Thouless phase transition at a temperature well below the bulk transition temperature. The superfluid low-temperature state is characterized by tightly bound vortex-antivortex pairs which at the Kosterlitz-Thouless temperature unbind and thereby disorder the superfluid state. The disordered state, at temperatures still below the bulk transition temperature, consists of a plasma of unbound vortices. We shall see in what way the dual theory, which gives a field theoretic description of the vortices, accounts for these phenomena.

The phase transition is an equilibrium transition, we can accordingly ignore any time dependence. The important fluctuations here, at temperatures below the bulk transition temperature, are phase fluctuations so that we can consider the London limit

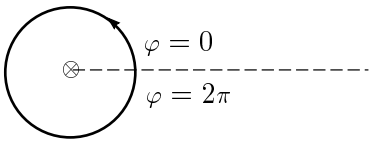


Figure 3.1: Vortex of unit winding number. The dashed line denotes the cut in the spatial plane along which φ makes a jump of 2π.

and take as Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\bar{\rho}_{\rm s}\mathbf{v}_{\rm s}^2, \tag{3.1}$$

where $\bar{\rho}_{\rm s}$ is the superfluid mass density which we assume to be constant and $\mathbf{v}_{\rm s}$ is the superfluid velocity

$$\mathbf{v}_{\rm s} = \frac{1}{m}(\nabla\varphi - \boldsymbol{\varphi}^{\rm P}). \tag{3.2}$$

Following Kleinert [17], we included a vortex gauge field $\boldsymbol{\varphi}^{\rm P}$ to account for possible vortices in the system. A vortex in two space dimensions is, as we mentioned in the introduction, a point-like object. It is characterized by the winding number of the map

$$\varphi(\mathbf{x}) : {\rm S}^1_{\mathbf{x}} \to {\rm S}^1 \tag{3.3}$$

of a circle ${\rm S}^1_{\mathbf{x}}$ around the vortex into the internal circle ${\rm S}^1$ parameterized by φ. When we circle an elementary vortex once, φ changes by 2π (see Fig. 3.1). In itself this is harmless because φ is a compact field with periodicity 2π, i.e., φ and $\varphi + 2\pi$ are identified. However, the jumps taking place along a cut in the spatial plane would turn the superfluid velocity into a discontinuous field. This is physically unacceptable. The field $\boldsymbol{\varphi}^{\rm P}$ is introduced with the purpose to compensate for these jumps. Its curl yields the vortex density $n_{\rm v}$ consisting of delta functions at the vortex positions $\mathbf{x}^\alpha$

$$\nabla \times \boldsymbol{\varphi}^{\rm P}(\mathbf{x}) = -2\pi\, n_{\rm v}(\mathbf{x}) = -2\pi \sum_\alpha w^\alpha \delta(\mathbf{x} - \mathbf{x}^\alpha), \tag{3.4}$$

leading to the vorticity

$$\nabla \times \mathbf{v}_{\rm s} = 2\pi \frac{n_{\rm v}}{m}. \tag{3.5}$$

The integer w_α is the winding number of the vortex located at $\mathbf{x}^\alpha$. We shall restrict ourselves to vortices of unit winding number, so that $w_\alpha = \pm 1$ for a vortex and antivortex, respectively.

The canonical partition function describing the equilibrium configuration of N_+ vortices and N_- antivortices in a superfluid ^{4}He film is given by

$$Z_N = \frac{1}{N_+!N_-!}\prod_\alpha \int_{\mathbf{x}^\alpha} \int {\rm D}\varphi\, \exp\left(-\beta \int_{\mathbf{x}} \mathcal{H}\right), \tag{3.6}$$

with $\mathcal{H}$ the Hamiltonian (3.1) and $N = N_+ + N_-$ the total number of vortices and antivortices. The factors $N_+!$ and $N_-!$ arise because the vortices and antivortices are

indistinguishable, and $\prod_\alpha \int_{\mathbf{x}^\alpha}$ denotes the integration over the positions of the vortices. The functional integral over φ is Gaussian and therefore easily carried out, with the result

$$Z_N = \frac{1}{N_+!N_-!} \prod_\alpha \int_{\mathbf{x}^\alpha} \exp\left[\pi \frac{\beta\bar{\rho}_\mathrm{s}}{m^2} \sum_{\alpha,\beta} w_\alpha w_\beta \ln\left(\frac{|\mathbf{x}^\alpha - \mathbf{x}^\beta|}{a}\right)\right]. \tag{3.7}$$

The constant a, with the dimension of a length, in the argument of the logarithm is included for dimensional reasons. Physically, it represents the vortex core diameter. Apart from an irrelevant normalization factor, Eq. (3.7) is the canonical partition function of a two-dimensional Coulomb gas with charges $q_\alpha = q w_\alpha = \pm q$, where

$$q = \sqrt{2\pi\bar{\rho}_\mathrm{s}}/m. \tag{3.8}$$

Let us rewrite the sum in the exponent appearing in (3.7) as

$$\begin{aligned}\sum_{\alpha,\beta} q_\alpha q_\beta \ln\left(\frac{|\mathbf{x}^\alpha - \mathbf{x}^\beta|}{a}\right) &= \sum_{\alpha,\beta} q_\alpha q_\beta \left[\ln\left(\frac{|\mathbf{x}^\alpha - \mathbf{x}^\beta|}{a}\right) - \ln(0)\right] \\ &\quad + \ln(0)\left(\sum_\alpha q_\alpha\right)^2,\end{aligned} \tag{3.9}$$

where we isolated the self-interaction in the last term at the right-hand side. Since $\ln(0) = -\infty$, the charges must add up to zero so as to obtain a nonzero partition function. From now on we will therefore assume overall charge neutrality, $\sum_\alpha q_\alpha = 0$, so that $N_+ = N_- = N/2$, where N must be an even integer. To regularize the remaining divergence, we replace $\ln(0)$ with an undetermined, negative constant $-c$. The exponent of (3.7) thus becomes

$$\frac{\beta}{2}\sum_{\alpha,\beta} q_\alpha q_\beta \ln\left(\frac{|\mathbf{x}^\alpha - \mathbf{x}^\beta|}{a}\right) = \frac{\beta}{2}\sum_{\alpha\neq\beta} q_\alpha q_\beta \ln\left(\frac{|\mathbf{x}^\alpha - \mathbf{x}^\beta|}{a}\right) - \beta\epsilon_\mathrm{c} N, \tag{3.10}$$

where $\epsilon_\mathrm{c} = cq^2/2$ physically represents the core energy, i.e., the energy required to create a single vortex. In deriving this we used the identity $\sum_{\alpha\neq\beta} q_\alpha q_\beta = -\sum_\alpha q_\alpha^2 = -Nq^2$ which follows from charge neutrality. Having dealt with the self-interaction, we limit the integrations $\prod_\alpha \int_{\mathbf{x}^\alpha}$ in (3.7) over the location of the vortices to those regions where they are more than a distance a apart, $|\mathbf{x}^\alpha - \mathbf{x}^\beta| > a$. The grand-canonical partition function of the system can now be cast in the form

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{[(N/2)!]^2} \prod_\alpha \int_{\mathbf{x}^\alpha} \exp\left[\frac{\beta}{2}\sum_{\alpha\neq\beta} q_\alpha q_\beta \ln\left(|\mathbf{x}^\alpha - \mathbf{x}^\beta|\right)\right], \tag{3.11}$$

where $z = \exp(-\beta\epsilon_\mathrm{c})$ is the fugacity. We suppressed an irrelevant dimensionful factor $a^{N(\beta q^2/2-1)}$. The system is known to undergo a phase transition at the Kosterlitz-Thouless temperature [130, 131]

$$T_\mathrm{KT} = \frac{1}{4}q^2 = \frac{\pi}{2}\frac{\bar{\rho}_\mathrm{s}}{m^2}, \tag{3.12}$$

triggered by the unbinding of vortex-antivortex pairs. It follows from this equation that the two-dimensional superfluid mass density $\bar{\rho}_{\rm s}(T)$, which varies from sample to sample, terminates on a line with universal slope as T approaches the Kosterlitz-Thouless temperature from below [132].

To derive the dual theory we note that $\ln(|\mathbf{x}|)$ is the inverse of the Laplace operator ∇^2,

$$\frac{1}{2\pi}\nabla^2 \ln(|\mathbf{x}|) = \delta(\mathbf{x}). \tag{3.13}$$

This allows us to represent the exponential function in (3.11) as a functional integral over an auxiliary field ϕ:

$$\exp\left[\frac{\beta}{2}\sum_{\alpha\neq\beta} q_\alpha q_\beta \ln\left(|\mathbf{x}^\alpha - \mathbf{x}^\beta|\right)\right] = \int \mathrm{D}\phi \exp\left\{-\int_{\mathbf{x}}\left[\frac{1}{4\pi\beta}(\nabla\phi)^2 + i\rho\phi\right]\right\}, \tag{3.14}$$

where $\rho(\mathbf{x}) = \sum_\alpha q_\alpha \delta(\mathbf{x}-\mathbf{x}^\alpha)$ is the charge density. In this way, the partition function becomes

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{[(N/2)!]^2} \prod_{\alpha=1}^{N}\int_{\mathbf{x}^\alpha} \int \mathrm{D}\phi \exp\left\{-\int_{\mathbf{x}}\left[\frac{1}{4\pi\beta}(\nabla\phi)^2 + i\rho\phi\right]\right\}. \tag{3.15}$$

In a mean-field treatment, the functional integral over the auxiliary field introduced in (3.14) is approximated by the saddle point determined by the field equation

$$iT\nabla^2\phi = -2\pi\rho. \tag{3.16}$$

When we introduce the scalar variable $\Phi := iT\phi$, this equation becomes formally Gauss' law, with Φ the electrostatic scalar potential. The auxiliary field introduces in (3.14) may therefore be thought of as representing the scalar potential of the equivalent Coulomb gas [17].

On account of charge neutrality, we have the identity

$$\left[\int_{\mathbf{x}}\left(\mathrm{e}^{iq\phi(\mathbf{x})} + \mathrm{e}^{-iq\phi(\mathbf{x})}\right)\right]^N = \frac{N!}{[(N/2)!]^2}\prod_{\alpha=1}^{N}\int_{\mathbf{x}^\alpha} \mathrm{e}^{-i\sum_\alpha q_\alpha \phi(\mathbf{x}^\alpha)}, \tag{3.17}$$

where we recall that N is an even number. The factor $N!/[(N/2)!]^2$ is the number of charge-neutral terms contained in the binomial expansion of the left-hand side. The partition function (3.15) may thus be written as [17]

$$\begin{aligned} Z &= \sum_{N=0}^{\infty}\frac{(2z)^N}{N!}\int \mathrm{D}\phi \exp\left[-\int_{\mathbf{x}}\frac{1}{4\pi\beta}(\nabla\phi)^2\right]\left[\cos\left(\int_{\mathbf{x}} q\phi\right)\right]^N \\ &= \int \mathrm{D}\phi \exp\left\{-\int_{\mathbf{x}}\left[\frac{1}{4\pi\beta}(\nabla\phi)^2 - 2z\cos(q\phi)\right]\right\}, \end{aligned} \tag{3.18}$$

where in the final form we recognize the sine-Gordon model. This is the dual theory we were seeking. Contrary to the original formulation (3.6), which contains the vortices as singular objects, the dual formulation has no singularities. To see how the vortices and the Kosterlitz-Thouless phase transition are represented in the dual theory we note that the field equation of the auxiliary field now reads

$$iT\nabla^2\phi = 2\pi z q \left(\mathrm{e}^{iq\phi} - \mathrm{e}^{-iq\phi}\right). \tag{3.19}$$

On comparison with the previous field equation (3.16), it follows that the right-hand side represents the charge density of the Coulomb gas. In terms of the scalar potential Φ, Eq. (3.19) becomes the Poisson-Boltzmann equation

$$\nabla^2\Phi = -2\pi q \left(z\,\mathrm{e}^{-\beta q\Phi} - z\,\mathrm{e}^{\beta q\Phi}\right), \tag{3.20}$$

describing, at least for temperatures above the Kosterlitz-Thouless temperature, a plasma of positive and negative charges with density $n_\pm$,

$$n_\pm = z\,\mathrm{e}^{\mp\beta q\Phi}, \tag{3.21}$$

respectively. The fugacity z is the density at zero scalar potential. (It is to recalled that we suppress factors of a denoting the diameter of the vortex cores.) Equation (3.20) is a self-consistent equation for the scalar potential Φ giving the spatial distribution of the charges via (3.21). It follows from this argument that the interaction term $2z\cos(q\phi)$ of the sine-Gordon model represents a plasma of vortices.

The renormalization group applied to the sine-Gordon model reveals that at the Kosterlitz-Thouless temperature $T_{\mathrm{KT}} = \frac{1}{4}q^2$ there is a phase transition between a low-temperature phase of tightly bound neutral pairs and a high-temperature plasma phase of unbound vortices [133]. In the low-temperature phase, the (renormalized) fugacity scales to zero in the large-scale limit so that the interaction term, representing the plasma of unbound vortices, is suppressed. The long-distance behavior of the low-temperature phase is therefore well described by the free theory $(\nabla\phi)^2/4\pi\beta$, representing the gapless Kosterlitz-Thouless mode. This is the superfluid state. The expectation value of a single vortex vanishes because in this gapless state its energy diverges in the infrared.

An important characteristic of a charged plasma is that it has no gapless excitations, the photon being transmuted into a massive plasmon. To see this we assume that $q\Phi << T$, so that $\sinh(\beta q\Phi) \approx \beta q\Phi$. In this approximation, the Poisson-Boltzmann equation (3.20) can be linearized to give

$$(\nabla^2 - m_{\mathrm{D}}^2)\Phi = 0, \quad m_{\mathrm{D}}^2 = 4\pi\beta z q^2. \tag{3.22}$$

This shows us that, in contradistinction to the low-temperature phase, in the high-temperature phase, the scalar potential describes a massive mode—the plasmon. In other words, the Kosterlitz-Thouless mode acquires an energy gap m_{D}. Since it provides the high-temperature phase with an infrared cutoff, isolated vortices have a finite energy now and accordingly a finite probability to be created. This Debeye mechanism of mass generation for the photon should be distinguished from the Higgs mechanism which operates in superconductors (see below) and also generates a photon mass.

Another property of a charged plasma is that it screens charges. This so-called Debeye screening may be illustrated by adding an external charge to the system. The linearized Poisson-Boltzmann equation (3.22) then becomes

$$(\nabla^2 - m_{\rm D}^2)\Phi(\mathbf{x}) = -2\pi q_0 \delta(\mathbf{x}), \tag{3.23}$$

with q_0 the external charge which we have placed at the origin. The solution of this equation is given by $\Phi(\mathbf{x}) = q_0 K_0(m_{\rm D}|\mathbf{x}|)$ with K_0 a modified Bessel function. The mass term in (3.23) is (2π times) the charge density induced by the external charge, i.e.,

$$\rho_{\rm ind}(\mathbf{x}) = -\frac{1}{2\pi} q_0 m_{\rm D}^2 K_0(m_{\rm D}|\mathbf{x}|). \tag{3.24}$$

By integrating this density over the entire system, we see that the total induced charge $\int_{\mathbf{x}} \rho_{\rm ind} = -q_0$ completely screens the external charge—at least in the linear approximation we are using here. The inverse of the plasmon mass is the screening length—the so-called Debeye screening length.

To see that the sine-Gordon model gives a dual description of a ^{4}He film we cast the field equation (3.16) in the form

$$iT\nabla^2\phi = -mq\nabla \times \mathbf{v}_{\rm s}, \tag{3.25}$$

where we employed Eq. (3.5). On integrating this equation, we obtain up to an irrelevant integration constant

$$iT\partial_i\phi = -q\epsilon_{ij}(\partial_j\varphi - \varphi_j^{\rm P}). \tag{3.26}$$

This relation, involving the antisymmetric Levi-Civita symbol, is a typical one between dual variables. It also nicely illustrates that although the dual variable ϕ is a regular field, it nevertheless contains the information about the vortices which in the original formulation are described via the singular vortex gauge field $\boldsymbol{\varphi}^{\rm P}$.

Given this observation it is straightforward to calculate the current-current correlation function $\langle g_i(\mathbf{k}) g_j(-\mathbf{k})\rangle$, with

$$\mathbf{g} = \bar{\rho}_{\rm s}\mathbf{v}_{\rm s} \tag{3.27}$$

the mass current. We find

$$\langle g_i(\mathbf{k}) g_j(-\mathbf{k})\rangle = -\frac{1}{2\pi\beta^2}\bar{\rho}_{\rm s}\epsilon_{ik}\epsilon_{jl}k_k k_l \langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle, \tag{3.28}$$

where the average is to be taken with respect to the partition function

$$Z_0 = \int {\rm D}\phi \exp\left[-\frac{1}{4\pi\beta}\int_{\mathbf{x}} (\nabla\phi)^2\right], \tag{3.29}$$

which is obtained from (3.18) by setting the interaction term to zero. We obtain in this way the standard expression for a superfluid

$$\langle g_i(\mathbf{k}) g_j(-\mathbf{k})\rangle = -\frac{\bar{\rho}_{\rm s}}{\beta}\frac{1}{\mathbf{k}^2}\left(\delta_{ij}\mathbf{k}^2 - k_i k_j\right). \tag{3.30}$$

The $1/\mathbf{k}^2$ reflects the gaplessness of the ϕ-field in the low-temperature phase, while the combination $\delta_{ij}\mathbf{k}^2 - k_i k_j$ arises because the current is divergent free, $\nabla\cdot\mathbf{g}(\mathbf{x}) = 0$, or $\mathbf{k}\cdot\mathbf{g}(\mathbf{k}) = 0$.

3.2 Two-dimensional Superconductor

We now turn to the dual description of a superconducting film. We thereto minimally couple the model of the preceding section to a magnetic field described by the magnetic vector potential $\mathbf{A}$. For the time being we ignore vortices by setting the vortex gauge field $\boldsymbol{\varphi}^{\mathrm{P}}$ to zero. The partition function of the system then reads

$$Z = \int \mathrm{D}\varphi \int \mathrm{D}\mathbf{A}\, \Xi(\mathbf{A})\, \exp\left(-\beta \int_{\mathbf{x}} \mathcal{H}\right), \tag{3.31}$$

where $\Xi(\mathbf{A})$ is a gauge-fixing factor for the gauge field $\mathbf{A}$, and $\mathcal{H}$ is the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\bar{\rho}_{\mathrm{s}}\mathbf{v}_{\mathrm{s}}^2 + \tfrac{1}{2}(\nabla \times \mathbf{A})^2 \tag{3.32}$$

with

$$\mathbf{v}_{\mathrm{s}} = \frac{1}{m}(\nabla\varphi - 2e\mathbf{A}). \tag{3.33}$$

The double charge $2e$ stands for the charge of the Cooper pairs which are formed at the bulk transition temperature. The functional integral over φ in (3.31) is easily carried out with the result

$$Z = \int \mathrm{D}\mathbf{A}\, \Xi(\mathbf{A}) \exp\left\{-\frac{\beta}{2}\int_{\mathbf{x}}\left[(\nabla\times\mathbf{A})^2 + m_A^2 A_i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)A_j\right]\right\}, \tag{3.34}$$

where the last term, with $m_A^2 = 4e^2\bar{\rho}_{\mathrm{s}}/m^2$, is a gauge-invariant, albeit nonlocal mass term for the gauge field generated by the Higgs mechanism. The number of degrees of freedom does not change in the process. This can be seen by noting that a gapless gauge field in two dimensions represents no physical degrees of freedom. (In Minkowski spacetime, this is easily understood by recognizing that in $1+1$ dimensions there is no transverse direction.) Before the Higgs mechanism took place, the system therefore contains only a single physical degree of freedom described by φ. This equals the number of degrees of freedom contained in (3.34).

We next introduce an auxiliary field $\tilde{h}$ to linearize the first term in (3.34),

$$\exp\left[-\frac{\beta}{2}\int_{\mathbf{x}}(\nabla\times\mathbf{A})^2\right] = \int \mathrm{D}\tilde{h}\exp\left[-\frac{1}{2\beta}\int_{\mathbf{x}}\tilde{h}^2 + i\int_{\mathbf{x}}\tilde{h}(\nabla\times\mathbf{A})\right], \tag{3.35}$$

and integrate out the gauge-field fluctuations [with a gauge-fixing term $(1/2\alpha)(\nabla\cdot\mathbf{A})^2$]. The result is a manifestly gauge-invariant expression for the partition function in terms of a massive scalar field $\tilde{h}$, representing the single degree of freedom contained in the theory:

$$Z = \int \mathrm{D}\tilde{h}\exp\left\{-\frac{1}{2\beta}\int_{\mathbf{x}}\left[\frac{1}{m_A^2}(\nabla\tilde{h})^2 + \tilde{h}^2\right]\right\}. \tag{3.36}$$

To understand the physical significance of this field, we note from (3.35) that it satisfies the field equation

$$\tilde{h} = i\beta\nabla\times\mathbf{A}. \tag{3.37}$$

That is, the fluctuating field $\tilde{h}$ represents the local magnetic induction, which is a scalar in two space dimensions. Equation (3.36) shows that the magnetic field has a finite penetration depth $\lambda_{\rm L} = 1/m_A$. In contrast to the original description where the functional integral runs over the gauge potential, the integration variable in (3.36) is the physical field.

We next include vortices. The penetration depth $\lambda_{\rm L}$ provides the system with an infrared cutoff so that a single magnetic vortex in the charged theory has a finite energy. Vortices can therefore be thermally activated. This is different from the superfluid phase of the neutral model, where the absence of an infrared cutoff permits only tightly bound vortex-antivortex pairs to exist. We expect, accordingly, the superconducting phase to describe a plasma of vortices, each carrying one magnetic flux quantum $\pm\pi/e$. The partition function now reads

$$Z = \sum_{N_+,N_-=0}^{\infty} \frac{z^{N_++N_-}}{N_+!\,N_-!} \prod_\alpha \int_{\mathbf{x}^\alpha} \int \mathrm{D}\varphi \int \mathrm{D}\mathbf{A}\, \Xi(\mathbf{A})\, \exp\left(-\beta \int_{\mathbf{x}} \mathcal{H}\right) \tag{3.38}$$

where z is the fugacity, i.e., the Boltzmann factor associated with the vortex core energy. The velocity appearing in the Hamiltonian (3.32) now includes the vortex gauge field

$$\mathbf{v}_{\rm s} = \frac{1}{m}(\nabla\varphi - 2e\mathbf{A} - \boldsymbol{\varphi}^{\rm P}). \tag{3.39}$$

The vortex gauge field $\boldsymbol{\varphi}^{\rm P}$ can be shifted from the first to the second term in the Hamiltonian (3.32) by applying the transformation $\mathbf{A} \to \mathbf{A} - \boldsymbol{\varphi}^{\rm P}/2e$. This results in the shift

$$\nabla \times \mathbf{A} \to \nabla \times \mathbf{A} - B^{\rm P}, \tag{3.40}$$

with the plastic field

$$B^{\rm P} = -\Phi_0 \sum_\alpha w_\alpha\, \delta(\mathbf{x} - \mathbf{x}^\alpha) \tag{3.41}$$

representing the magnetic flux density. Here, $\Phi_0 = \pi/e$ is the elementary flux quantum. Repeating the steps of the previous paragraph we now obtain instead of (3.36)

$$\begin{aligned} Z = \sum_{N_\pm=0}^{\infty} \frac{z^{N_++N_-}}{N_+!\,N_-!} \prod_\alpha \int_{\mathbf{x}^\alpha} \int \mathrm{D}\tilde{h} \\ \times \exp\left\{-\frac{1}{2\beta}\int_{\mathbf{x}} \left[\frac{1}{m_A^2}(\nabla\tilde{h})^2 + \tilde{h}^2\right] + i\int_{\mathbf{x}} B^{\rm P}\tilde{h}\right\}, \end{aligned} \tag{3.42}$$

where $\tilde{h}$ represents the physical local magnetic induction h

$$\tilde{h} = i\beta(\nabla \times \mathbf{A} - B^{\rm P}) = i\beta h. \tag{3.43}$$

The field equation for $\tilde{h}$ obtained from (3.42) yields for the magnetic induction:

$$-\nabla^2 h + m_A^2 h = m_A^2 B^{\rm P}, \tag{3.44}$$

which is the familiar equation in the presence of magnetic vortices.

The last term in (3.42) shows that the charge g with which a magnetic vortex couples to the fluctuating $\tilde{h}$-field is the product of an elementary flux quantum (contained in the definition of $B^{\rm P}$) and the inverse penetration depth $m_A = 1/\lambda_{\rm L}$,

$$g = \Phi_0 m_A. \tag{3.45}$$

For small fugacities the summation indices N_+ and N_- can be restricted to the values $0, 1$ and we arrive at the partition function of the massive sine-Gordon model [134]

$$Z = \int {\rm D}\tilde{h} \exp\left(- \int_{\mathbf{x}} \left\{ \frac{1}{2\beta} \left[\frac{1}{m_A^2} (\nabla \tilde{h})^2 + \tilde{h}^2 \right] - 2z \cos\left(\Phi_0 \tilde{h} \right) \right\} \right). \tag{3.46}$$

This is the dual formulation of a two-dimensional superconductor. The magnetic vortices of unit winding number $w_\alpha = \pm 1$ turned the otherwise free theory (3.36) into an interacting one.

The final form (3.46) demonstrates the rationales for going over to a dual theory. First, it is a formulation directly in terms of a physical field representing the local magnetic induction. There is no redundancy in this description and therefore no gauge invariance. Second, the magnetic vortices are accounted for in a nonsingular fashion. This is different from the original formulation of the two-dimensional superconductor where the local magnetic induction is the curl of an unphysical gauge potential $\mathbf{A}$, and where the magnetic vortices appear as singular objects.

Up to this point we have discussed a genuine two-dimensional superconductor. As a model to describe superconducting films this is, however, not adequate. The reason is that the magnetic interaction between the vortices takes place mostly not through the film but through free space surrounding the film where the photon is gapless. This situation is markedly different from a superfluid film. The interaction between the vortices there is mediated by the Kosterlitz-Thouless mode which is confined to the film. A genuine two-dimensional theory therefore gives a satisfactory description of a superfluid film.

To account for the fact that the magnetic induction is not confined to the film and can roam in outer space, the field equation (3.44) is modified in the following way [135, 136]

$$-\nabla^2 h(\mathbf{x}_\perp, x_3) + \frac{1}{\lambda_\perp} \delta_d(x_3) h(\mathbf{x}_\perp, x_3) = \frac{1}{\lambda_\perp} \delta_d(x_3) B^{\rm P}(\mathbf{x}). \tag{3.47}$$

Here, $1/\lambda_\perp = d m_A^2$, with d denoting the thickness of the superconducting film, is an inverse length scale, $\mathbf{x}_\perp$ denotes the coordinates in the plane, h the component of the induction field perpendicular to the film, and $\delta_d(x_3)$ is a smeared delta function of thickness d along the x_3-axis

$$\delta_d(x_3) \begin{cases} = 0 & \text{for} \quad |x_3| > d/2 \\ \neq 0 & \text{for} \quad |x_3| \leq d/2 \end{cases} . \tag{3.48}$$

The reason for including the smeared delta function at the right-hand side of (3.47) is that the vortices are confined to the film. The delta function in the second-term at the left-hand side is included because this term is generated by screening currents which are also confined to the film.

To be definite, we consider a single magnetic vortex located at the origin. The induction field found from (3.47) reads

$$h(\mathbf{x}_\perp, 0) = \frac{\Phi_0}{2\pi} \int_0^\infty \mathrm{d}q \frac{q}{1 + 2\lambda_\perp q} J_0(q|\mathbf{x}_\perp|), \tag{3.49}$$

with J_0 the 0th Bessel function of the first kind. At small distances from the vortex core ($\lambda_\perp q >> 1$)

$$h(\mathbf{x}_\perp, 0) \sim \frac{\Phi_0}{4\pi\lambda_\perp |\mathbf{x}_\perp|}, \tag{3.50}$$

while far away ($\lambda_\perp q << 1$)

$$h(\mathbf{x}_\perp, 0) \sim \frac{\Phi_0 \lambda_\perp}{\pi |\mathbf{x}_\perp|^3}. \tag{3.51}$$

This last equation shows that the field does not exponentially decay as would be the case in a genuine two-dimensional system. The reason for the long range is that most of the magnetic interaction takes place in free space outside the film where the photon is gapless. If, as is often the case, the length $\lambda_\perp = 1/dm_A^2$ is much larger than the sample size, it can be effectively set to infinity. In this limit, the effect of the magnetic interaction, as can be seen from (3.50), diminishes and the vortices behave as in a superfluid film. One therefore expects a superconducting film to also undergo a Kosterlitz-Thouless transition at some temperature T_{KT} characterized by an unbinding of vortex-antivortex pairs. The first experiment to study this possibility was carried out in Ref. [137]. Because the transition temperature T_{KT} is well below the bulk temperature T_{c} where the Cooper pairs form, the energy gap of the fermions remains finite at the critical point [138]. This prediction has been corroborated by experiments performed by Hebard and Palaanen on superconducting films [139]. For temperatures $T_{\mathrm{KT}} \leq T \leq T_{\mathrm{c}}$, there is a plasma of magnetic vortices which disorder the superconducting state. At T_{KT} vortices and antivortices bind into pairs and algebraic long-range order sets in.

3.3 Bosonization

The two dual models (3.36) and (3.46) we encountered in the preceding section describing a superconducting film without and with vortices included are reminiscent of the bosonized massless and massive Schwinger model, respectively. The massless Schwinger model [116] describes gapless fermions interacting with an electromagnetic field in one space and one time dimension. It is defined by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\tilde{\not\partial} - e\not{A})\psi - \tfrac{1}{4}F_{\mu\nu}^2, \tag{3.52}$$

where $\not{A} = A_\mu \gamma_\mu$ with γ_μ the Dirac matrices (2.68), ψ is a two-component Grassmann field describing the fermions, and $\bar{\psi} = \psi^\dagger \beta$. Since the theory is bilinear in ψ, the functional integral over the Grassmann fields in the partition function

$$Z = \int \mathrm{D}\bar{\psi}\mathrm{D}\psi \int \mathrm{D}A_\mu \Xi(A_\mu) \, \exp\left(i \int_x \mathcal{L} \right), \tag{3.53}$$

with $\Xi(A_\mu)$ a gauge-fixing factor, can be easily carried out. It yields a functional determinant

$$\int \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[i\int_x \bar{\psi}(i\tilde{\partial\!\!\!/} - eA\!\!\!/)\psi\right] = \mathrm{Det}\,(p\!\!\!/ - eA\!\!\!/) \tag{3.54}$$

which can be evaluated in closed form using, for example, the derivative expansion [140]. One finds

$$\mathrm{Det}\,(p\!\!\!/ - eA\!\!\!/) = \exp\left[\frac{ie^2}{2\pi}\int_x A_\mu\left(g_{\mu\nu} - \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\partial^2}\right)A_\nu\right]. \tag{3.55}$$

The partition function (3.53) can now be written in a form,

$$\begin{aligned} Z = &\int \mathrm{D}A_\mu \Xi(A_\mu) \\ &\times \exp\left\{i\int_x\left[-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_A^2 A_\mu\left(g_{\mu\nu} - \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\partial^2}\right)A_\nu\right]\right\}, \end{aligned} \tag{3.56}$$

which is the Minkowski analog of the partition function (3.34) describing a superconducting film without vortices. Equation (3.56) shows us that the photon has acquired a mass m_A, given by $m_A^2 = e^2/\pi$, where it should be borne in mind that in one spatial dimension the electric charge e has mass dimension one. This mass generation due to gapless fermions is called the Schwinger mechanism.

We have seen that the partition function (3.34) could be equivalently represented by the massive scalar field theory (3.36). The same holds for the Schwinger model. Rather than linearizing the Maxwell term in (3.56), which would be the analog of what we did in (3.35), we represent the term obtained in (3.55) as a functional integral over a Bose field [141]:

$$\begin{aligned} &\int \mathrm{D}\bar{\psi}\mathrm{D}\psi \exp\left[i\int_x \bar{\psi}(i\tilde{\partial\!\!\!/} - eA\!\!\!/)\psi\right] = \\ &\qquad \int \mathrm{D}\phi \exp\left\{i\int_x\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{e}{\sqrt{\pi}}\epsilon_{\mu\nu}A_\mu\tilde{\partial}_\nu\phi\right]\right\}. \end{aligned} \tag{3.57}$$

We recognize here two of the bosonization rules (2.78) we employed before, viz.:

$$\begin{aligned} \bar{\psi}i\tilde{\partial\!\!\!/}\psi &\rightarrow \tfrac{1}{2}(\partial_\mu\phi)^2 \\ j_\mu &\rightarrow \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\tilde{\partial}_\nu\phi, \end{aligned} \tag{3.58}$$

where $j_\mu = \bar{\psi}\gamma_\mu\psi$ is the electromagnetic current. The integral over the gauge field can now be easily carried out, say in the Lorentz gauge $\tilde{\partial}_\mu A_\mu = 0$, to yield the bosonized form of the massless Schwinger model

$$Z = \int \mathrm{D}\phi \exp\left\{\frac{i}{2}\int_x\left[(\partial_\mu\phi)^2 - m_A^2\phi^2\right]\right\}. \tag{3.59}$$

This shows that the model is equivalent to a massive scalar theory. It is the Minkowski analog of the dual theory (3.36) of a superconducting film without taking into account vortices.

To understand the physical origin of the scalar field ϕ and why it is massive, it should be noted that the massless Schwinger model has besides the local U(1) gauge symmetry

$$\psi(x) \to \mathrm{e}^{ie\alpha(x)}\psi(x); \quad A_\mu(x) \to A_\mu(x) + \tilde{\partial}_\mu \alpha(x) \tag{3.60}$$

also a global U(1) chiral symmetry. Under chiral transformations

$$\psi \to \mathrm{e}^{i\lambda\gamma_5}\psi, \tag{3.61}$$

where λ is the transformation parameter and γ_5 denotes the matrix $\gamma_5 = i\gamma_0\gamma_1$. The corresponding Noether current is the axial current

$$j^5_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi. \tag{3.62}$$

At the quantum level, the chiral U(1) symmetry is spontaneously broken by a fermion condensate [142]

$$\langle\bar{\psi}\psi\rangle = -\frac{\mathrm{e}^\gamma}{2\pi}m_A, \tag{3.63}$$

with $\gamma \approx 0.577216$ Euler's constant. By virtue of the identity

$$i\gamma_\mu\gamma_5 = \epsilon_{\mu\nu}\gamma_\nu, \tag{3.64}$$

the axial and electromagnetic current are related via

$$j^5_\mu = \epsilon_{\mu\nu}j_\nu. \tag{3.65}$$

With the bosonization rule (3.58) we then obtain the correspondence

$$j^5_\mu \to -\frac{1}{\sqrt{\pi}}\tilde{\partial}_\mu\phi, \tag{3.66}$$

identifying ϕ as the Goldstone field of the broken U(1) chiral symmetry. This can, however, not be the end of the story [143]. Goldstone modes are by definition gapless, but Eq. (3.59) shows that ϕ has acquired a mass, meaning that the chiral symmetry is no longer an exact symmetry of the bosonized theory. The axial current, which is classically conserved, has an anomaly at the quantum level as can be seen by invoking the field equation obtained from (3.59),

$$\tilde{\partial}_\mu j^5_\mu = \frac{m_A^2}{\sqrt{\pi}}\phi. \tag{3.67}$$

The destruction of the chiral symmetry at the quantum level can also be seen directly in the fermionic formulation, where it can be attributed to the effect of one-loop graphs. The anomaly (3.67) follows there from integrating out the fermionic degrees of freedom. Using, for example, the derivative expansion [140], one finds

$$\tilde{\partial}_\mu j^5_\mu = -\frac{m_A}{\sqrt{\pi}}\epsilon_{\mu\nu}\tilde{\partial}_\mu A_\nu. \tag{3.68}$$

On comparing the two expressions for the anomaly, we see that the Bose field ϕ represents the dual field strength $\tilde{F} := \epsilon_{\mu\nu}\tilde{\partial}_\mu A_\nu$,

$$\phi = -\frac{1}{m_A}\tilde{F}. \tag{3.69}$$

This is the Minkowski analog of the relation (3.37) we found in the context of a superconducting film.

We next consider possible vortex contributions. Let us go back to the functional determinant appearing in (3.55) and evaluate it in the presence of a vortex. To this end we go over to Euclidean spacetime where a vortex becomes an instanton. There exists a powerful index theorem [144] relating the zero eigenvalues of the massless Dirac operator $i\tilde{\partial\!\!\!/} - eA\!\!\!/$ to the so-called Pontryagin index

$$Q = \frac{e}{4\pi}\int_x \epsilon_{\mu\nu}F_{\mu\nu}, \tag{3.70}$$

which is the winding number of the vortex. The theorem states that the index of the Dirac operator, defined by the number n_+ of zero-modes with positive chirality minus the number n_- of zero-modes with negative chirality, is given by the Pontryagin index,

$$n_+ - n_- = Q. \tag{3.71}$$

In the presence of vortices $Q \neq 0$, so that there are always zero-modes. Since the determinant of an operator is the product of its eigenvalues, the determinant appearing in (3.55) is zero when vortices are included. Hence, in the massless Schwinger model, vortices do not contribute to the partition function. The suppression of instanton contributions by gapless fermions is a common feature [77].

The situation is different in the massive model, defined by the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\tilde{\partial\!\!\!/} - m - eA\!\!\!/)\psi - \tfrac{1}{4}F_{\mu\nu}^2. \tag{3.72}$$

First of all, the mass term $-m\bar{\psi}\psi$ in the Lagrangian explicitly breaks the chiral symmetry. So, even if there were no anomaly, we expect a massive bosonized theory. Second, because the massive Dirac operator contains no zero-modes, the index theorem is of no relevance here and vortices should be included in (3.59). The bosonized partition function then becomes, cf. (3.42),

$$\begin{aligned} Z = \sum_{N_\pm=0}^{\infty} \frac{z^{N_+ + N_-}}{N_+!\,N_-!}\prod_\alpha \int_{x^\alpha}\int \mathrm{D}\phi \\ \times\exp\left\{\frac{i}{2}\int_x\left[(\partial_\mu\phi)^2 - m_A^2\phi^2\right] + i\int_x \tilde{F}^{\mathrm{P}}\phi\right\}, \end{aligned} \tag{3.73}$$

where $z = \exp(iS_{\mathrm{c}})$, with S_{c} the vortex core action. The plastic field $\tilde{F}^{\mathrm{P}}$ describes the vortices,

$$\tilde{F}^{\mathrm{P}} = -\Phi_0\sum_\alpha w_\alpha\,\delta(x - x^\alpha), \tag{3.74}$$

where the sum is over all vortex locations x^α and w_α is the winding number of the αth vortex. The elementary flux quantum in the Schwinger model is $\Phi_0 = 2\pi/e$, which is twice the value in a superconductor because the electric charge of a Cooper pair is twice that of an electron. We shall consider only vortices of unit winding number $w_\alpha = \pm 1$. For z small we can restrict ourselves to configurations having zero or one vortex. The partition function then becomes the Minkowski analog of (3.46)

$$Z = \int \mathrm{D}\phi \exp\left(i \int_x \left\{ \frac{1}{2} \left[(\partial_\mu \phi)^2 - m_A^2 \phi^2 \right] - 2z \cos\left(g\phi\right) \right\} \right), \tag{3.75}$$

with $g = \Phi_0 m_A$ as in (3.45). From this we see that the inclusion of the mass term in the Schwinger model resulted in the cosine term in the bosonized theory. That is, we have the correspondence

$$m\bar{\psi}\psi \to 2z \cos\left(\sqrt{4\pi}\phi\right), \tag{3.76}$$

where we used that

$$g = \Phi_0 m_A = \sqrt{4\pi} \tag{3.77}$$

in the Schwinger model. Surprisingly, (3.76) is the remaining bosonization rule of (2.78). Instanton contributions, which were suppressed in the massless model, generate the interaction term in the bosonized form of the massive Schwinger model. We note that even in the absence of the anomaly, implying that the mass term in (3.75) would be absent, the bosonized theory would still be massive. This follows from noting that the expansion of the cosine term contains a term quadratic in ϕ.

3.4 Ginzburg-Landau Model

In this section, we give arguments as to why the Ginzburg-Landau model of a three-dimensional superconductor is not well suited to establish the critical properties of a type-II superconductor at zero external field.

The model is defined by the Hamiltonian

$$\mathcal{H} = |(\nabla - 2ie\mathbf{A})\phi|^2 + m_\phi^2 |\phi|^2 + \lambda |\phi|^4 + \frac{1}{2}(\nabla \times \mathbf{A})^2 + \frac{1}{2\alpha}(\nabla \cdot \mathbf{A})^2, \tag{3.78}$$

where the coefficients $2e$ and m_ϕ are the electric charge and mass of the complex ϕ-field, while λ is a coupling constant. The mass term changes sign at the critical temperature T_c. We have added a gauge-fixing term with parameter α. To acquire a physical understanding of what the Hamiltonian (3.78) describes, we bring to the attention the well-know fact [17] that a $|\phi|^4$-theory gives a field theoretic description of a gas of closed loops with contact repulsion. This equivalence is based on Feynman's observation [145] that the Green function

$$G(\mathbf{x}) = \int_\mathbf{k} \frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m_\phi^2} \tag{3.79}$$

of the free theory with positive mass term,

$$\mathcal{H}_0 = |\nabla\phi|^2 + m_\phi^2 |\phi|^2, \tag{3.80}$$

can be expressed as a path integral. (In contrast to the previous two chapters, we in this and the following chapter denote the free Green function by G without an index 0.) To this end, the Schwinger proper-time method is invoked to write the right-hand side of (3.79) as an integral over the proper time τ [146]:

$$\begin{aligned} G(\mathbf{x}) &= \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau m_\phi^2} \int_{\mathrm{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{e}^{-\tau \mathbf{k}^2} \\ &= \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau m_\phi^2} \left(\frac{1}{4\pi\tau}\right)^{3/2} \mathrm{e}^{-\frac{1}{4}\mathbf{x}^2/\tau}, \end{aligned} \tag{3.81}$$

where we used the identity

$$\frac{1}{a} = \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\tau a}. \tag{3.82}$$

According to the path-integral formulation of quantum mechanics [147], the right-hand side of (3.81) can be represented as a sum over all paths of a relativistic particle with mass m_ϕ running from 0 at imaginary time 0 to $\mathbf{x}$ at time τ:

$$G(\mathbf{x}) = \int_0^\infty \mathrm{d}\tau \int_{\mathbf{x}(0)=0}^{\mathbf{x}(\tau)=\mathbf{x}} \mathcal{D}\mathbf{x}(\tau') \mathrm{e}^{-S_0}. \tag{3.83}$$

The (Euclidean) action

$$S_0 = \int_0^\tau \mathrm{d}\tau' \left[\tfrac{1}{4}\dot{\mathbf{x}}^2(\tau') + m_\phi^2\right], \tag{3.84}$$

with $\dot{\mathbf{x}}(\tau) = \mathrm{d}\mathbf{x}(\tau)/\mathrm{d}\tau$, is an elusive representation of (m_ϕ times) the length L of the path

$$m_\phi L = m_\phi \int_0^\tau \mathrm{d}\tau' \sqrt{\dot{\mathbf{x}}^2(\tau')}, \tag{3.85}$$

with m_ϕ now being interpreted as the line tension. To establish this connection we consider the canonical momentum obtained from (3.85)

$$\mathbf{p} = m_\phi \frac{\dot{\mathbf{x}}}{\sqrt{\dot{\mathbf{x}}^2}} \tag{3.86}$$

which is seen to satisfy the constraint

$$\mathbf{p}^2 = m_\phi^2. \tag{3.87}$$

To incorporate this constraint we go over to the Hamilton formalism and write instead of (3.85)

$$m_\phi L \to \int_0^\tau \mathrm{d}\tau' \left[\mathbf{p}\cdot\dot{\mathbf{x}} - \alpha(\mathbf{p}^2 - m_\phi^2)\right] \tag{3.88}$$

where the constraint is implemented with the help of the Lagrange multiplier α. The path integral is now over phase space. Because the action is invariant under reparameterization, i.e., under the transformation $\tau \to \bar{\tau}(\tau)$, this multiplier can be given a fixed

value; $\alpha = 1$, for example. The integral over $\mathbf{p}$ is Gaussian and immediately yields the action (3.84).

The partition function (with fields and coupling constants rescaled such that no explicit temperature dependence appears in the Boltzmann factor)

$$Z_0 = \int \mathrm{D}\phi^* \mathrm{D}\phi \, \exp\left(- \int_{\mathbf{x}} \mathcal{H}_0 \right), \tag{3.89}$$

involves only closed paths:

$$\begin{aligned} \ln(Z_0) &= -\ln[\mathrm{Det}(\mathbf{p}^2 + m_\phi^2)] = -\mathrm{Tr}\ln(\mathbf{p}^2 + m_\phi^2) \\ &= \int_0^\infty \frac{d\tau}{\tau} \mathrm{e}^{-\tau m_\phi^2} \int_{\mathbf{k}} \mathrm{e}^{-\tau \mathbf{k}^2} = \int_0^\infty \frac{d\tau}{\tau} \oint \mathcal{D}\mathbf{x}(\tau') \mathrm{e}^{-S_0}, \end{aligned} \tag{3.90}$$

with S_0 the action (3.84).

The $|\phi|^4$-interaction in the Ginzburg-Landau model results in an additional term

$$S_\lambda = -\lambda \int_0^\tau \mathrm{d}\tau' \mathrm{d}\tau'' \, \delta\left[\mathbf{x}(\tau') - \mathbf{x}(\tau'')\right] \tag{3.91}$$

in the action, which gives an extra weight each time two loops—one parameterized by τ' and one by τ''—intersect. Physically, it represents a repulsive contact interaction between loops. Finally, the coupling of the field ϕ to the magnetic vector potential $\mathbf{A}$ via the electric current

$$\mathbf{j}_e = -2ei\phi^* \overleftrightarrow{\nabla} \phi - 2(2e)^2 \mathbf{A}|\phi|^2 \tag{3.92}$$

with a charge $2e$, results in the additional term

$$S_e = 2ie \int_0^\tau \mathrm{d}\tau' \, \dot{\mathbf{x}} \cdot \mathbf{A}[\mathbf{x}(\tau')], \tag{3.93}$$

showing that the loops described by the Ginzburg-Landau model are electric current loops.

On entering the superconducting phase, characterized by a negative mass term ($m_\phi^2 < 0$), the electric current loops proliferate and $|\phi|$ develops a vacuum expectation value. In the London limit, the fluctuations in the modulus of ϕ are neglected and the field is written as

$$\phi(\mathbf{x}) = \frac{\bar{\phi}}{\sqrt{2}} \mathrm{e}^{i\varphi(\mathbf{x})}, \tag{3.94}$$

with $\bar{\phi}$ a constant. Formally, this corresponds to taking the limit $m_\phi^2 \to -\infty$ and $\lambda \to \infty$ in the Hamiltonian such that

$$|\bar{\phi}|^2 = -\frac{m_\phi^2}{\lambda} \tag{3.95}$$

is finite. Physically, $|\bar{\phi}|^2$ denotes the mass density $\bar{\rho}_\mathrm{s}$ of the superconducting electrons.

Let us first define an order parameter describing the superconducting state. The field ϕ as it stands can not play this role because it is not gauge invariant. But it can be used to construct a gauge-invariant Mandelstam-like operator in the following way

$$W(L_{\mathbf{z}}) = \exp\left\{ i \left[\varphi(\mathbf{z}) + \int_{\mathbf{x}} \mathbf{A}(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x}) \right] \right\}, \tag{3.96}$$

where φ is the phase of ϕ and $\mathbf{J}$ describes an external current line originating in $\mathbf{z}$,

$$\nabla \cdot \mathbf{J}(\mathbf{x}) = 2e\delta(\mathbf{x} - \mathbf{z}), \tag{3.97}$$

or

$$J_i(\mathbf{x}) = 2e \int_{L_{\mathbf{z}}} \mathrm{d}y_i \delta(\mathbf{x} - \mathbf{y}). \tag{3.98}$$

Here, $L_{\mathbf{z}}$ denotes a path running from $\mathbf{z}$ to infinity. The second term in (3.96) is incorporated to render the operator gauge invariant. Indeed, under a gauge transformation

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}(\mathbf{x}) + \nabla\alpha(\mathbf{x}), \quad \varphi(\mathbf{z}) \to \varphi(\mathbf{z}) + 2e\alpha(\mathbf{z}), \tag{3.99}$$

the operator $W(L_{\mathbf{z}})$ is invariant

$$W(L_{\mathbf{z}}) \to W(L_{\mathbf{z}}) \exp\left[i\alpha(\mathbf{z}) + i \int_{\mathbf{x}} \nabla\alpha(\mathbf{x}) \cdot \mathbf{J}(\mathbf{x}) \right] = W(L_{\mathbf{z}}), \tag{3.100}$$

where in the last step we performed an integration by parts. The gauge invariance of $W(L_{\mathbf{z}})$ can be made more explicit by writing it in the equivalent form

$$W(L_{\mathbf{z}}) = \exp\left[-\frac{i}{2e} \int_{\mathbf{x}} (\nabla\varphi - 2e\mathbf{A}) \cdot \mathbf{J} \right]. \tag{3.101}$$

To show that this operator indeed is the order parameter of the superconducting state, we calculate the correlation function

$$\langle W(L_{\mathbf{z}}) W^*(L_{\bar{\mathbf{z}}}) \rangle = \int \mathbf{D}\mathbf{A}\mathbf{D}\varphi \, W(L_{\mathbf{z}}) W^*(L_{\bar{\mathbf{z}}}) \, \exp\left(- \int_{\mathbf{x}} \mathcal{H} \right), \tag{3.102}$$

obtained by introducing besides an external current line originating in $\mathbf{z}$ also one terminating at $\bar{\mathbf{z}}$. The Hamiltonian $\mathcal{H}$ is the Ginzburg-Landau Hamiltonian (3.78) in the London limit. Since both integrations are Gaussian, they are easily carried out with the result [148]

$$\begin{aligned} &\langle W(L_{\mathbf{z}}) W^*(L_{\bar{\mathbf{z}}}) \rangle = \\ &\exp\left\{ -\frac{1}{2} \int_{\mathbf{x},\mathbf{y}} \left[\frac{1}{m_A^2} \rho(\mathbf{x}) \, G(\mathbf{x} - \mathbf{y}) \, \rho(\mathbf{y}) + J_i(\mathbf{x}) \, G(\mathbf{x} - \mathbf{y}) \, J_i(\mathbf{y}) \right] \right\}, \end{aligned} \tag{3.103}$$

where $\rho(\mathbf{x}) = 2e[\delta(\mathbf{x} - \mathbf{z}) - \delta(\mathbf{x} - \bar{\mathbf{z}})]$ denotes the electric current source and sink, and

$$G(\mathbf{x}) = \int_{\mathbf{k}} \frac{\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + m_A^2} = \frac{1}{4\pi} \frac{\mathrm{e}^{-m_A|\mathbf{x}|}}{|\mathbf{x}|} \tag{3.104}$$

is the scalar Green function, with

$$m_A = 2e|\bar{\phi}| \tag{3.105}$$

the photon mass.

In the superconducting phase, the electric current lines proliferate and carry no energy; they are of no physical significance, only the endpoints are. The external current $\mathbf{J}$ can then be written as a gradient of a potential U,

$$\mathbf{J} = -\nabla U, \tag{3.106}$$

with $\nabla^2 U(\mathbf{x}) = -\rho(\mathbf{x})$. In this way, the correlation function becomes

$$\langle W(\mathbf{z}) W^*(\bar{\mathbf{z}})\rangle = \exp\left[-\frac{1}{2m_A^2}\int_{\mathbf{x},\mathbf{y}} \rho(\mathbf{x}) G_0(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})\right], \tag{3.107}$$

where $G_0(\mathbf{x}-\mathbf{y})$ is the gapless scalar Green function obtained from the massive one (3.104) by taking the limit $m_A \to 0$. For $\mathbf{x} = \mathbf{y}$ we have a diverging self-interaction which is irrelevant and can be eliminated by defining a renormalized operator $W_{\rm r}$ via

$$W_{\rm r}(\mathbf{z}) = W(\mathbf{z}) \exp\left[\frac{1}{2|\bar{\phi}|^2} G(0)\right]. \tag{3.108}$$

We then find for this operator

$$\langle W_{\rm r}(\mathbf{z}) W_{\rm r}^*(\bar{\mathbf{z}})\rangle = \exp\left(\frac{1}{4\pi|\bar{\phi}|^2}\frac{1}{|L_{\mathbf{z}\bar{\mathbf{z}}}|}\right), \tag{3.109}$$

where $L_{\mathbf{z}\bar{\mathbf{z}}}$ is a path connecting the source at $\mathbf{z}$ with the sink at $\bar{\mathbf{z}}$, and $|L_{\mathbf{z}\bar{\mathbf{z}}}|$ is the length of the path. For large separation this correlation function tends to one, implying that the operator $W_{\rm r}(\mathbf{z})$ develops an expectation value in the superconducting phase.

It is important to realize that this operator is gauge invariant. Frequently, the superconducting phase is referred to as a phase of spontaneously broken gauge symmetry [149]. This should, however, not be taken too literally. A celebrated theorem due to Elitzur [150] states that a local gauge symmetry can never be spontaneously broken. An operator that by developing a nonzero vacuum expectation value would spontaneously break the gauge symmetry does not exist. To break the symmetry, the operator must transform under the gauge group, but then it cannot develop an expectation value because only gauge-invariant objects can.

In the normal phase, electric current lines carry energy and therefore become physical. The form (3.106) is then no longer applicable and we have to use Eq. (3.98) instead. We shall assume that the current lines have a certain width of the order $1/|m_\psi|$, where the mass m_ψ will be identified later on in this chapter (see page 106). To prevent infrared divergences, we give the magnetic vector potential $\mathbf{A}$ a small mass μ. The correlation function (3.103) then becomes after renormalization

$$\langle W_{\rm r}(L_{\mathbf{z}}) W_{\rm r}^*(L_{\bar{\mathbf{z}}})\rangle = \exp(-M_W|L_{\mathbf{z}\bar{\mathbf{z}}}|)\exp\left[\frac{1}{4\pi}\left(\frac{2e}{\mu}\right)^2 \frac{{\rm e}^{-\mu|L_{\mathbf{z}\bar{\mathbf{z}}}|}}{|L_{\mathbf{z}\bar{\mathbf{z}}}|}\right], \tag{3.110}$$

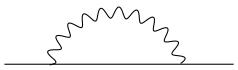


Figure 3.2: One-loop mass correction. The straight and wiggly lines represent the ϕ- and $\mathbf{A}$-field Green functions, respectively.

where now $L_{\mathbf{z}\bar{\mathbf{z}}}$ denotes the shortest path connecting the source at $\mathbf{z}$ with the sink at $\bar{\mathbf{z}}$, while $|L_{\mathbf{z}\bar{\mathbf{z}}}|$ denotes the length and M_W the line tension of the current line,

$$M_W = \frac{(2e)^2}{4\pi} \ln\left(\frac{|m_\psi|}{\mu}\right). \tag{3.111}$$

We see that in the limit $|L_{\mathbf{z}\bar{\mathbf{z}}}| \to \infty$, the expectation value vanishes. Although the operator W distinguishes the superconducting phase from the normal phase, having a zero expectation value in the normal phase and a nonzero one in the superconducting phase, it is not an order parameter in the strict sense of Landau in that it makes no symmetry statement.

A few remarks are in order. First, the line tension M_W diverges when the artificially introduced photon mass μ is taken to zero. This means that the current line connecting the source and sink becomes infinitely heavy. Isolated sources and sinks can therefore not exist in the normal phase. They are confined in neutral configurations of tightly bound pairs. In Minkowski spacetime, where an electric current line becomes the worldline of a charged particle, the energy (3.111) denotes the infrared-diverging selfenergy [151] of that particle obtained from evaluating the Feynman graph depicted in Fig. 3.2. Second, the alert reader might wonder how the result (3.109) obtained in the normal phase is connected with Eq. (3.110) obtained in the superconducting phase. In particular, the line tension M_W of the current line seems not to vanish at the critical temperature, implying that the first exponential function in (3.110) would survive the transition to the superconducting phase, and that the operator W would vanish there. This is in apparent contradiction with previous statements that in the superconducting phase, current lines carry no energy and proliferate. This issue will be addressed below (see page 106) after we applied a duality transformation to the Ginzburg-Landau model.

We continue to investigate the phase transition by taking into account fluctuations. The one-loop effective potential is readily calculated in the London limit (3.94). In the gauge $\nabla \cdot \mathbf{A} = 0$, one obtains

$$\mathcal{V}_{\rm eff}(\bar{\phi}) = \int_{k<\Lambda} \ln(k^2 + m_A^2) = -\frac{4}{3\pi} e^3 |\bar{\phi}|^3. \tag{3.112}$$

We regularized the integral by introducing an ultraviolet cutoff Λ. An irrelevant ultraviolet divergence of the type $\Lambda^3 \log \Lambda$ is ignored, while a divergence of the type $\Lambda|\bar{\phi}|^2$ is absorbed into a renormalization of the mass parameter, as usual. The effective potential $\mathcal{V}_{\rm eff}(\bar{\phi})$ has to be added to the tree potential $\mathcal{V}_0 = \frac{1}{2} m_\phi^2 |\bar{\phi}|^2 + \frac{1}{4}\lambda |\bar{\phi}|^4$. It results in a 1st-order phase transition taking place at a temperature T_1 *above* the temperature $T_{\rm c}$ where the mass term of the Ginzburg-Landau model changes sign (see Fig. 3.3). The (positive) value of m_ϕ^2 at the 1st-order transition point, obtained by equating the sum

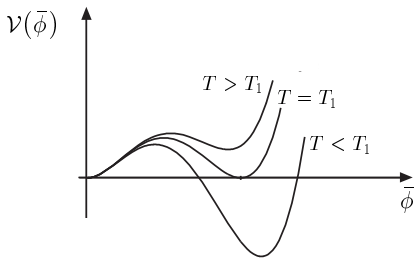


Figure 3.3: Effective potential up to one-loop order.

of the tree and the one-loop potential and also its first derivative with respect to $\bar{\phi}$ to zero, is given by [152]

$$m_\phi^2 = \frac{32}{9\pi^2}\frac{e^6}{\lambda}. \tag{3.113}$$

At small values of the so-called Ginzburg-Landau parameter $\kappa_{\rm GL}$, which is defined as the ratio of the two mass scales in the theory,

$$\kappa_{\rm GL} = \frac{|m_\phi|}{m_A} = \frac{\lambda_{\rm L}}{\xi} = \frac{\lambda}{e^2}, \tag{3.114}$$

the reliability of this perturbative result has been verified numerically [153]. It has been argued by Kleinert [154] that for larger values this result is, however, not meaningful anymore. At some critical value of the Ginzburg-Landau parameter $\kappa_{\rm GL}$, the 1st-order transition goes over into a 2nd-order transition.

Let us investigate the superconducting-to-normal transition further by applying renormalization-group analysis to the Ginzburg-Landau model. This was first done by Halperin, Lubensky, and Ma [155] in an $\epsilon(=4-d)$-expansion around the upper critical dimension $d=4$. We shall instead use the fixed-dimension approach of Parisi [156, 157]. The model (3.78) involves only a single complex field. However, for reasons that will become clear when we proceed, it is often generalized to contain $2n$ real components:

$$\phi \to \boldsymbol{\phi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \vdots \\ \phi_{2n-1} + i\phi_{2n} \end{pmatrix}. \tag{3.115}$$

The quantities appearing in (3.78) should be given an index 0 to indicate that they are bare quantities which will change in the process of renormalization. The renormalized fields and parameters are related to the bare ones via

$$\mathbf{A} = Z_A^{-1/2}\mathbf{A}^0, \quad e = Z_A^{1/2}e_0, \quad \boldsymbol{\phi} = Z_\phi^{-1/2}\boldsymbol{\phi}^0, \quad \lambda = Z_\lambda^{-1}Z_\phi^2\,\lambda_0. \tag{3.116}$$

We shall work at criticality and to that end set the mass m_ϕ to zero. To avoid infrared

divergences, diagrams are evaluated at finite external momentum κ. The renormalization factors can be used to define the following renormalization-group functions

$$\gamma_A = \kappa\frac{\partial}{\partial\kappa}\ln(Z_A)\bigg|_0, \quad \gamma_\phi = \kappa\frac{\partial}{\partial\kappa}\ln(Z_\phi)\bigg|_0, \tag{3.117}$$

where the subscript 0 is to indicate that the bare coupling constants e_0, λ_0 are kept fixed. The dimensional parameter κ plays the role of renormalization group scale parameter. The critical points $\hat{e}_*^2$ and $\hat{\lambda}_*$ are determined by the zeros of the renormalization-group beta functions,

$$\beta_{e^2} = \kappa\frac{\partial}{\partial\kappa}\hat{e}^2\bigg|_0, \quad \beta_\lambda = \kappa\frac{\partial}{\partial\kappa}\hat{\lambda}\bigg|_0, \tag{3.118}$$

where $\hat{e}^2 = e^2\kappa^{d-4}$ and $\hat{\lambda} = \lambda\kappa^{d-4}$ are the dimensionless coupling constants. When evaluated at the critical point, the renormalization-group functions γ_A and γ_ϕ yield the anomalous dimension η_A and η_ϕ of the **A**- and ϕ-field, respectively.

The various renormalization factors are readily evaluated in arbitrary dimension $2 < d < 4$, using the fixed-dimension approach of Parisi [156, 157]. To the one-loop order we find

$$Z_A = 1 - n\frac{c(d)}{d-1}\hat{e}^2 \tag{3.119}$$

$$Z_\phi = 1 + c(d)[(d-1) - \alpha(d-3)]\hat{e}^2 \tag{3.120}$$

$$Z_\lambda = 1 + c(d)\left[(2n+8)\hat{\lambda} - 2\alpha(d-3)\hat{e}^2 + \tfrac{1}{4}d(d-1)\hat{e}^4/\hat{\lambda}\right], \tag{3.121}$$

where $c(d)$ stands for the one-loop integral

$$c(d) = \int_{\mathbf{k}} \frac{1}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2}\bigg|_{\mathbf{q}^2=1} = \frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{(4\pi)^{d/2}\Gamma(d-2)}. \tag{3.122}$$

In deriving this we used analytic regularization to handle the ultraviolet divergences. With the one-loop expressions for the renormalization factors, we obtain for the beta functions

$$\beta_{e^2} = (d-4)\left[\hat{e}^2 - n\frac{c(d)}{d-1}\hat{e}^4\right] \tag{3.123}$$

$$\beta_\lambda = (d-4)\left\{\hat{\lambda} - c(d)\left[(2n+8)\hat{\lambda}^2 - 2(d-1)\hat{\lambda}\hat{e}^2 + \tfrac{1}{4}d(d-1)\hat{e}^4\right]\right\}. \tag{3.124}$$

Note that while Z_λ and Z_ϕ depend on the gauge-fixing parameter α, the function β_λ extracted from them is independent thereof. We expect this cancelation of gauge-dependence in β_λ to persist in all orders of perturbation theory. Although the beta functions are gauge independent, the locations of the fixed points are not universal. They depend on the regularization chosen as well as on the renormalization prescription. In the limit $d \to 4 - \epsilon$ our formulas reduce to the know form [155, 158].

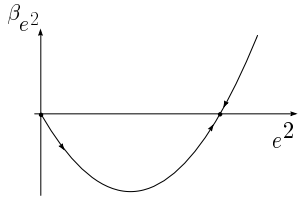


Figure 3.4: Schematic representation of the β_{e^2}-function.

Remarkably, β_{e^2} is independent of the self-coupling λ. By itself it gives rise to two fixed points (see Fig. 3.4):

$$\hat{e}_*^2 = \begin{cases} 0 \\ (d-1)/nc(d). \end{cases} \tag{3.125}$$

The neutral fixed point is ultraviolet stable, while the charged fixed point is infrared stable in the e^2-direction. Since also Z_A as given in Eq. (3.119) and thus γ_A are independent of λ, the anomalous dimension η_A of the gauge field can be calculated without any knowledge of the location of the fixed points along the λ-direction. For the infrared-stable (IR) fixed point one finds at the one-loop order [155]

$$\eta_A = \gamma_A\big|_{\hat{e}_*^2} = 4 - d. \tag{3.126}$$

It was pointed out by Herbut and Tešanović [159] that this is an exact result which follows directly from the definition of the β_{e^2}-function in terms of the renormalization factor Z_A

$$\beta_{e^2} = \kappa \frac{\partial}{\partial \kappa} \hat{e}^2\big|_0 = \hat{e}^2[(d-4) + \gamma_A]. \tag{3.127}$$

That is to say, there are no higher-order corrections to the one-loop result (3.126). We have checked this explicitly to the two-loop order in the $\epsilon(= 4 - d)$-expansion. The anomalous dimension is such that the dimension d_A of the gauge field, which by naive power counting equals $\frac{1}{2}(d-2)$, is exactly unity,

$$d_A = 1 \qquad \text{for} \qquad 2 < d < 4. \tag{3.128}$$

It is as if we were in the upper critical dimension $d = 4$.

Since the result is independent of the number $2n$ of field components contained in the theory, this conclusion can be confirmed in the $1/n$-expansion. To leading order in $1/n$, the inverse gauge field propagator reads in the gauge $\nabla \cdot \mathbf{A} = 0$ and for $2 < d < 4$

$$\Pi_{ij}(\mathbf{q}) = -\frac{c(d)}{d-1} P_{ij}(\mathbf{q}) |\mathbf{q}|^{d-2}, \tag{3.129}$$

with $P_{ij}(\mathbf{q})$ the transverse projection operator

$$P_{ij}(\mathbf{q}) = \delta_{ij} - \frac{q_i q_j}{\mathbf{q}^2}. \tag{3.130}$$

The exponent $d-2$ of the momentum in (3.129) equals $d-2d_A$, so that we again arrive at the conclusion that $d_A = 1$ in $2 < d < 4$.

It implies that near the charged fixed point, the electric charge scales with the correlation length ξ in a way one expects from naive power counting

$$e^2 = Z_A e_0^2 \sim \xi^{d-4} \tag{3.131}$$

since Z_A scales as $\xi^{-\eta_A}$.

The exact result $d_A = 1$ has far reaching consequences as it implies that the term $(\nabla \times \mathbf{A})^2$ in the Hamiltonian, which by naive power counting is marginal, has dimension 4. It is therefore an irrelevant operator for all d below the upper critical dimension and can be omitted. Formally, this corresponds to taking the limit $e \to \infty$ in the Ginzburg-Landau model as can be seen after a rescaling of the gauge field $\mathbf{A} \to \mathbf{A}/e$. Without the term $(\nabla \times \mathbf{A})^2$, the Ginzburg-Landau model resembles the CP^{n-1} model,

$$\mathcal{L}_{\mathrm{CP}} = |(\nabla - i\mathbf{A})\boldsymbol{\phi}|^2 , \tag{3.132}$$

which has the additional constraint

$$|\boldsymbol{\phi}|^2 = 1. \tag{3.133}$$

As regards to critical exponents this difference is however irrelevant. Indeed, the well-studied O(n) $\boldsymbol{\phi}^4$-model and its nonlinear version, the O(n) nonlinear sigma model where the fields satisfy a condition like (3.133), are generally accepted to be in the same universality class [160]. Whereas the linear model is usually investigated in an ϵ-expansion around the upper critical dimension ($d = 4$), the nonlinear model is often treated in an expansion around the lower critical dimension ($d = 2$), so a direct comparison is not possible. But the $1/n$-expansion, being the same for both theories and applicable in arbitrary dimension $2 < d < 4$, can be used to bridge these two regions. The results obtained in this way for d close to the upper and lower critical dimension are identical to those obtained in the respective ϵ-expansions for large n.

The $1/n$-expansion has also been employed to argue that the Ginzburg-Landau model with $2n$-components and the CP^{n-1} model, too, are in the same universality class [161, 162]. Our conclusion that below the upper critical dimension, the term $(\nabla \times \mathbf{A})^2$ in the Ginzburg-Landau model is irrelevant gives additional support to this conjecture.

For a superconductor, having one complex field ($n = 1$), this implies that the relevant model is the CP^0 model. Although the $\epsilon(= d-2)$-expansion [161] predicts a nontrivial 2nd-order transition, this model has no degrees of freedom. Physically, this means that the charged degrees of freedom are irrelevant at the phase transition [163]. In itself this is no reason to reject the Ginzburg-Landau model as a basis for studying the phase transition—the "unphysical" limit $n \to 0$ of the O(n) model describes the long-distance behavior of a flexible polymer. But what is more worrying is that vortices, which we saw to be of paramount importance in two dimensions, are not accounted for in the discussion so far. In addition, there are technical problems with using the Ginzburg-Landau model to study the critical properties of the phase transition.

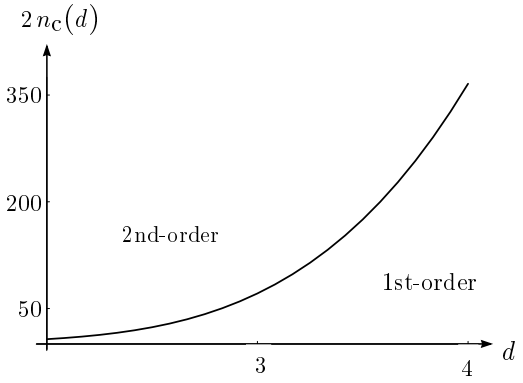


Figure 3.5: The critical number of components $2n_c$ as a function of the dimensionality d below which the one-loop calculation predicts a 1st-order transition.

Namely, the two beta functions yield no IR fixed point unless the number of field components is taken large enough. Below a critical number $n_c(d)$, determined by the condition that

$$\Delta_d = 2\sqrt{n^2 - 2(d-2)(d-1)^2(d+1)n - 4(d-1)^3(d+1)} \tag{3.134}$$

be real, the system has no IR fixed point and the one-loop calculation predicts a 1st-order transition. In Fig. 3.5 this number is plotted as function of the dimensionality d. We see that the number decreases when the dimensionality is reduced. More precisely,

$$2n_{\rm c}(4) = 12(15 + 4\sqrt{15}) \approx 365.9 \tag{3.135}$$

$$2n_{\rm c}(3) = 16(2 + \sqrt{6}) \approx 71.2 \tag{3.136}$$

$$2n_{\rm c}(2) = 4\sqrt{3} \approx 6.9, \tag{3.137}$$

where the $d = 4$-result is due to Halperin, Lubensky, and Ma [155]. The figure also shows that for a fixed number of field components, 2nd-order behavior is favored when $d \to 2$, while in the opposite limit, $d \to 4$, 1st-order behavior is favored. However, even close to the lower critical dimension, where the CP^{n-1} model predicts a 2nd-order phase transition all the way down to $n = 1$ [161], the Ginzburg-Landau model still requires more than $2\sqrt{3} \approx 3.5$ complex fields in order to have an IR fixed point. The status of the results obtained from the Ginzburg-Landau model in low-order perturbation theory is therefore not clear.

Given the two-dimensional results, showing the importance of vortices, it is natural to consider the dual theory of the three-dimensional Ginzburg-Landau model as an alternative to study the phase transition. As before, the dual formulation focuses on the vortices in the system, which in three dimensions are line defects. It will again turn out that the dual formulation is one directly in terms of physical variables and for that reason involves only a global, not a local, symmetry. We will see that the phase transition of a type-II superconductor at zero external field becomes more tractable in the dual formulation than in the original Ginzburg-Landau formulation.

3.5 Dual Ginzburg-Landau Model

In this section, we apply a duality transformation to the Ginzburg-Landau model. The basic idea of this approach originates from three-dimensional lattice studies carried out two decades ago [164, 165, 166, 167]. These studies were instigated by the success of the Kosterlitz-Thouless theory of the phase transition in a superfluid film [130, 131], which we discussed in Sec. 3.1. The three-dimensional lattice studies of the neutral [164] and charged xy-model [165, 166] were aimed at obtaining a dual description in terms of vortices. Following a suggestion by Helfrich and Müller [168], Dasgupta and Halperin [169] carried out a lattice simulation of the superconductor-to-normal phase transition in the dual formulation. Their study revealed that for small values of the electric charge e, which implies large values for the Ginzburg-Landau parameter (3.114), the transition was 2nd-order with xy-exponents. A detailed account of these matters as well as an extensive list of references to the literature can be found in Ref. [17].

Another development underscoring the importance of the dual approach to the Ginzburg-Landau model was initiated in Ref. [170]. The basic observation was that since a local gauge symmetry can never be broken [150], a local gauge description of a phase transition is not feasible. It was argued that the three-dimensional Ginzburg-Landau theory contains, in addition to the local gauge symmetry, another *global* U(1) symmetry. When considered in $2+1$ space-time dimensions with a Minkowski metric, this symmetry is generated by the magnetic flux operator. It was demonstrated that this symmetry is broken in the normal phase, while it is unbroken in the superconducting phase. A genuine order parameter in the sense of Landau was given, and it was shown that the massless photon of the normal phase is the Goldstone mode associated with the broken flux symmetry.

To see in which regime magnetic vortices or Abrikosov flux tubes are important, we introduce a magnetic monopole at some point $\mathbf{z}$ inside the system. A monopole is a source of magnetic flux. Due to the Meissner effect, the flux lines emanating from the monopole are squeezed into a flux tube. In this way we have created a magnetic vortex at zero external field. Electrodynamics in the presence of a monopole was first described by Dirac [171] who argued that the combination

$$\nabla \times \mathbf{A}(\mathbf{x}) - \mathbf{B}^{\mathrm{P}}(\mathbf{x}) \tag{3.138}$$

is the physical local magnetic induction $\mathbf{h}$. The subtracted plastic field

$$B_i^{\mathrm{P}}(\mathbf{x}) = \Phi_0 \int_{L_{\mathbf{z}}} \mathrm{d}y_i \, \delta(\mathbf{x}-\mathbf{y}), \tag{3.139}$$

with $\Phi_0 = \pi/e$ the magnetic flux quantum, removes the field of the so-called Dirac string running along some path $L_{\mathbf{z}}$ from the location $\mathbf{z}$ of the monopole to infinity. On account of Stokes' theorem, the plastic field satisfies the equation

$$\nabla \cdot \mathbf{B}^{\mathrm{P}}(\mathbf{x}) = \Phi_0 \, \delta(\mathbf{x}-\mathbf{z}). \tag{3.140}$$

In the presence of the monopole, the Ginzburg-Landau Hamiltonian becomes in the

London limit and after integrating out the phase field φ

$$\mathcal{H}^{\mathrm{P}} = \frac{1}{2}(\nabla \times \mathbf{A} - \mathbf{B}^{\mathrm{P}})^2 + \frac{1}{2}m_A^2 A_i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) A_j + \frac{1}{2\alpha}(\nabla \cdot \mathbf{A})^2, \quad (3.141)$$

where we gave $\mathcal{H}$ the superscript P to indicate the presence of the monopole. Let us first verify Dirac's assertion that the combination (3.138) describes a point monopole with its Dirac string removed and consider the field equation for the gauge field,

$$A_i(\mathbf{x}) = \int_{\mathbf{y}} G_{ij}(\mathbf{x}-\mathbf{y}) \left[\nabla \times \mathbf{B}^{\mathrm{P}}(\mathbf{y})\right]_j . \quad (3.142)$$

The gauge-field Green function G_{ij} appearing here is

$$Gij(\mathbf{x}) = \int_{\mathbf{k}} \left(\frac{\delta_{ij} - (k_i k_j)/\mathbf{k}^2}{\mathbf{k}^2 + m_A^2} + \alpha \frac{k_i k_j}{\mathbf{k}^4}\right) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}. \quad (3.143)$$

The local magnetic induction corresponding to the classical solution given in (3.142) is

$$\nabla \times \mathbf{A}(\mathbf{x}) - \mathbf{B}^{\mathrm{P}}(\mathbf{x}) = \Phi_0 \nabla G(\mathbf{x}-\mathbf{z}) - m_A^2 \int_{\mathbf{y}} G(\mathbf{x}-\mathbf{y}) \mathbf{B}^{\mathrm{P}}(\mathbf{y}), \quad (3.144)$$

where G is the Green function (3.104). The first term at the right-hand side corresponds to a screened Coulomb force generated by the monopole. The last term, which is only present in the superconducting phase where the Meissner effect is operating and $m_A \neq 0$, describes the magnetic vortex. If we calculate from the right-hand side of (3.144) the magnetic flux through a plane perpendicular to the Dirac string, we find that precisely one flux quantum pierces the surface in the negative direction

$$\int \mathrm{d}^2 x_i \left[\Phi_0 \partial_i G(\mathbf{x}-\mathbf{z}) - m_A^2 \int_{\mathbf{y}} G(\mathbf{x}-\mathbf{y}) B_i^{\mathrm{P}}(\mathbf{y})\right] = -\Phi_0. \quad (3.145)$$

Here, $\mathrm{d}^2 x_i$ denotes an element of the surface orthogonal to the Dirac string. Equation (3.145) confirms Dirac's assertion that the magnetic flux emanating from a monopole must be supplied by an infinitesimally thin string of magnetic dipoles and that in order to obtain the true local magnetic induction of a genuine point monopole, this string has to be subtracted. Whence, the $\mathbf{B}^{\mathrm{P}}$-term at the left-hand side of (3.144). While the Dirac string is immaterial in the normal phase, the last term at the right-hand side of (3.144) shows that it acquires physical relevance in the superconducting phase where it serves as the core of the Abrikosov flux tube [172].

The energy E_V of this configuration is obtained by substituting the solution (3.142) back into the Hamiltonian (3.141). It is divergent in the ultraviolet because in the London limit, where the mass $|m_\phi|$ of the ϕ-field is taken to be infinite, the vortices are considered to be ideal lines. For a finite mass, a vortex core has a typical width of the order of the coherence length $\xi = 1/|m_\phi|$. This mass therefore provides a natural ultraviolet cutoff to the theory. Omitting the (diverging) monopole self-interaction, one finds [172]

$$E_V = \frac{1}{2} g^2 \int_{L_{\mathbf{z}}} \mathrm{d}x_i \int_{L_{\mathbf{z}}} \mathrm{d}y_i \, G(\mathbf{x}-\mathbf{y}) = M_V \, |L_{\mathbf{z}}| \, , \quad (3.146)$$

with $g = \Phi_0 m_A$ a combination we also encountered in the two-dimensional charged models, $|L_{\mathbf{z}}|$ the (infinite) length of the flux tube, and [79]

$$M_V = \frac{1}{8\pi} g^2 \ln\left(\frac{|m_\phi|^2}{m_A^2}\right) = \frac{1}{4\pi} g^2 \ln(\kappa_{\rm GL}) \tag{3.147}$$

the line tension. The value $\kappa_{\rm GL} = 1/\sqrt{2}$ for the Ginzburg-Landau parameter (3.114) separates the type-II regime ($\kappa_{\rm GL} > 1/\sqrt{2}$), where isolated vortices can exist, from the type-I regime ($\kappa_{\rm GL} < 1/\sqrt{2}$), where a partial penetration of an external field is impossible. Remembering that $L_{\mathbf{z}}$ was the Dirac string, we see from (3.146) that it indeed becomes the core of a vortex in the superconducting phase.

To identify the regime where vortices are important, we convert the line tension M_V into a magnetic field via

$$H_{\rm c_1} = \frac{M_V}{\Phi_0}, \tag{3.148}$$

and compare it with the critical field

$$H_{\rm c} = \frac{1}{2\sqrt{2}\pi} g |m_\phi|, \tag{3.149}$$

obtained by equating the tree potential

$$\mathcal{V}_0 = \tfrac{1}{2} m_\phi^2 |\bar{\phi}|^2 + \tfrac{1}{4}\lambda |\bar{\phi}|^4 \tag{3.150}$$

to $-\frac{1}{2} H_{\rm c}^2$. The critical field $H_{\rm c_1}$ physically denotes the smallest value of an external field needed to excite vortices. The field $H_{\rm c}$, on the other hand, is a measure of the condensation energy which in turn sets the energy scale. It physically denotes the value of an external field at which a type-I superconductor can no longer resist the magnetic pressure and reverts to the normal phase characterized by a perfect penetration of the field. The ratio of the two fields

$$\frac{H_{\rm c_1}}{H_{\rm c}} = \frac{1}{\sqrt{2}} \frac{\ln(\kappa_{\rm GL})}{\kappa_{\rm GL}} \tag{3.151}$$

shows that for increasing $\kappa_{\rm GL}$, vortices become easier to excite. In other words, the deeper one enters the type-II regime, the more important vortex excitations become. Since these have been ignored in the calculation of the effective potential (3.112), the prediction of a 1st-order transition is only reliable at small $\kappa_{\rm GL}$ and breaks down at larger values of the Ginzburg-Landau parameter.

The above construct of inserting a monopole into the system can be used to define a so-called disorder parameter [17]. In contrast to the order parameter (3.96), a disorder parameter should develop an expectation value in the normal, not in the superconducting phase. The operator $V(L_{\mathbf{z}})$ describing the monopole with its emerging flux tube is easily obtained by noting that in the functional-integral approach, a given field configuration is weighted with a Boltzmann factor $\exp\left(-\int_{\mathbf{x}} \mathcal{H}^{\rm P}\right)$, where the Hamiltonian is given by (3.141). From this we infer that the explicit form of the operator is

$$V(L_{\mathbf{z}}) = \exp\left\{\int_{\mathbf{x}} \left[(\nabla \times \mathbf{A}) \cdot \mathbf{B}^{\rm P} - \tfrac{1}{2}\left(\mathbf{B}^{\rm P}\right)^2\right]\right\}. \tag{3.152}$$

We are interested in the correlation function $\langle V(L_{\mathbf{z}})V^*(L_{\bar{\mathbf{z}}})\rangle$, where $V^*(L_{\bar{\mathbf{z}}})$ describes an additional antimonopole brought into the system at $\bar{\mathbf{z}}$, with $L_{\bar{\mathbf{z}}}$ being the accompanying Dirac string running from infinity to $\bar{\mathbf{z}}$. Since all the integrals involved are Gaussian, this expectation value can be evaluated directly. However, we proceed in an indirect way to reveal some aspects of the nature of the dual theory and first linearize the functional integral over the gauge field by introducing an auxiliary field $\tilde{\mathbf{h}}$. In the gauge $\nabla \cdot \mathbf{A} = 0$, which corresponds to setting $\alpha = 0$, we find

$$\langle V(L_{\mathbf{z}})V^*(L_{\bar{\mathbf{z}}})\rangle = \int \mathrm{D}\mathbf{A}\mathrm{D}\tilde{\mathbf{h}} \exp\left\{\int_{\mathbf{x}}\left[-\frac{1}{2}\tilde{\mathbf{h}}^2 + i\tilde{\mathbf{h}}\cdot(\nabla\times\mathbf{A} - \mathbf{B}^{\mathrm{P}}) - \frac{m_A^2}{2}\mathbf{A}^2\right]\right\}, \tag{3.153}$$

where now $\nabla\cdot\mathbf{B}^{\mathrm{P}}(\mathbf{x}) = \Phi_0[\delta(\mathbf{x}-\mathbf{z}) - \delta(\mathbf{x}-\bar{\mathbf{z}})]$. To appreciate the physical relevance of the auxiliary field, let us consider its field equation

$$\tilde{\mathbf{h}} = i(\nabla\times\mathbf{A} - \mathbf{B}^{\mathrm{P}}) = i\mathbf{h}. \tag{3.154}$$

It tells us that apart from a factor i, $\tilde{\mathbf{h}}$ can be thought of as representing the local magnetic induction $\mathbf{h}$.

The integral over the vector potential is easily carried out by substituting the field equation for $\mathbf{A}$,

$$\mathbf{A} = \frac{i}{m_A^2}\nabla\times\tilde{\mathbf{h}}, \tag{3.155}$$

back into (3.153), with the result

$$\langle V(L_{\mathbf{z}})V^*(L_{\bar{\mathbf{z}}})\rangle = \int \mathrm{D}\tilde{\mathbf{h}}\, \exp\left\{-\frac{1}{2}\int_{\mathbf{x}}\left[\frac{1}{m_A^2}(\nabla\times\tilde{\mathbf{h}})^2 + \tilde{\mathbf{h}}^2\right] - i\int_{\mathbf{x}}\tilde{\mathbf{h}}\cdot\mathbf{B}^{\mathrm{P}}\right\}. \tag{3.156}$$

This shows that magnetic vortices described by a plastic field $\mathbf{B}^{\mathrm{P}}$ couple to the fluctuating massive vector field $\tilde{\mathbf{h}}$, with a coupling constant given by $g = \Phi_0 m_A = 2\pi|\bar{\phi}|$ as in two dimensions. As the temperature approaches the critical temperature from below, $\bar{\phi}$ tends to zero, so that the vortices decouple from $\tilde{\mathbf{h}}$. The finite penetration depth in the superconducting phase is reflected by the mass term of the $\tilde{\mathbf{h}}$-field (3.156).

After carrying out the integral over $\tilde{\mathbf{h}}$ in (3.156), we obtain for the correlation function

$$\langle V(L_{\mathbf{z}})V^*(L_{\bar{\mathbf{z}}})\rangle = \exp\left\{-\frac{1}{2}\int_{\mathbf{x},\mathbf{y}}\big[\,\rho_{\mathrm{m}}(\mathbf{x})\,G(\mathbf{x}-\mathbf{y})\,\rho_{\mathrm{m}}(\mathbf{y}) + m_A^2 B_i^{\mathrm{P}}(\mathbf{x})\,G(\mathbf{x}-\mathbf{y})\,B_i^{\mathrm{P}}(\mathbf{y})\big]\right\}, \tag{3.157}$$

where $\rho_{\mathrm{m}}(\mathbf{x}) = \Phi_0[\delta(\mathbf{x}-\mathbf{z}) - \delta(\mathbf{x}-\bar{\mathbf{z}})]$ is the monopole density. The first term in the argument of the exponential function contains a diverging monopole self-interaction for $\mathbf{x} = \mathbf{y}$. This divergence is irrelevant and can be eliminated by defining a renormalized operator

$$V_{\mathrm{r}}(L_{\mathbf{z}}) = V(L_{\mathbf{z}})\exp\left[\tfrac{1}{2}\Phi_0^2 G(0)\right]. \tag{3.158}$$

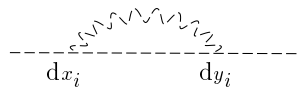


Figure 3.6: Biot-Savart interaction (wiggly line) between two line elements $\mathrm{d}x_i$ and $\mathrm{d}y_i$ of a magnetic vortex (straight line).

The second term in the argument is the most important one for our purposes. It represents a Biot-Savart interaction between two line elements $\mathrm{d}x_i$ and $\mathrm{d}y_i$ of the magnetic vortex (see Fig. 3.6). For the renormalized operators we find

$$\langle V_{\mathrm{r}}(L_{\mathbf{z}})V_{\mathrm{r}}^*(L_{\bar{\mathbf{z}}})\rangle = \exp(-M_V\,|L_{\mathbf{z}\bar{\mathbf{z}}}|)\exp\left(\frac{\Phi_0^2}{4\pi}\frac{\mathrm{e}^{-m_A|L_{\mathbf{z}\bar{\mathbf{z}}}|}}{|L_{\mathbf{z}\bar{\mathbf{z}}}|}\right), \tag{3.159}$$

where $L_{\mathbf{z}\bar{\mathbf{z}}}$ is the flux tube connecting the monopole at $\mathbf{z}$ with the antimonopole at $\bar{\mathbf{z}}$, and $|L_{\mathbf{z}\bar{\mathbf{z}}}|$ is its length. Initially, the two Dirac strings may run to any point at infinity. Due to the string tension, however, they join on the shortest path $L_{\mathbf{z}\bar{\mathbf{z}}}$ between the monopoles.

The result (3.159) is central to our line of arguments. It shows that the correlation function $\langle V_{\mathrm{r}}(L_{\mathbf{z}})V_{\mathrm{r}}^*(L_{\bar{\mathbf{z}}})\rangle$ behaves differently in the two phases [170, 173]. In the superconducting phase, where because of the Higgs mechanism $m_A \neq 0$, the first factor, which amounts to a confining linear potential between the monopole and antimonopole, dominates. As a result, the correlation function decays exponentially for distances larger than $1/M_V$:

$$\langle V_{\mathrm{r}}(L_{\mathbf{z}})V_{\mathrm{r}}^*(L_{\bar{\mathbf{z}}})\rangle \to 0. \tag{3.160}$$

This behavior is typical for an operator in a phase without gapless excitations. On the other hand, in the high-temperature phase, where $m_A = 0$, the confinement factor in the correlation function (3.159) disappears, while the argument of the second exponential turns into a pure Coulomb potential. The correlation function remains, consequently, finite for large distances:

$$\langle V_{\mathrm{r}}(L_{\mathbf{z}})V_{\mathrm{r}}^*(L_{\bar{\mathbf{z}}})\rangle \to 1. \tag{3.161}$$

By the cluster property of correlation functions this implies that the operator describing the finite vortex develops a vacuum expectation value. This signals a proliferation of magnetic vortices. Indeed, according to (3.147) the line tension M_V of a vortex vanishes at the transition point, where $\bar{\phi} \to 0$. It should be noted that it is the high-temperature phase and not the superconducting phase where $V_{\mathrm{r}}(L_{\mathbf{z}})$ develops an expectation value. Hence, it is indeed a disorder parameter.

It is interesting to consider the limit $m_A \to 0$ in (3.156), where the magnetic vortex decouples from the massive vector field. This limit yields the constraint $\nabla \times \tilde{\mathbf{h}} = 0$ which can be solved by setting $\tilde{\mathbf{h}} = \nabla\gamma$. The correlation function then takes the simple form

$$\langle V_{\mathrm{r}}(\mathbf{z})V_{\mathrm{r}}^*(\bar{\mathbf{z}})\rangle = \int \mathrm{D}\gamma \exp\left[-\frac{1}{2}\int_{\mathbf{x}}(\nabla\gamma)^2 + i\int_{\mathbf{x}}\gamma\rho_{\mathrm{m}}\right]. \tag{3.162}$$

In the absence of monopoles, the theory reduces to that of a free gapless mode γ that may be thought of as representing the magnetic scalar potential. This follows from combining the physical interpretation of the vector field $\mathbf{h}$ (3.154) with the equation $\tilde{\mathbf{h}} = \nabla\gamma$. Specifically,

$$\nabla\gamma = i(\nabla \times \mathbf{A} - \mathbf{B}^{\mathrm{P}}). \tag{3.163}$$

Inserting the explicit form for the monopole density, which is given by $\rho_{\mathrm{m}}(\mathbf{x}) = \Phi_0[\delta(\mathbf{x}-\mathbf{z}) - \delta(\mathbf{x}-\bar{\mathbf{z}})]$, we see that in terms of the field γ, the correlation function reads

$$\langle V_{\mathrm{r}}(\mathbf{z}) V_{\mathrm{r}}^*(\bar{\mathbf{z}}) = \left\langle \mathrm{e}^{i\Phi_0[\gamma(\mathbf{z})-\gamma(\bar{\mathbf{z}})]} \right\rangle. \tag{3.164}$$

This demonstrates that the operator $V_{\mathrm{r}}(L_{\mathbf{z}})$ describing the finite vortex, which was introduced in (3.152) via the singular plastic field $\mathbf{B}^{\mathrm{P}}$ (3.139), is now represented as an ordinary field. Since we are in the normal phase, where V_{r} develops a nonzero expectation value, the presence of the phase γ indicates that this expectation value breaks a global U(1) symmetry, with γ the ensuing Goldstone field. This point will be further clarified below.

We note that Eq. (3.164) reveals also that in the normal phase, the Dirac string looses its physical relevance, the right-hand side depending only on the end points $\mathbf{z}$ and $\bar{\mathbf{z}}$, not on the path $L_{\mathbf{z}\bar{\mathbf{z}}}$ connecting these points. The notion of a magnetic vortex is of no relevance in this phase because the vortices proliferate and carry no energy. There is also no nontrivial topology to assure their stability. This is the reason for omitting any reference to vortex lines in the argument of V in Eqs. (3.162) and (3.164).

We are now in a position to write down the dual theory of a three-dimensional superconductor. It features a grand-canonical ensemble of fluctuating closed magnetic vortices of arbitrary shape and length having a repulsive contact interaction. The loop gas can be described by a complex disorder $|\psi|^4$-theory. In addition, as our study of a single external vortex revealed, the magnetic vortices couple to the fluctuating vector field $\tilde{\mathbf{h}}$ with a coupling constant g. Hence, the dual theory is given by [174, 175, 17, 163, 148]

$$Z = \int \mathrm{D}\tilde{\mathbf{h}}\mathrm{D}\psi^*\mathrm{D}\psi \, \exp\left(-\int_{\mathbf{x}} \mathcal{H}_\psi\right) \tag{3.165}$$

with the Hamiltonian

$$\mathcal{H}_\psi = \frac{1}{2m_A^2}(\nabla \times \tilde{\mathbf{h}})^2 + \frac{1}{2}\tilde{\mathbf{h}}^2 + |(\nabla - i\Phi_0\tilde{\mathbf{h}})\psi|^2 + m_\psi^2|\psi|^2 + u|\psi|^4, \tag{3.166}$$

where the field ψ is minimally coupled to the massive vector field $\tilde{\mathbf{h}}$. Equation (3.166) is a description of the superconducting state in terms of physical variables: the field $\mathbf{n}$ describes the local magnetic induction, whereas ψ accounts for the loop gas of magnetic vortices. There are no other physical entities present in a superconductor. The dual theory has no local gauge symmetry because the vector field $\tilde{\mathbf{h}}$ is massive.

Although (3.165) was derived starting from the London limit, it is also relevant near the phase transition. The point is that integrating out the size fluctuations of the scalar field ϕ would only generate higher-order interaction terms. But these modifications do not alter the critical behavior of the theory.

The line tension M_V (3.147) appears in the dual theory as a one-loop on-shell mass correction to the mass m_ψ stemming from the graph depicted in Fig. 3.6, which we now interpret as a Feynman graph. The straight and wiggly lines represent the ψ- and $\tilde{\mathbf{h}}$-field correlation functions, respectively. We have used dashed lines to distinguish the Feynman graphs of the dual theory from those in the Ginzburg-Landau model.

A measure for the interaction strength of a massive vector field in three dimensions is given by the dimensionless parameter equal to the square of the coupling constant multiplied by the range of the interaction. For the dual theory this factor is $g^2/m_A \sim m_A/e^2$, which is the inverse of the strength of the electromagnetic gauge field $\mathbf{A}$ in the superconducting phase. This is a common feature of theories which are dual to each other.

Another notable property of the dual theory is that in the limit $e \to 0$ it changes into a local gauge theory [17],

$$\mathcal{H}_\psi \to \frac{1}{2}(\nabla \times \tilde{\mathbf{h}})^2 + |(\nabla - ig\tilde{\mathbf{h}})\psi|^2 + m_\psi^2|\psi|^2 + u|\psi|^4, \tag{3.167}$$

as can be checked by rescaling the dual field $\tilde{\mathbf{h}}$ in the Hamiltonian (3.166). By taking the limit $e \to 0$ in the Ginzburg-Landau model, we obtain a $|\phi|^4$-theory, which describes a superfluid, with a decoupled gauge field.

We next investigate what happens with the dual theory as we approach the critical temperature. Remember that $\bar{\phi}$ and therefore m_A tends to zero as T approaches the critical temperature from below. From the first term in the Hamiltonian (3.166) it again follows that $\nabla \times \tilde{\mathbf{h}} \to 0$ in this limit, so that we can write once more $\tilde{\mathbf{h}} = \nabla\gamma$, and (3.166) becomes

$$\mathcal{H}_\psi = \tfrac{1}{2}(\nabla\gamma)^2 + |(\nabla - i\Phi_0\nabla\gamma)\psi|^2 + m_\psi^2|\psi|^2 + u|\psi|^4. \tag{3.168}$$

This equation shows that γ, representing the magnetic scalar potential, cannot be distinguished from the phase of the disorder field. Indeed, let $\Phi_0\theta$ be this phase. Then, the canonical transformation $\theta \to \theta + \gamma$ absorbs the scalar potential into the phase of ψ; the first term in (3.168) decouples from the theory and yields a trivial contribution to the partition function. In this way, the dual theory reduces to a pure $|\psi|^4$-theory

$$\mathcal{H}_\psi = |\nabla\psi|^2 + m_\psi^2|\psi|^2 + u|\psi|^4. \tag{3.169}$$

At the transition temperature, the magnetic vortices proliferate and the field ψ develops an expectation value. The transition is triggered by a change in sign of m_ψ^2. In the London limit, the Hamiltonian (3.169) then takes the simple form

$$\mathcal{H}_\psi = \tfrac{1}{2}|\bar{\psi}|^2\Phi_0^2(\nabla\gamma)^2, \tag{3.170}$$

with $|\bar{\psi}|$ the expectation value of the disorder field, $|\bar{\psi}|/\sqrt{2} = \langle|\psi|\rangle$, and where we now represented the phase of ψ by $\Phi_0\gamma$ to bring out the fact that γ describes the magnetic scalar potential. From a symmetry point of view [170], γ is the Goldstone mode of the spontaneously broken global U(1) symmetry of the $|\psi|^4$-theory. Whereas in the Ginzburg-Landau formulation a magnetic vortex is described by a singular plastic field

$\mathbf{B}^{\mathrm{P}}$, in the dual formulation it is represented by the Noether current associated with the global U(1) symmetry,

$$\mathbf{j}_g = -ig\psi^* \overset{\leftrightarrow}{\nabla} \psi + 2g^2\tilde{\mathbf{h}}|\psi|^2. \tag{3.171}$$

This follows from comparing the terms coupling linearly to the fluctuating $\tilde{\mathbf{h}}$ field. In the normal phase, where ψ develops a vacuum expectation value, the Noether current becomes in the London limit $\mathbf{j} = \nabla\gamma$. This is the usual relation between the current of a spontaneously broken symmetry and the ensuing Goldstone field.

As we will demonstrate next, $\bar{\psi}$ has the value $|\bar{\psi}| = 1/\Phi_0$ [170] of an inverse flux quantum, so that with our normalization choice of the phase of the ψ-field, Eq. (3.170) takes the canonical form. Let us introduce a closed vortex L in the dual theory (3.170) by minimally coupling the magnetic scalar potential to a vortex gauge field $\boldsymbol{\varphi}^{\mathrm{P}}$ and consider the expectation value of this configuration

$$\langle W(L)\rangle = \int \mathrm{D}\gamma \exp\left[-\frac{1}{2}|\bar{\psi}|^2 \int_{\mathbf{x}} \left(\Phi_0\nabla\gamma - \boldsymbol{\varphi}^{\mathrm{P}}\right)^2\right] \tag{3.172}$$

We linearize the theory by introducing an auxiliary vector field $\mathbf{b}$ via

$$\begin{aligned}\exp\left[-\frac{1}{2}|\bar{\psi}|^2 \int_{\mathbf{x}} \left(\Phi_0\nabla\gamma - \boldsymbol{\varphi}^{\mathrm{P}}\right)^2\right] = \\ \int \mathrm{D}\mathbf{b} \exp\left\{-\int_{\mathbf{x}}\left[\frac{1}{2}\mathbf{b}^2 + i\,|\bar{\psi}|\mathbf{b}\cdot\left(\Phi_0\nabla\gamma - \boldsymbol{\varphi}^{\mathrm{P}}\right)\right]\right\}.\end{aligned} \tag{3.173}$$

This amounts to a duality transformation again. The integral over γ now yields the constraint $\nabla\cdot\mathbf{b} = 0$, demanding $\mathbf{b}$ to be the rotation of a vector field, $\mathbf{b} = \nabla\times\mathbf{A}$. This gives

$$\langle W(L)\rangle = \int \mathrm{D}\mathbf{A} \exp\left\{-\int_{\mathbf{x}}\left[\frac{1}{2}(\nabla\times\mathbf{A})^2 - i\mathbf{A}\cdot\mathbf{J}\right]\right\}, \tag{3.174}$$

where $J_i(\mathbf{x}) := 2\pi|\bar{\psi}| \oint_L \mathrm{d}y_i\delta(\mathbf{x}-\mathbf{y})$ describes the closed vortex. It is natural to interpret the fluctuating gapless gauge field $\mathbf{A}$ as the magnetic vector potential, and the closed vortex as an electric current loop. This justifies the use of the symbol W in (3.172) which was first introduced in (3.96). (The analogy between vortices and electric currents was first pointed out by von Helmholtz.) In this way, we get back the Ginzburg-Landau model with just one external electric current loop. The full model (3.78) is recovered when we consider a loop gas of these defects and describe it by a $|\phi|^4$-theory, provided we make the identification [170]

$$2\pi|\bar{\psi}| = 2e, \tag{3.175}$$

or $|\bar{\psi}| = 1/\Phi_0$. It links the expectation value $\bar{\psi}$ of the disorder field ψ describing the magnetic vortex loops to the coupling constant $2e$ of the Ginzburg-Landau model. It is the exact analog of the relation between the expectation value $\bar{\phi}$ of the ϕ-field describing the electric current loops and the coupling constant $g = \Phi_0 m_A$ of the dual theory:

$$2\pi|\bar{\phi}| = g. \tag{3.176}$$

Since $\bar{\psi}$ vanishes as T approaches the critical temperature from above, the coupling constant $2e$ goes to zero at the critical point and the electric current loops decouple from $\mathbf{A}$. Moreover, Eq. (3.111), giving the tension of electric current lines in the normal phase, shows that indeed these lines carry no energy in the superconducting phase where they proliferate. This solves the apparent contradiction we mentioned on page 92. Precisely the same things happened with the magnetic vortices: the coupling constant g of the dual theory vanishes as the critical temperature is approached from below, so that the magnetic vortices decouple from the local magnetic induction and proliferate. We thus have a complete duality here between magnetic vortices and electric current loops in complementary phases.

At this stage we can also clarify the physical relevance of the ultraviolet cutoff $|m_\psi|$ introduced in the calculation of the tension (3.111) of a current loop. This calculation was performed in the London limit where current lines are considered to be infinitely thin. Outside the London limit, vortices of the dual theory have a typical width of the order of the coherence length $1/|m_\psi|$, which therefore provides a natural ultraviolet cutoff.

The line tension (3.111) we previously calculated in the framework of the Ginzburg-Landau model can also be obtained in the dual theory directly from (3.172). The integration over γ can be carried out by substituting the field equation of the Goldstone field

$$\gamma(\mathbf{x}) = -\frac{1}{2\Phi_0}\int_{\mathbf{y}} G_0(\mathbf{x}-\mathbf{y})\nabla\cdot\boldsymbol{\varphi}_\mu^{\mathrm{P}}(\mathbf{y}), \tag{3.177}$$

where G_0 is the Green function (3.104) with $m_A = 0$. This yields an expression

$$\langle W(L)\rangle = \exp\left\{-\frac{1}{2\Phi_0^2}\int_{\mathbf{x}}\int_{\mathbf{y}}[\nabla\times\boldsymbol{\varphi}^{\mathrm{P}}(\mathbf{x})]_i G_0(\mathbf{x}-\mathbf{y})\,[\nabla\times\boldsymbol{\varphi}^{\mathrm{P}}(\mathbf{y})]_i\right\}, \tag{3.178}$$

very similar to the expression for the disorder parameter obtained in Eq. (3.157) with the monopole density set to zero since

$$[\nabla\times\boldsymbol{\varphi}^{\mathrm{P}}(\mathbf{x})]_i = 2\pi\int_L \mathrm{d}y_i\,\delta(\mathbf{x}-\mathbf{y}). \tag{3.179}$$

This immediately gives

$$\langle W(L)\rangle = \mathrm{e}^{-M_W|L|}, \tag{3.180}$$

with $|L|$ the vortex length and M_W the line tension (3.111).

The field equation we derive from (3.174),

$$\nabla\times\mathbf{h} = i\mathbf{J} \tag{3.181}$$

is one of the two basic equations of magnetostatics. It should be noted, however, that an additional factor of i shows up here. As a result, the Biot-Savart law yields opposite signs from what is usually the case. Two parallel currents repel instead of attract each other so that a single current loop prefers to crumple. It follows that a state where these current loops proliferate has zero magnetization—as should be the case in the superconducting phase of the Ginzburg-Landau model.

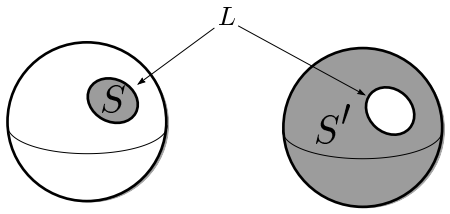


Figure 3.7: Two different surfaces S and S' spanning the loop L. The flux through the surfaces differ by a factor 4π.

Because of the imaginary current, the local magnetic induction generated by a current loop L is also purely imaginary as follows from Ampere's law,

$$\mathbf{h}(\mathbf{x}) = i\frac{1}{2\Phi_0}\oint_L \mathrm{d}\mathbf{y} \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} = i\frac{1}{2\Phi_0}\nabla\Omega(\mathbf{x}), \tag{3.182}$$

where $\Omega(\mathbf{x})$ is the solid angle that the loop subtends at $\mathbf{x}$. The same result can also be directly derived from the dual theory. Rewriting the field equation for γ obtained from (3.172), we can relate the Goldstone field to the solid angle in the following way [17]

$$\gamma(\mathbf{x}) = \frac{1}{2\Phi_0}\int_S \mathrm{d}^2 y_i \frac{(\mathbf{x}-\mathbf{y})_i}{|\mathbf{x}-\mathbf{y}|^3} = -\frac{1}{2\Phi_0}\Omega(\mathbf{x}), \tag{3.183}$$

where $\mathrm{d}^2 y_i$ is an element of the surface S spanned by the current loop L. Together with the observation that the local magnetic induction can, apart from a factor i, be identified with the gradient of the phase variable γ

$$\mathbf{h} = -i\nabla\gamma, \tag{3.184}$$

[see (3.163)], this yields the previous result (3.182).

As a side remark we note that the magnetic moment density, or magnetization, is represented in the dual theory by $\boldsymbol{\varphi}^{\mathrm{P}}$. This follows from (3.172) showing that an electric current loop couples to the magnetic field $\nabla\gamma$ via $\boldsymbol{\varphi}^{\mathrm{P}}$.

With this physical identification of the Goldstone mode it follows that, in the normal state, the disorder parameter $V_{\mathrm{r}}(\mathbf{x})$ essentially measures the solid angle $\Omega(\mathbf{x})$

$$V_{\mathrm{r}}(\mathbf{x}) = \mathrm{e}^{i\Phi_0\gamma(\mathbf{x})} = \mathrm{e}^{i\Omega(\mathbf{x})/2}. \tag{3.185}$$

Since the operator $V(\mathbf{x})$ was constructed by putting a magnetic monopole at $\mathbf{x}$, we see that $\frac{1}{2}\Omega(\mathbf{x})$ is the magnetic flux emanated by the monopole through the electric current loop L. As a last remark we note that one can chose two topologically different surfaces spanning the loop L (see Fig. 3.7). Both lead, however, to the same phase factor because Ω differs only by a factor of 4π.

Because the dual Ginzburg-Landau model involves only physical degrees of freedom and a global U(1) symmetry, it is ideally suited as basis to study the critical properties of the superconducting-to-normal phase transition [176]. As has been argued

above, at the mean-field level the massive induction field decouples from the theory as the critical temperature is approached from below. The resulting theory is a $|\psi|^4$-theory which is known to undergo a 2nd-order phase transition with xy-exponents. This can also be seen by integrating out the massive induction field in the low-temperature phase. This only leads to changes in the coefficients of the $|\psi|^4$-theory. Explicitly [148],

$$\mathcal{H}_{\psi,\mathrm{eff}} = |\nabla\psi|^2 + \left(m_\psi^2 - g^2\frac{m_A}{2\pi}\right)|\psi|^2 + \left(u - \frac{g^4}{4\pi m_A}\right)|\psi|^4. \tag{3.186}$$

In deriving this we used dimensional regularization and (irrelevant) higher-order terms are omitted. We note that all contributions stemming from the vector field $\tilde{\mathbf{h}}$ vanish in the limit where T approaches T_c from below, so that $\mathcal{H}_{\psi,\mathrm{eff}}$ reduces to a pure $|\psi|^4$-theory in this limit.

A one-loop renormalization-group analysis of the dual theory carried out in Ref. [176] also led to the conclusion that the disorder field behaves as in a pure $|\psi|^4$-theory with xy-exponents. In addition, the exponents of the magnetic induction field where shown to retain their Gaussian values because it decouples from the theory. The xy-behavior of the heat capacity was established experimentally in [177], while experiments on YBaCuO [178] seem to confirm the prediction of the Gaussian value for the divergence of the magnetic penetration depth. The confirmation relies, however, on a delicate finite-size analysis, so that there is still room for a reinterpretation of the data and the possibility of suggesting a different critical behavior as has been done by Herbut [179] who concluded that the penetration depth has an xy-exponent. One recent lattice simulation directly of the Ginzburg-Landau model corroborated the Gaussian value for the divergence of the penetration depth [180], while an other found an xy-exponent [181].

Chapter 4

Quantum Phase Transitions

In this chapter, we discuss various continuous phase transitions at the absolute zero of temperature—so-called quantum phase transitions. Unlike in their classical counterparts taking place at finite temperature and being an equilibrium phenomenon, time plays an important role in quantum phase transitions. Put differently, whereas the critical behavior of classical 2nd-order phase transitions is governed by thermal fluctuations, that of 2nd-order quantum transitions is controlled by quantum fluctuations. These transitions, which have attracted much attention in recent years (for an introductory review, see Ref. [182]), are triggered by varying not the temperature, but some other parameter in the system, like the applied magnetic field or the amount of disorder. The natural language to describe these transitions is quantum field theory. In addition to a diverging correlation length ξ, quantum phase transitions also have a diverging correlation time ξ_t. They indicate, respectively, the distance and time period over which the order parameter characterizing the transition fluctuates coherently. The way the diverging correlation time relates to the diverging correlation length,

$$\xi_t \sim \xi^z, \tag{4.1}$$

defines the so-called dynamic exponent z. It is a measure for the asymmetry between the time and space directions and tells us how long it takes for information to propagate across a distance ξ. The traditional scaling theory of classical 2nd-order phase transitions is easily extended to include the time dimension [183] because relation (4.1) implies the presence of only one independent diverging scale. The critical behavior of a phase transition at finite temperature is still controlled by the quantum critical point provided $T < 1/\xi_t$. This is what makes quantum phase transitions experimentally accessible.

We start in the next section discussing the so-called superfluid-to-Mott-insulating phase transition in the pure case, while in Sec. 4.2 we include (quenched) impurities. In Sec. 4.3 we discuss the effective theory describing the fractional quantized Hall effect and argue that, in principle, it can be employed to describe the quantum phase transitions in quantum Hall systems. In Sec. 4.4 we then apply renormalization-group analysis to this theory. In Sec. 4.5 we discuss scaling and hyperscaling theory applied

to the systems under study and in Sec. 4.6 we discuss various experiments probing quantum phase transitions.

4.1 Repulsively Interacting Bosons

The first quantum phase transition we wish to investigate is the superfluid-to-Mott-insulating transition of repulsively interacting bosons in the absence of impurities [184]. The transition is described by the nonrelativistic $|\phi|^4$-theory (1.164), which becomes critical at the absolute zero of temperature at some (positive) value $\mu_{\rm c}$ of the renormalized chemical potential. The Mott insulating phase is destroyed and makes place for the superfluid phase as μ increases. Whereas in the superfluid phase the single-particle (Bogoliubov) spectrum is gapless and the system compressible, the single-particle spectrum of the insulating phase has an energy gap and the compressibility κ vanishes here.

The nature of the insulating phase can be best understood by putting the theory on a lattice. The lattice model is defined by the Hamiltonian

$$H_{\rm H} = -t \sum_j (\hat{a}_j^\dagger \hat{a}_{j+1} + {\rm h.c.}) + \sum_j (-\mu_{\rm L} \hat{n}_j + U \hat{n}_j^2), \tag{4.2}$$

where the sum $\sum_j$ is over all lattice sites. The operator $\hat{a}_j^\dagger$ creates a boson at site j and $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ is the particle number operator at that site; t is the hopping parameter, U the interparticle repulsion, and $\mu_{\rm L}$ is the chemical potential on the lattice. The zero-temperature phase diagram is as follows [184]. In the limit $t/U \to 0$, each site is occupied by an integer number n of bosons which minimizes the on-site energy (see Fig. 4.1)

$$\epsilon(n) = -\mu_{\rm L} n + U n^2. \tag{4.3}$$

It follows that within the interval $2n-1 < \mu_{\rm L}/U < 2n+1$, each site is occupied by exactly n bosons. When the chemical potential is negative, $n = 0$. The intervals become smaller when t/U increases. Within such an interval, where the particles are pinned to the lattice sites, the single-particle spectrum has an energy gap, and the system is in the insulating phase with zero compressibility, $\kappa = n^{-2} \partial n / \partial \mu_{\rm L} = 0$. Outside these intervals, the particles delocalize and can hop through the lattice. Being at zero temperature, the delocalized bosons condense in a superfluid state. The single-particle spectrum is gapless here and the system compressible ($\kappa \neq 0$).

As t/U increases, the gap in the single-particle spectrum as well as the width of the intervals decrease and eventually vanish at some critical value $t_{\rm c}$. For values $t > t_{\rm c}$ of the hopping parameter, the superfluid phase is the only phase present (see Fig. 4.2). The continuum model (1.164), with renormalized chemical potential $\mu > \mu_{\rm c}$ describes the condensed delocalized lattice bosons which are present when the density deviates from integer values (see Fig. 4.1). In the limit $\mu \to \mu_{\rm c}$ from above, the number of delocalized bosons decreases and eventually becomes zero at the phase boundary $\mu = \mu_{\rm c}$ between the superfluid and insulating phases.

Various quantum phase transitions belong to the universality class defined by the zero-density transition of repulsively interacting bosons. For example, itinerant quan-

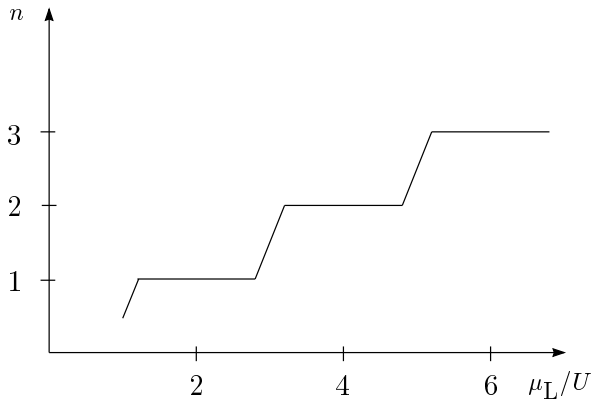


Figure 4.1: Schematic representation of the average number n of particles per site as function of the chemical potential $\mu_{\rm L}$ at some finite value of the hopping parameter $t < t_{\rm c}$ [191].

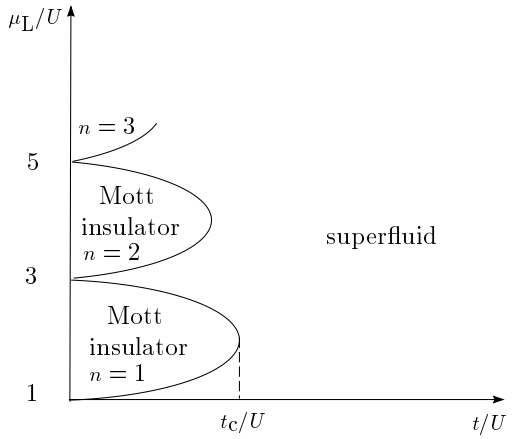


Figure 4.2: Schematic representation of the phase diagram of the lattice model (4.2) at the absolute zero of temperature [184].

tum antiferromagnets [185, 186, 187] as well as lower-dimensional (clean) superconductors belong to this universality class. As we have seen in Sec. 1.4, Cooper pairs become tightly bound composite particles in the strong-coupling limit, which are described by the nonrelativistic $|\phi|^4$-theory with a weak repulsive interaction. For $\mu > \mu_{\rm c}$, the field ϕ now describes the condensed delocalized Cooper pairs. When the chemical potential decreases, the condensate diminishes, and the system again becomes insulating for $\mu < \mu_{\rm c}$ [138]. By continuity, we expect also the superconductor-to-insulator transition of a (clean) weakly interacting BCS superconductor to be in this universality class. The restriction to lower dimensions is necessary for two different reasons. First, only for $d \leq 2$ the penetration depth is sufficiently large [see, for example, below (3.47)], so that it is appropriate to work in the limit $\lambda_{\rm L} \rightarrow \infty$ with no fluctuating gauge field [188]. Second, in lower dimensions, the energy gap which the fermionic excitations face remains finite at the critical point, so that it is appropriate to ignore these degrees of freedom. Moreover, since also the coherence length remains finite at the critical point, the Cooper pairs look like point particles on the scale of the diverging correlation length associated with the phase fluctuations, even in the weak-coupling limit [138].

The nonrelativistic $|\phi|^4$-theory is also of importance for the description of the fractional quantized Hall effect (FQHE) (see Sec. 4.3). As function of the applied magnetic field, this two-dimensional system undergoes a zero-temperature transition between a so-called quantum Hall liquid, where the Hall conductance is quantized in odd fractions of $e^2/2\pi$, or, reinstalling Planck's constant, e^2/h, and an insulating phase. Here, the nonrelativistic $|\phi|^4$-theory describes—after coupling to a Chern-Simons term—the original electrons contained in the system bound to an odd number of flux quanta. The Hall liquid corresponds to the phase with $\mu > \mu_{\rm c}$, while the other phase again describes the insulating phase. In this picture, the Hall liquid is characterized by a condensate of composite particles.

It should be noted however that in most of these applications of the nonrelativistic $|\phi|^4$-theory mentioned here, impurities play an important role; this will be the subject of the succeeding section.

The critical properties of the zero-density transition of the nonrelativistic $|\phi|^4$-theory were first studied by Uzunov [189]. To facilitate the discussion let us make use of the fact that in nonrelativistic theories the mass is—as far as critical phenomena concerned—an irrelevant parameter which can be transformed away. This transformation changes, however, the scaling dimensions of the ϕ-field and the coupling constant which is of relevance to the renormalization-group theory. The engineering dimensions are

$$[\mathbf{x}] = -1, \quad [t] = -2, \quad [\mu_0] = 2, \quad [\lambda_0] = 2 - d, \quad [\phi] = \tfrac{1}{2}d, \qquad (4.4)$$

with d the number of space dimensions. In two space dimensions the coupling constant λ_0 is dimensionless, showing that the $|\phi|^4$-term is a marginal operator, and $d_{\rm c} = 2$ the upper critical space dimension. Uzunov showed that below the upper critical dimension there appears a non-Gaussian IR fixed point. He computed the corresponding critical exponents to all orders in perturbation theory and showed them to have Gaussian values, $\nu = \frac{1}{2},\ z = 2,\ \eta = 0$. Here, ν characterizes the divergence of the correlation length, z is the dynamic exponent, and η is the correlation-function exponent which

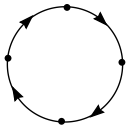


Figure 4.3: A closed oriented loop.

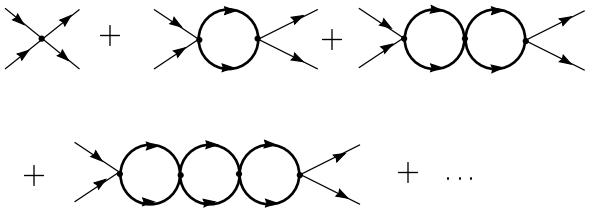


Figure 4.4: Ring diagrams renormalizing the vertex function of the neutral $|\phi|^4$-theory.

determines the anomalous dimension of the field ϕ. The unexpected conclusion that a non-Gaussian fixed point has nevertheless Gaussian exponents is rooted in the analytic structure of the nonrelativistic propagator at zero (bare) chemical potential ($\mu_0 = 0$):

$$\overset{k_\mu}{\longrightarrow} = G(k) = \frac{i\mathrm{e}^{ik_0\eta}}{k_0 - \frac{1}{2}\mathbf{k}^2 + i\eta}, \tag{4.5}$$

where, as before, η is a small positive constant that has to be taken to zero after the loop integrations over the energies have been carried out. By setting $\mu_0 = 0$, we are considering the system at criticality. The rule $k_0 \to k_0 + i\eta$ in (4.5) expresses the fact that in this nonrelativistic theory particles propagate only forward in time. In Feynman diagrams involving loops with more than one propagator, the integrals over the loop energy are convergent and can be evaluated by contour integration with the contour closed in either the upper or the lower half plane. If a diagram contains a loop which has all its poles in the same half plane, it consequently vanishes. Pictorially, such a loop has all its arrows, representing the Green functions contained in the loop, oriented in a clockwise or anticlockwise direction [190] (see Fig. 4.3). We will refer to them as closed oriented loops. Owing to this property most diagrams are zero. In particular, all self-energy diagrams vanish. The only surviving ones are the so-called ring diagrams which renormalize the vertex (see Fig. 4.4). Because this class of diagrams constitute a geometric series, the one-loop result is already exact. The vertex renormalization leads to a non-Gaussian fixed point in $d < 2$, while the vanishing of all the self-energy diagrams asserts that the exponents characterizing the transition are not affected by quantum fluctuations and retain their Gaussian values [189]. These results

Table 4.1: The upper critical space dimension d_c of a nonrelativistic (NR) and a relativistic (R) quantum theory with a $|\phi|^{2k}$ interaction term.

k	d_c(NR)	d_c(R)
2	2	3
3	1	2
∞	0	1

have been confirmed by numerical simulations in $d = 1$ [191] and also by general scaling arguments [184].

We have seen that $d_c = 2$ is the upper critical dimension of the nonrelativistic $|\phi|^4$-theory. Dimensional analysis shows that for an interaction term of the form

$$\mathcal{L}_{\rm int} = -g_0|\phi|^{2k} \tag{4.6}$$

the upper critical dimension is

$$d_c = \frac{2}{k-1}. \tag{4.7}$$

The two important physical cases are $d_c = 2$, $k = 2$ and $d_c = 1$, $k = 3$, while $d_c \to 0$ when $k \to \infty$. For space dimensions $d > 2$ only the quadratic term, $|\phi|^2$, is relevant so that here the critical behavior is well described by a Gaussian theory.

In the corresponding relativistic theory, the scaling dimensions of t and $\mathbf{x}$ are, of course, equal $[t] = [\mathbf{x}] = -1$ and $[\phi] = \frac{1}{2}(d-1)$. This leads to different upper critical (space) dimensions, viz.,

$$d_c = \frac{k+1}{k-1} = \frac{2}{k-1} + 1, \tag{4.8}$$

instead of (4.7). The two important physical cases are here $d_c = 3$, $k = 2$ and $d_c = 2$, $k = 3$, while $d_c \to 1$ when $k \to \infty$. On comparison with the nonrelativistic results, we see that the nonrelativistic theory has an upper critical space dimension which is one lower than that of the corresponding relativistic theory (see Table 4.1). Heuristically, this can be understood by noting that in a nonrelativistic context the time dimension counts double in that it has a scaling dimension twice that of a space dimension [see Eq. (4.4)], thereby increasing the *effective* spacetime dimensionality by one.

From this analysis it follows that for a given number of space dimensions the critical properties of a nonrelativistic theory are unrelated to those of the corresponding relativistic extension.

In closing this section we recall that in a one-dimensional relativistic theory—corresponding to the lowest upper critical dimension ($d_c = 1$)—a continuous symmetry cannot be spontaneously broken. However, the theory can nevertheless have a phase transition of the so-called Kosterlitz-Thouless type. Given the connection between the relativistic and nonrelativistic theories discussed above, it seems interesting to study the nonrelativistic theory at zero space dimension ($d = 0$) to see if a similar

rich phenomenon as the Kosterlitz-Thouless transition occurs in the quantum theory. This may be of relevance to so-called quantum dots.

4.2 Including Quenched Impurities

In the preceding section, we saw that in the absence of impurities repulsively interacting bosons will undergo a 2nd-order quantum phase transition. As was pointed out there, this universality class is of relevance to various condensed-matter systems. However, in most of these systems, as well in ^{4}He in porous media, impurities play an essential if not decisive role. For example, the two-dimensional superconductor-to-insulator transition investigated by Hebard and Palaanen [192] is driven by impurities. This means that, e.g., the correlation length ξ diverges as $|\hat{\Delta}^* - \hat{\Delta}|^{-\nu}$ when the parameter $\hat{\Delta}$ characterizing the disorder approaches the critical value $\hat{\Delta}^*$. Hence, a realistic description of the critical behavior of these systems should include impurities.

Some years ago, it has been argued that upon including quenched impurities, the quantum critical behavior of the nonrelativistic $|\phi|^4$-theory becomes unstable [193]. Only after introducing an artificial high-energy cutoff, an IR fixed point was found by Weichman and Kim [194]. However, as was pointed out by the authors, such a cutoff is difficult to justify as it would imply that time is discrete. So, it is widely accepted that the random nonrelativistic $|\phi|^4$-theory has no perturbatively accessible IR fixed point, if any at all [184]. The absence of an IR fixed point in the nonrelativistic $|\phi|^4$-theory theory would imply that also the quantum critical behavior of the systems mentioned in the preceding section, is unstable with respect to impurity influences. Because of its implications for the description of the critical behavior of these systems we have revisited the problem. Below, it will be shown that, contrary to general conviction, the random nonrelativistic $|\phi|^4$-theory does have a new IR fixed point. The calculations are performed without introducing a (physically unnatural) high-energy cutoff.

To account for impurities, we add to the nonrelativistic $|\phi|^4$-theory (1.164) a term [183]

$$\mathcal{L}_\Delta = \psi(\mathbf{x})\,|\phi|^2, \tag{4.9}$$

with $\psi(\mathbf{x})$ a random field. As we have seen in the preceding section, the theory becomes critical in the limit where the bare chemical potential tends to zero, $\mu_0 \to 0$. We shall study the random theory in the symmetrical state where the chemical potential is negative and the global U(1) symmetry unbroken. We therefore set $\mu_0 = -r_0$ again, with $r_0 > 0$. We leave the number of space dimensions d unspecified for the moment.

The random field $\psi(\mathbf{x})$ is assumed to be Gaussian distributed [183]:

$$P(\psi) = \exp\left[-\frac{1}{\Delta_0}\int_{\mathbf{x}} \psi^2(\mathbf{x})\right], \tag{4.10}$$

characterized by the disorder strength Δ_0. The engineering dimension of the random field is the same as that of the chemical potential which is one, $[\psi] = 1$, while that of the parameter Δ_0 is $[\Delta_0] = 2 - d$ so that the exponent in (4.10) is dimensionless. The quantity

$$Z[\psi] = \int \mathrm{D}\phi^*\mathrm{D}\phi\, \exp\left(i\int_x \mathcal{L}\right), \tag{4.11}$$

where $\mathcal{L}$ now stands for the Lagrangian (1.164) with the term (4.9) added, is the zero-temperature partition function for a given impurity configuration ψ. In the case of quenched impurities, the average of an observable $O(\phi^*, \phi)$ is obtained as follows

$$\langle O(\phi^*, \phi) \rangle = \int \mathrm{D}\psi P(\psi) \langle O(\phi^*, \phi) \rangle_\psi, \tag{4.12}$$

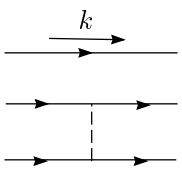
where $\langle O(\phi^*, \phi) \rangle_\psi$ indicates the grand-canonical average for a given impurity configuration, i.e., taken with respect to (4.11). In other words, first the ensemble average is taken, and only after that the averaging over the random field is carried out.

Since $\psi(\mathbf{x})$ depends only on the d spatial dimensions, the impurities it describes should be considered as grains randomly distributed in space. When—as is required for the study of quantum critical phenomena—time is included, the static grains trace out straight worldlines. That is to say, the impurities are line-like. It has been shown by Dorogovtsev [195] that the critical properties of systems with extended defects must be studied in a double epsilon expansion, otherwise no IR fixed point is found. The method differs from the usual epsilon expansion, in that it also includes an expansion in the defect dimensionality ϵ_d. To carry out this program in the present context, where the defect dimensionality is determined by the dimensionality of time, the theory has to be formulated in ϵ_d time dimensions. The case of interest is $\epsilon_\mathrm{d} = 1$, while in the opposite limit, $\epsilon_\mathrm{d} \to 0$, the random nonrelativistic $|\phi|^4$-theory reduces to the classical spin model with random (point-like) impurities. Hence, ϵ_d is a parameter with which quantum fluctuations can be suppressed. An expansion in ϵ_d is a way to perturbatively include the effect of quantum fluctuations on the critical behavior. Ultimately, we will be interested in the case $\epsilon_\mathrm{d} = 1$.

To calculate the quantum critical properties of the random theory, which have been first studied in [193], we will not employ the replica method [196], but instead follow Lubensky [197]. In this approach, the averaging over impurities is carried out for each Feynman diagram separately. The upshot is that only those diagrams are to be included which remain connected when Δ_0, the parameter characterizing the Gaussian distribution of the impurities, is set to zero [198]. To obtain the relevant Feynman rules of the random theory we average the interaction term (4.9) over the distribution (4.10):

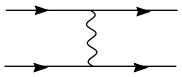
$$\begin{aligned} \int \mathrm{D}\psi \, P(\psi) \exp\left[i^{\epsilon_\mathrm{d}} \int \mathrm{d}^{\epsilon_\mathrm{d}} t \, \mathrm{d}^d x \, \psi(\mathbf{x}) \, |\phi(x)|^2 \right] = \\ \exp\left[\tfrac{1}{4} i^{2\epsilon_\mathrm{d}} \Delta_0 \int \mathrm{d}^{\epsilon_\mathrm{d}} t \, \mathrm{d}^{\epsilon_\mathrm{d}} t' \, \mathrm{d}^d x \, |\phi(t, \mathbf{x})|^2 |\phi(t', \mathbf{x})|^2 \right]. \end{aligned} \tag{4.13}$$

The randomness is seen to result in a quartic interaction term which is nonlocal in time. The factor i^{ϵ_d} appearing in (4.13) arises from the presence of ϵ_d time dimensions, each of which is accompanied by a factor of i. The Feynman rules of the random theory are now easily obtained



$$= \frac{-i^{-\epsilon_\mathrm{d}} \mathrm{e}^{i(\omega_1 + \omega_2 + \cdots + \omega_{\epsilon_\mathrm{d}})\eta}}{\omega_1 + \omega_2 + \cdots + \omega_{\epsilon_\mathrm{d}} - \mathbf{k}^2 - r_0 + i\eta}$$

$$= -4 i^{\epsilon_\mathrm{d}} \lambda_0 \tag{4.14}$$



$$= \quad i^{\epsilon_{\rm d}}(2\pi)^{\epsilon_{\rm d}}\delta^{\epsilon_{\rm d}}(\omega_1+\omega_2+\cdots+\omega_{\epsilon_{\rm d}})\Delta_0,$$

where we note that the Lagrangian in $\epsilon_{\rm d}$ time dimensions involves instead of just one time derivative, a sum of $\epsilon_{\rm d}$ derivatives: $\partial_t \to \partial_{t_1}+\partial_{t_2}+\cdots+\partial_{t_{\epsilon_{\rm d}}}$. The integral (4.17) has an additional convergence factor $\exp(i\omega\eta)$ for each of the $\epsilon_{\rm d}$ energy integrals. This factor, which is typical for nonrelativistic quantum theories [28], is to be included in self-energy diagrams containing only one ϕ-propagator.

Following Weichman and Kim [194], we evaluate the integrals over loop energies assuming that all energies are either positive or negative. This allows us to employ Schwinger's propertime representation of propagators [146], which is based on the integral representation of the gamma function,

$$\frac{1}{a^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{{\rm d}\tau}{\tau}\tau^\alpha {\rm e}^{-\tau a}. \tag{4.15}$$

The energy integrals we encounter to the one-loop order can be carried out with the help of the equations

$$\int' \frac{{\rm d}^{\epsilon_{\rm d}}\omega}{(2\pi)^{\epsilon_{\rm d}}}\frac{1}{\omega_1+\omega_2+\cdots+\omega_{\epsilon_{\rm d}}-x\pm i\eta} = -\frac{\Gamma(1-\epsilon_{\rm d})}{(2\pi)^{\epsilon_{\rm d}}}{\rm sgn}(x)|x|^{\epsilon_{\rm d}-1}\left({\rm e}^{\pm i\,{\rm sgn}(x)\pi\epsilon_{\rm d}}+1\right), \tag{4.16}$$

$$\int' \frac{{\rm d}^{\epsilon_{\rm d}}\omega}{(2\pi)^{\epsilon_{\rm d}}}\frac{{\rm e}^{i(\omega_1+\omega_2+\cdots+\omega_{\epsilon_{\rm d}})\eta}}{\omega_1+\omega_2+\cdots+\omega_{\epsilon_{\rm d}}-x+ix\eta} = \frac{i\pi}{(2\pi)^{\epsilon_{\rm d}}\Gamma(\epsilon_{\rm d})}(i|x|)^{\epsilon_{\rm d}-1}\left[\sin(\tfrac{1}{2}\pi\epsilon_{\rm d})-\frac{{\rm sgn}(x)}{\sin(\frac{1}{2}\pi\epsilon_{\rm d})}\right], \tag{4.17}$$

where η is again an infinitesimal positive constant which is to be taken to zero after the energy integrals have been carried out. The prime on the integrals is to remind the reader that the energy integrals are taken over only two domains with either all energies positive or negative. The energy integrals have been carried out by using again the integral representation (4.15) of the gamma function. In doing so, the integrals are regularized and—as is always the case with analytic regularizations—irrelevant divergences suppressed.

By differentiation with respect to x, Eq. (4.16) can, for example, be employed to calculate integrals involving integrands of the form $1/(\omega_1+\omega_2+\cdots+\omega_{\epsilon_{\rm d}}-x+i\eta)^2$. Is is easily checked that in the limit $\epsilon_{\rm d}\to 1$, where the energy integral can be performed with help of contour integration, Eqs. (4.16) and (4.17) reproduce the right results. When considering the limit of zero time dimensions ($\epsilon_{\rm d}\to 0$), it should be remembered that the energy integrals were taken over two separate domains with all energies either positive or negative. Each of these domains is contracted to a single point in the limit $\epsilon_{\rm d}\to 0$, so that one obtains a result which is twice that obtained by simply purging any reference to the time dimensions.

Before studying the random theory, let us briefly return to the repulsively interacting bosons in the absence of impurities. In this case, there is no need for an $\epsilon_{\rm d}$-expansion and the formalism outlined above should yield results for arbitrary time

dimensions $0 \leq \epsilon_{\rm d} \leq 1$, interpolating between the classical and quantum limit. After the energy integrals have been performed with the help of Eqs. (4.16) and (4.17), the standard technique of integrating out a momentum shell can be applied to obtain the renormalization-group equations. For the correlation-length exponent ν we obtain in this way [201]

$$\nu = \frac{1}{2}\left[1 + \frac{\epsilon}{2}\frac{m+1}{(m+4)-(m+3)\epsilon_{\rm d}}\cos^2(\tfrac{1}{2}\pi\epsilon_{\rm d})\right]. \tag{4.18}$$

Here, $\epsilon = 4 - 2\epsilon_{\rm d} - d$ is the deviation of the *effective* spacetime dimensionality from 4, where it should be noted that in (canonical) nonrelativistic theories, time dimensions have an engineering dimension twice that of space dimensions. (This property is brought out by the Gaussian value $z = 2$ for the dynamic exponent z.) For comparison we have extended the theory (1.164) to include m complex ϕ fields instead of just one field. In the classical limit, Eq. (4.18) gives the well-known one-loop result for a classical spin model with $2m$ real components [183],

$$\nu \to \frac{1}{2}\left(1 + \frac{\epsilon}{2}\frac{m+1}{m+4}\right), \tag{4.19}$$

while in the quantum limit it gives the result $\nu \to \frac{1}{2}$, as required.

The exponent (4.18), and also the location of the fixed point, diverges when the number of time dimensions becomes $\epsilon_{\rm d} \to (m+4)/(m+3)$. Since this value is always larger than one, the singularity is outside the physical domain $0 \leq \epsilon_{\rm d} \leq 1$. This simple example illustrates the viability of the formalism developed here to generate results interpolating between the classical and quantum limit.

We continue with the random theory. After the energy integrals have been carried out, it is again straightforward to derive the renormalization-group equations by integrating out a momentum shell $\Lambda/b < k < \Lambda$, where Λ is a high-momentum cutoff and $b = \exp(l)$, with l infinitesimal. Defining the dimensionless variables

$$\hat{\lambda} = \frac{K_d}{(2\pi)^{\epsilon_{\rm d}}}\lambda\Lambda^{-\epsilon}; \quad \hat{\Delta} = K_d\Delta\Lambda^{d-4}; \quad \hat{r} = r\Lambda^{-2}, \tag{4.20}$$

where K_d given in (1.249) is the area of a unit sphere in d spatial dimensions divided by $(2\pi)^d$, we find [201]

$$\begin{aligned}
\frac{{\rm d}\hat{\lambda}}{{\rm d}l} &= \epsilon\hat{\lambda} - 8\left[\Gamma(1-\epsilon_{\rm d}) + (m+3)\Gamma(2-\epsilon_{\rm d})\right]\cos(\tfrac{1}{2}\pi\epsilon_{\rm d})\hat{\lambda}^2 + 6\hat{\Delta}\hat{\lambda} \\
\frac{{\rm d}\hat{\Delta}}{{\rm d}l} &= (\epsilon + 2\epsilon_{\rm d})\hat{\Delta} + 4\hat{\Delta}^2 - 16(m+1)\Gamma(2-\epsilon_{\rm d})\cos(\tfrac{1}{2}\pi\epsilon_{\rm d})\hat{\lambda}\hat{\Delta} \\
\frac{{\rm d}\hat{r}}{{\rm d}l} &= 2\hat{r} + 4\pi\frac{m+1}{\Gamma(\epsilon_{\rm d})}\frac{\cos^2(\frac{1}{2}\pi\epsilon_{\rm d})}{\sin(\frac{1}{2}\pi\epsilon_{\rm d})}\hat{\lambda} - \hat{\Delta}.
\end{aligned} \tag{4.21}$$

These results are to be trusted only for small values of $\epsilon_{\rm d}$. For illustrative purposes we have, however, kept the full $\epsilon_{\rm d}$ dependence. The set of equations yields the fixed point

$$\hat{\lambda}^* = \frac{1}{16\cos(\frac{1}{2}\pi\epsilon_{\rm d})\Gamma(1-\epsilon_{\rm d})}\,\frac{\epsilon + 6\epsilon_{\rm d}}{2m(1-\epsilon_{\rm d}) - 1} \tag{4.22}$$

$$\hat{\Delta}^* = \frac{1}{4}\frac{m(1-\epsilon_{\rm d})(2\epsilon_{\rm d}-\epsilon)+2\epsilon_{\rm d}(4-3\epsilon_{\rm d})+\epsilon(2-\epsilon_{\rm d})}{2m(1-\epsilon_{\rm d})-1},$$

and the critical exponent

$$\nu = \frac{1}{2}+\frac{\epsilon+2\epsilon_{\rm d}}{16}+\frac{m+1}{16}\frac{(6\epsilon_{\rm d}+\epsilon)[\epsilon_{\rm d}+\cos(\pi\epsilon_{\rm d})]}{2m(1-\epsilon_{\rm d})-1}. \tag{4.23}$$

The dynamic exponent is given by $z = 2+\hat{\Delta}^*$. We see that both $\hat{\lambda}^*$ and $\hat{\Delta}^*$ diverge when $\epsilon_{\rm d} \to 1-1/2m$. At this point, the fixed point becomes unphysical. The singularity separates the quantum regime $\epsilon_{\rm d} \lesssim 1$ from the classical regime $\epsilon_{\rm d} \gtrsim 0$ about which perturbation theory is to be carried out. When the equations are expanded to first order in $\epsilon_{\rm d}$, we recover the IR fixed point found by Weichman and Kim [194] using an high-energy cutoff:

$$\hat{\lambda}^* = \frac{1}{16}\frac{\epsilon+6\epsilon_{\rm d}}{2m-1}; \quad \hat{\Delta}^* = \frac{1}{4}\frac{(2-m)\epsilon+2(m+4)\epsilon_{\rm d}}{2m-1}, \tag{4.24}$$

with the critical exponent

$$\nu = \frac{1}{2}\left[1+\frac{1}{8}\frac{3m\epsilon+(5m+2)2\epsilon_{\rm d}}{2m-1}\right]. \tag{4.25}$$

We thereby provide support for the existence of this fixed point.

The value of the critical exponent (4.25) should be compared with that of the classical spin model with $2m$ components in the presence of random impurities of dimension $\epsilon_{\rm d}$ [195]:

$$\nu = \frac{1}{2}\left[1+\frac{1}{8}\frac{3m\epsilon+(5m+2)\epsilon_{\rm d}}{2m-1}\right]. \tag{4.26}$$

Taking into account that in a nonrelativistic quantum theory, time dimensions count double as compared to space dimensions, we see that both results are equivalent. As to the dynamic exponent, we mention that a remarkable simple scaling argument (see Sec. 4.5) using the quadratic terms in the effective theory (1.177) predicts the exact value $z = d$ for $\epsilon_{\rm d} = 1$ [184]. The perturbative result $z = 2+\hat{\Delta}^*$, with $\hat{\Delta}^*$ given by (4.24), is seen to be far away from this.

The limit of interest to us, corresponding to $\epsilon_{\rm d} = 1$, is probably difficult to reach by low-order perturbation theory for the quantum regime is separated by a singularity from the classical regime where perturbation theory applies. Although this might be an artifact of the one-loop calculation, it is unlikely that by including a few more loops, the quantum regime becomes accessible via the classical regime. We note that the singularity moves towards $\epsilon_{\rm d} = 1$ when the number of field components increases.

If the IR fixed point is to be of relevance to the systems mentioned in Sec. 4.1, the impurities have to lead to localization in the superfluid phase. For the model at hand, albeit in $d = 3$, this connection has been established by Huang and Meng [199]. They showed that in this phase, the impurities give rise to an additional depletion of the condensate as well as of the superfluid mass density. (The latter is defined, as we discussed in Sec. 1.7, by the response of the system to an externally imposed velocity

field as specified by the expression for the momentum density, or mass current **g**.) They found that the depletion of the superfluid mass density is larger than that of the condensate, indicating that part of the zero-momentum states belongs to the normal fluid rather than to the condensate. They interpreted this as implying that these states are trapped by impurities.

This situation should be contrasted to the one in the absence of impurities, where the condensate is depleted only due to the interparticle repulsion. Despite the depletion, *all* the particles were nevertheless seen in Sec. 1.7 to participate in the superflow motion at zero temperature. In other words, the normal fluid is dragged along by the condensate. This conclusion was based on the observation that the momentum density was given by $\mathbf{g} = mn\mathbf{v}_{\rm s}$, where n is the *total* particle number density and $\mathbf{v}_{\rm s}$ the velocity with which the condensate moves, i.e., the superfluid velocity. (For clarity, we have reintroduced the mass parameter m here.)

In terms of the shifted field (1.166), the random term (4.9) reads

$$\mathcal{L}_\Delta = \psi(\mathbf{x})(|\bar{\phi}|^2 + |\tilde{\phi}|^2 + \bar{\phi}\tilde{\phi}^* + \bar{\phi}^*\tilde{\phi}). \tag{4.27}$$

The first two terms lead to an irrelevant change in the chemical potential, so that we only have to consider the last two terms, which we can cast in the form

$$\mathcal{L}_\Delta = \psi(\mathbf{x})\,\bar{\Phi}^\dagger\tilde{\Phi}, \qquad \bar{\Phi} = \begin{pmatrix} \bar{\phi} \\ \bar{\phi}^* \end{pmatrix}. \tag{4.28}$$

The integral over $\tilde{\Phi}$ is Gaussian in the Bogoliubov approximation and therefore easily performed to yield an additional term to the effective action

$$S_\Delta = -\frac{1}{2}\int_{x,y} \psi(\mathbf{x})\bar{\Phi}^\dagger\, G_0(x-y)\bar{\Phi}\psi(\mathbf{y}), \tag{4.29}$$

where the propagator G_0 is the inverse of the matrix M_0 introduced in (1.170) with the field $U(x)$ set to zero. Let us first Fourier transform the fields,

$$G_0(x-y) = \int_k \mathrm{e}^{-ik\cdot(x-y)}\, G_0(k) \tag{4.30}$$

$$\psi(\mathbf{x}) = \int_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{k}). \tag{4.31}$$

The contribution to the effective action then appears in the form

$$S_\Delta = -\frac{1}{2}\int_{\mathbf{k}} |\psi(\mathbf{k})|^2\bar{\Phi}^\dagger G(0,\mathbf{k})\bar{\Phi}. \tag{4.32}$$

Since the random field is Gaussian distributed [see (4.10)], the average over this field representing quenched impurities yields,

$$\langle|\psi(\mathbf{k})|^2\rangle = \tfrac{1}{2}V\Delta_0. \tag{4.33}$$

The remaining integral over the loop momentum in (4.32) is readily carried out to yield in arbitrary space dimensions

$$\langle\mathcal{L}_\Delta\rangle = \frac{1}{2}\Gamma(1-d/2)\left(\frac{m}{2\pi}\right)^{d/2}|\bar{\phi}|^2(6\lambda_0|\bar{\phi}|^2-\mu_0)^{d/2-1}\Delta_0. \tag{4.34}$$

Because this is a one-loop result, we may to this order replace the bare parameters with the (one-loop) renormalized ones.

In Sec. 1.6 we saw that due to the interparticle repulsion, not all the particles reside in the condensate. We expect that the random field causes an additional depletion of the condensate. To obtain this, we differentiate (4.34) with respect to the chemical potential. This gives [200, 201]

$$\bar{n}_\Delta = \frac{\partial \langle \mathcal{L}_\Delta \rangle}{\partial \mu} = \frac{2^{d/2-5}\Gamma(2-d/2)}{\pi^{d/2}} m^{d/2} \lambda^{d/2-2} \bar{n}_0^{d/2-1} \Delta_0, \tag{4.35}$$

where $\bar{n}_0$ denotes the number density of particles residing in the condensate. The divergence in the limit $\lambda \to 0$ for $d < 4$ signals the collapse of the system when the interparticle repulsion is removed.

We next calculate the mass current $\mathbf{g}$ to determine superfluid mass density, i.e., the mass density flowing with the superfluid velocity $\mathbf{v}_{\rm s}$. As we have seen in the preceding section, in the absence of impurities and at zero temperature all the particles participate in the superflow and move on the average with the velocity $\mathbf{v}_{\rm s}$. We expect this no longer to hold in the presence of impurities. To determine the change in the superfluid mass density due to impurities, we replace μ_0 with $\mu_{\rm eff}$ as defined in (1.236) and $i\partial_0$ with $i\partial_0 - (\mathbf{u}-\mathbf{v}_{\rm s})\cdot(-i\nabla)$ in the contribution (4.32) to the effective action, and differentiate it with respect to the externally imposed velocity, $-\mathbf{u}$. We find to linear order in the difference $\mathbf{u} - \mathbf{v}_{\rm s}$:

$$\mathbf{g} = \rho_{\rm s}\mathbf{v}_{\rm s} + \rho_{\rm n}\mathbf{u}, \tag{4.36}$$

with the superfluid and normal mass density [201]

$$\rho_{\rm s} = m\left(\bar{n} - \frac{4}{d}\bar{n}_\Delta\right), \quad \rho_{\rm n} = \frac{4}{d} m \bar{n}_\Delta. \tag{4.37}$$

We see that the normal density is a factor $4/d$ larger than the mass density $m\bar{n}_\Delta$ knocked out of the condensate by the impurities. (For $d = 3$ this gives the factor $\frac{4}{3}$ first found in Ref. [199].) Apparently, part of the zero-momentum states belongs for $d < 4$ not to the condensate, but to the normal fluid. Being trapped by the impurities, this fraction of the zero-momentum states are localized. This shows that the phenomenon of localization can be accounted for in the Bogoliubov theory of superfluidity by including a random field.

An other realistic modification of the nonrelativistic $|\phi|^4$-theory is to include a $1/r$ Coulomb repulsion. Using very general scaling arguments (see Sec. 4.5) in the context of the effective theory, Fisher, Grinstein, and Girvin [188] predicted that in the presence of impurities the inclusion of this interaction changes the value of the dynamic exponent to $z = 1$, which is exact again. This prediction has been confirmed in experiments on the superconductor-to-insulator transition in two-dimensional films [192]. The same value $z = 1$ has also been found in the quantum Hall transitions, and—more recently—in metal-to-insulating transitions observed in dilute two-dimensional electron systems in silicon MOSFETS [202, 203, 204, 205]. This suggests that the $1/r$ Coulomb interaction plays an important role in these systems too. However, let us at this stage mention the renormalization-group results obtained by Giamarchi and Schulz [206] who studied fermions with *short-range* interactions in a random one-dimensional system. They

found first of all that the superconductor-to-insulator transition these fermions undergo is indeed in the universality class of repulsively interacting bosons in the presence of impurities. Moreover, their result for the conductivity is consistent with the value $z = 1$ [188], implying that already a local repulsive interaction as used in the nonrelativistic $|\phi|^4$-theory can lead to this value for the dynamic exponent.

4.3 CSGL Theory

We now turn to the fractional quantized Hall effect (FQHE) which is the hallmark of a new, intrinsically two-dimensional condensed-matter state—the quantum Hall liquid. Many aspects of this state are well understood in the framework of the quantum-mechanical picture developed by Laughlin [207]. Considerable effort has nevertheless been invested in formulating an effective field theory which captures the essential low-energy, small-momentum features of the liquid. A similar approach in the context of superconductors has proven most successful. Initially, only the phenomenological model proposed by Ginzburg and Landau [75] in 1950 was known here. Most of the fundamental properties of the superconducting state such as superconductivity—the property that gave this condensed-matter state its name, Meissner effect, magnetic flux quantization, Abrikosov flux lattice, and Josephson effect, can be explained by the model. The microscopic theory was given almost a decade later by Bardeen, Cooper, and Schrieffer [24]. Shortly here after, Gorkov [76] made the connection between the two approaches by deriving the Ginzburg-Landau model from the microscopic BCS theory, thus giving the phenomenological model the status of an effective field theory.

A first step towards an effective field theory of the quantum Hall liquid was taken by Girvin and MacDonald [208] and has been developed further by Zhang, Hansson and Kivelson [209], who also gave an explicit construction starting from a microscopic Hamiltonian. Their formulation incorporates time dependence which is important for the study of quantum phase transitions. This approach has proven very successful (for a review see Ref. [210]). In this and the following section, we shall argue that the effective theory—the so-called Chern-Simons-Ginzburg-Landau (CSGL) theory—can also be employed to describe the quantum phase transition which a quantum Hall system undergoes as the applied magnetic field changes.

We shall in this section first recall some basic properties of the CSGL theory and show how it can be used to describe the field-induced Hall-liquid-to-insulator transition of a Hall liquid. In Sec. 4.4 we then apply renormalization group theory to the CSGL theory and study its critical properties.

An important ingredient for obtaining an effective theory of the FQHE was the identification by Girvin and MacDonald [208] of a bosonic operator ϕ which exhibits (algebraic) off-diagonal long-range order. The long-range order was found to be of a type known to exist in two-dimensional bosonic superfluids. They argued that this field should be viewed as an order parameter in terms of which the effective field theory should be formulated. To account for the incompressibility of the quantum Hall liquid they suggested to couple ϕ to a statistical gauge field a_μ. The gapless spectrum of the neutral system then changes into one with an energy gap [209], thus rendering the charged system incompressible.

Girvin and MacDonald assumed that the statistical gauge field is governed solely by a Chern-Simons term

$$\mathcal{L}_{\rm CS} = \frac{\theta}{2}\partial_0 \mathbf{a} \times \mathbf{a} - \theta a_0 \nabla \times \mathbf{a}, \tag{4.38}$$

with $\nabla \times \mathbf{a}$ the statistical magnetic field and θ a constant. Because of the absence of a kinetic term (the usual Maxwell term), the statistical gauge field does not represent a physical degree of freedom. In a relativistic setting, a Maxwell term is usually generated by quantum corrections so that the statistical gauge field becomes dynamical at the quantum level. The quantum theory then differs qualitatively from the classical theory. On the other hand, as we shall see below, this need not be the case in a nonrelativistic setting. That is to say, the *Ansatz* of the absence of a Maxwell term is here not necessarily obstructed by quantum corrections.

The CSGL theory is described by the Lagrangian [209]

$$\mathcal{L} = i\phi^* D_0 \phi - \frac{1}{2m}|\mathbf{D}\phi|^2 + \mu_0|\phi|^2 - \lambda_0|\phi|^4 + \mathcal{L}_{\rm CS}. \tag{4.39}$$

The covariant derivatives $D_0 = \partial_0 + ieA_0 + iea_0$ and $\mathbf{D} = \nabla - ie\mathbf{A} - ie\mathbf{a}$ give a minimal coupling to the applied magnetic and electric field described by the gauge field A_μ and also to the statistical gauge field. For definiteness we will assume that our two-dimensional sample is perpendicular to the applied magnetic field, defining the z-direction, and we choose the electric field to point in the x-direction. The charged field ϕ represents the Girvin-MacDonald order parameter describing the original electrons bound to an odd number of flux quanta. To see that it indeed does, let us consider the field equation for a_0:

$$|\phi|^2 = -e\theta \nabla \times \mathbf{a}. \tag{4.40}$$

The simplest solution of the CSGL Lagrangian is the uniform mean-field solution

$$|\phi|^2 = \bar{n}, \quad \mathbf{a} = -\mathbf{A}, \quad a_0 = -A_0 = 0, \tag{4.41}$$

where $\bar{n}$ indicates the constant fermion number density. The statistical gauge field is seen to precisely cancel the applied field. The constraint equation (4.40) then becomes

$$\bar{n} = e\theta H, \tag{4.42}$$

with H the applied magnetic field. Now, if we choose $\theta^{-1} = 2\pi(2l+1)$, it follows on integrating this equation that, as required, with every electron there is associated $2l+1$ flux quanta:

$$N = \frac{1}{2l+1} N_\otimes, \tag{4.43}$$

where $N_\otimes = \Phi/\Phi_0$, with $\Phi = \int_{\mathbf{x}} H$ the magnetic flux, indicates the number of flux quanta. Equation (4.42) implies an odd-denominator filling factor ν_H which is defined by

$$\nu_H = \frac{\bar{n}}{H/\Phi_0} = \frac{1}{2l+1}. \tag{4.44}$$

The coupling constant $\lambda_0\,(>0)$ in (4.39) is the strength of the repulsive contact interaction between the composite particles, and μ_0 is a chemical potential introduced to account for a finite number density of composite particles.

It is well known from anyon physics that the inclusion of the Chern-Simons term changes the statistics of the field ϕ to which the statistical gauge field is coupled [211]. If one composite particle circles another, it picks up an additional Aharonov-Bohm factor, representing the change in statistics. The binding of an odd number of flux quanta changes the fermionic character of the electrons into a bosonic one for the composite particles, allowing them to Bose condense. The algebraic off-diagonal long-range order of a quantum Hall liquid can in this picture be understood as resulting from this condensation. Conversely, a flux quantum carries $1/(2l+1)$th of an electron's charge [207], and also $1/(2l+1)$th of an electron's statistics [212].

The defining phenomenological properties of a quantum Hall liquid are easily shown to be described by the CSGL theory [209, 210]. From the lowest-order expression for the induced electromagnetic current one finds

$$ej_i = \frac{\delta\mathcal{L}}{\delta A_i} = -\frac{\delta\mathcal{L}_\phi}{\delta a_i} = \frac{\delta\mathcal{L}_{\rm CS}}{\delta a_i} = -e^2\theta\epsilon_{ij}(\partial_0 a_j - \partial_j a_0) = e^2\theta\epsilon_{ij}E_j, \qquad (4.45)$$

with $\mathbf{E}$ the applied electric field and where the Lagrangian (4.39) is written as a sum $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_{\rm CS}$. It follows that the Hall conductance σ_{xy} is quantized in odd fractions of $e^2/2\pi$, or, reinstalling Planck's constant, e^2/h. This result can also be understand in an intuitive way as follows. Since the composite particles carry a charge e, the applied electric field gives rise to an electric current

$$I = e\frac{{\rm d}N}{{\rm d}t} \qquad (4.46)$$

in the direction of $\mathbf{E}$, i.e., the x-direction. This is not the end of the story because the composite objects carry in addition to electric charge also $2l+1$ flux quanta. When the Goldstone field φ encircles $2l+1$ flux quanta, it picks up a factor 2π for each of them

$$\oint_\Gamma \nabla\cdot\varphi = 2\pi(2l+1). \qquad (4.47)$$

Now, consider two points across the sample from each other. Let the phase of these points initially be equal. As a composite particle moves downstream, and crosses the line connecting the two points, the relative phase $\Delta\varphi$ between them changes by $2\pi(2l+1)$. This phase slippage [9] leads to a voltage drop across the sample given by

$$V_{\rm H} = \frac{1}{e}\partial_0\Delta\varphi = (2l+1)\Phi_0\frac{{\rm d}N}{{\rm d}t}, \qquad (4.48)$$

where the first equation can be understood by recalling that due to minimal coupling $\partial_0\varphi \to \partial_0\varphi + eA_0$. For the Hall resistance we thus obtain the expected value

$$\rho_{xy} = \frac{V_{\rm H}}{I} = (2l+1)\frac{2\pi}{e^2}. \qquad (4.49)$$

If the CSGL theory is to describe an incompressible liquid, the spectrum of the single-particle excitations must have a gap. Without the coupling to the statistical gauge

field, the spectrum is given by the gapless Bogoliubov spectrum (1.174). To obtain the single-particle spectrum of the coupled theory, we integrate out the statistical gauge field. The integration over a_0 was shown to yield the constraint (4.40) which in the Coulomb gauge $\nabla \cdot \mathbf{a} = 0$ is solved by

$$a_i = \frac{1}{e\theta}\epsilon_{ij}\frac{\partial_j}{\nabla^2}|\phi|^2. \tag{4.50}$$

The integration over the remaining components of the statistical gauge field is now simply performed by substituting (4.50) back into the Lagrangian. The only nonzero contribution arises from the term $-e^2|\phi|^2\mathbf{a}^2/2m$. The spectrum of the charged system acquires as a result an energy gap $\omega_{\rm c}$

$$E(\mathbf{k}) = \sqrt{\omega_{\rm c}^2 + \epsilon^2(\mathbf{k}) + 2\mu_0\epsilon(\mathbf{k})}, \tag{4.51}$$

with $\omega_{\rm c} = \mu_0/2\theta m\lambda_0$. To lowest order, the gap equals the cyclotron frequency of a free charge e in a magnetic field H

$$\omega_{\rm c} = \frac{\bar{n}}{\theta m} = \frac{eH}{m}. \tag{4.52}$$

The presence of this energy gap results in dissipationless flow with $\sigma_{xx} = 0$.

These facts show that the CSGL theory captures the essentials of a quantum Hall liquid. Given this success, it is tempting to investigate if the theory can also be employed to describe the field-induced Hall-liquid-to-insulator transitions. It should however be borne in mind that both the $1/|\mathbf{x}|$-Coulomb potential as well as impurities should be incorporated into the theory in order to obtain a realistic description of the FQHE. The repulsive Coulomb potential is believed to play a decisive role in the formation of the the composite particles, while the impurities are responsible for the width of the Hall plateaus. As the magnetic field moves away from the magic filling factor, magnetic vortices will materialize in the system to make up the difference between the applied field and the magic field value. In the presence of impurities, these defects get pinned and do not contribute to the resistivities, so that both σ_{xx} and σ_{xy} are unchanged. Only if the difference becomes too large, the system reverts to an other quantum Hall state with a different filling factor.

For positive bare chemical potential $\mu_0 > 0$ the Girvin-MacDonald order parameter ϕ has a nonvanishing expectation value given by $|\phi|^2 = \mu_0/2\lambda_0$, implying that the composite particles are condensed. When $\mu_0 \to 0$, the condensate is drained of composite particles, and at $\mu_0 = 0$, it vanishes altogether. The system becomes critical here and reverts to the insulating phase characterized by a negative bare chemical potential.

In the spirit of Landau, we take a phenomenological approach towards the field-induced phase transition of the CSGL theory. And assume that when the applied magnetic field H is close to the upper critical field $H^+_{\nu_H}$ at which the quantum Hall liquid with filling factor ν_H is destroyed, the chemical potential of the composite particles depends linearly on H, i.e., $\mu_0 \sim eH^+_{\nu_H} - eH$. This state can of course also be destroyed by lowering the applied field. If the system is near the lower critical field $H^-_{\nu_H}$, we assume that the chemical potential is instead given by $\mu_0 \sim eH - eH^-_{\nu_H}$. This is the basic postulate of our approach.

In the CSGL Lagrangian (4.39) we have again transformed the mass m of the nonrelativistic $|\phi|^4$-theory away. In addition to the engineering dimensions (4.4), we have for the Chern-Simons field

$$[ea_i] = 1, \quad [ea_0] = 2, \quad [\theta] = 0. \tag{4.53}$$

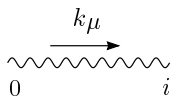
In two space dimensions the coupling constant λ_0 was seen to be dimensionless, implying that the $|\phi|^4$-term is a marginal operator. We see from (4.53) that also the Chern-Simons term is a marginal operator. Hence, the CSGL theory contains precisely those terms relevant to the description of the Hall-liquid-to-insulator transition in a quantized Hall system.

4.4 Renormalization Group

We now turn our attention to the critical properties of the CSGL theory. A remaining open problem in this context is the nature of the Hall-liquid-to-insulating transitions in the theory. Taking place at the absolute zero of temperature, these transitions are a pure quantum phenomenon. They are driven either by impurities, a periodic potential, or by the applied magnetic field. The impurity-driven phase transition has been considered in Ref. [213], while the transition driven by a periodic potential has been studied in Refs. [214, 215]. The latter problem was mapped onto a *relativistic* CSGL theory and it was concluded [215] that the transition induced by a periodic potential is 1st-order. We in this section shall discuss the transition induced by the magnetic field in the absence of both impurities and a periodic potential. We will be working with the standard nonrelativistic CSGL theory. As has been eluded to above, there are essential differences between relativistic and nonrelativistic Chern-Simons theories. There are features special to the nonrelativistic theory which in our view are important for understanding the field-induced transition, and which cannot be implemented in a relativistic formulation.

Experimentally, if the external field is changed so that the filling factor ν_H moves away from an odd-denominator value, the system eventually becomes critical and undergoes a transition to an insulating phase. We will argue that this feature is encoded in the CSGL theory. We shall show that the theory has an IR fixed point, implying that the phase transition is 2nd-order, and determine the critical exponents. They turn out to be universal and independent of the filling fraction.

We will be working in the Coulomb gauge. To this end we add a gauge-fixing term $(\nabla \cdot \mathbf{a})^2/2\alpha_{\rm gf}$ to the Lagrangian (4.39), and take the limit $\alpha_{\rm gf} \to 0$. In this gauge, the propagator of the statistical gauge field is purely off-diagonal



$$= G_{0i}(k_) = -G_{i0}(k) = i\alpha\epsilon_{ij}\frac{k_j}{\mathbf{k}^2}, \tag{4.54}$$

with $\alpha = 1/\theta$. The two propagators (4.5) and (4.54) of the CSGL theory are causing infrared divergences. We regularize these by introducing an infrared cutoff via the substitution $\mathbf{k}^2 \to \mathbf{k}^2 + 2r_0$ in both propagators. In the scalar propagator, this amounts to

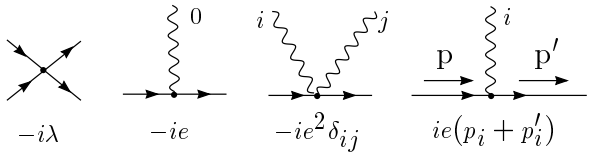


Figure 4.5: Feynman rules of the CSGL theory.

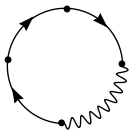


Figure 4.6: A closed oriented loop containing in addition a gauge field propagator.

taking a negative chemical potential, $\mu_0 = -r_0$, so that we are in the symmetrical state of the theory. Because we consider the theory in its upper critical dimension and at criticality, the parameter r_0 is the only dimensional parameter present. It is necessary to introduce a dimensional parameter in a theory not having any, not just to avoid infrared divergences, but also to be able to carry out the renormalization-group program—this parameter playing the role of renormalization-group scale parameter. There are various ways to do so, our proposal is one possibility. Another would be to take the renormalization point at nonzero external momenta, as has been done by, for example, Bergman and Lozano [216]. The critical properties are, of course, independent of the particular choice.

We evaluate Feynman diagrams according to the following computational scheme. First, we carry out the discrete sums over repeated indices as well as the integrations over loop energies. Then, to handle ultraviolet divergences, the resulting momentum integrals are analytically continued to arbitrary space dimensions d. It is to be noted that this step is taken after the discrete sums have been carried out and the antisymmetric tensor ϵ_{ij}—which makes sense only in $d = 2$—has disappeared from the expressions. Finally, the counter terms and the critical exponents are determined by employing the minimal subtraction scheme in which only those parts of the counter terms are considered which diverge in the limit $d \to 2$. The Feynman rules for the vertices are given in Fig. 4.5.

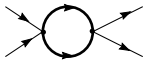
The reason we have chosen the Coulomb gauge is that in this gauge the statistical gauge field propagator does not depend on the energy variable k_0 [see Eq. (4.54)]. This means that the analytic structure of a given diagram is determined solely by its scalar propagators. As a result, the rule that closed oriented loops vanish is not corrupted if a loop contains in addition gauge field propagators. (See Fig. 4.6 for an example.) Hence, also in the coupled theory most diagrams vanish, making the quantum critical behavior of the CSGL theory tractable.

Figure 4.7: A one-loop and a two-loop example of vanishing diagrams which would otherwise renormalize the charge e.

In particular, all the diagrams which would renormalize the charge e vanish. (A one-loop and a two-loop example of this class of diagrams are depicted in Fig. 4.7.) Put differently, no Maxwell term for the statistical gauge field is generated at the quantum level so that the statistical gauge field propagator (4.54) remains purely off-diagonal. This is an important characteristic of the nonrelativistic CSGL theory not shared by the corresponding relativistic extension where a Maxwell term is automatically generated at the quantum level. Since the Chern-Simons term is also not renormalized, the statistical gauge field propagator (4.54) is not effected at all by quantum corrections.

As in the neutral case, the boson self-energy Σ also vanishes at every loop order in the charged theory, and the only object that renormalizes is the self-coupling parameter λ_0. Consequently, if an IR fixed point is found in the CSGL theory, the 2nd-order phase transition described by it has Gaussian exponents.

In order to examine the presence of an IR fixed point, we compute the beta function using the scheme outlined in [217]. To this end we evaluate the diagrams:

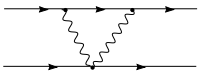


$$\begin{aligned} &= 8\lambda^2 \int_k \frac{1}{-k_0 - \frac{1}{2}\mathbf{k}^2 - r + i\eta} \; \frac{1}{k_0 - \frac{1}{2}\mathbf{k}^2 - r + i\eta} \\ &= 8i\lambda^2 \int_{\mathbf{k}} \frac{1}{\mathbf{k}^2 + 2r} = 8i\lambda^2 \, I_d, \end{aligned} \tag{4.55}$$

with λ and r denoting the renormalized parameters,

$$I_d = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \frac{1}{(2r)^{1-d/2}}, \tag{4.56}$$

and



$$\begin{aligned} &= \alpha^2 \epsilon_{jl} \, \epsilon_{jm} \int_k \frac{k_l}{\mathbf{k}^2 + 2r} \; \frac{1}{k_0 - \frac{1}{2}\mathbf{k}^2 - r + i\eta} \; \frac{k_m}{\mathbf{k}^2 + 2r} \\ &= -\tfrac{1}{2} i\alpha^2 \int_{\mathbf{k}} \frac{\mathbf{k}^2}{(\mathbf{k}^2 + 2r)^2} \\ &= -\tfrac{1}{2} i\alpha^2 \Omega_d \left[\Gamma\left(1 - \frac{d}{2}\right) - \Gamma\left(2 - \frac{d}{2}\right) \right], \end{aligned} \tag{4.57}$$

where we introduced the abbreviation

$$\Omega_d = \frac{1}{(4\pi)^{d/2}} \frac{1}{(2r)^{1-d/2}}. \tag{4.58}$$

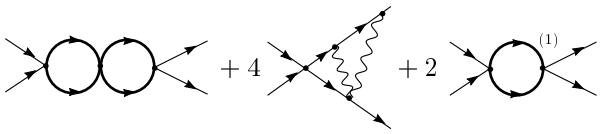
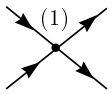


Figure 4.8: The relevant two-loop diagrams.

There are three other diagrams of the form (4.57), one having the right external legs crossed and two having the triangle turned upside-down. To account for these, the result (4.57) must be included with an additional factor of 4. The diagrams (4.55) and (4.57) are found to diverge in the limit $d \to 2$. The divergences are cancelled by including the one-loop counter term



$$= -i4\Omega_2 \frac{1}{\epsilon}\left(4\lambda^2 - \alpha^2\right), \tag{4.59}$$

with $\epsilon = 2 - d$ the deviation from the upper critical dimension and $\Omega_2 = 1/4\pi$. This leads in the usual way to the one-loop beta function in the upper critical dimension ($d = 2$ in our case)

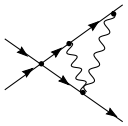
$$\beta(\lambda) = -\left(1 - \lambda\frac{\partial}{\partial\lambda} - \alpha\frac{\partial}{\partial\alpha}\right) a_1 = \frac{1}{\pi}\left(4\lambda^2 - \alpha^2\right), \tag{4.60}$$

where a_1 is (i times) the residue of the simple pole in the counter term (4.59). The result is in accordance with previous studies by Lozano and Bergman [216] who used a momentum cutoff to regularize the ultraviolet divergences.

We continue with the two-loop calculation. The relevant diagrams are given in Fig. 4.8. The factor 4 in the second diagram is to account for the related diagram having the 4-vertex to the right rather than to the left and for the diagrams having the triangles turned upside-down. The factor 2 in the last diagram accounts for the fact that there are two vertices which can be replaced with the counter term (4.59). The first diagram is readily shown to yield

$$= -16i\lambda^3 I_d^2 \tag{4.61}$$

whereas the second gives

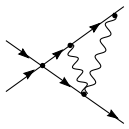


$$= -2i\lambda\alpha^2 \int_{\mathbf{k},\mathbf{l}} \frac{1}{\mathbf{k}^2 + 2r} \frac{1}{\mathbf{l}^2 + 2r} \frac{(\mathbf{k} - \mathbf{l}) \cdot \mathbf{l}}{(\mathbf{k} - \mathbf{l})^2 + 2r}. \tag{4.62}$$

To evaluate the remaining integrals, we employ the identity

$$\int_{\mathbf{k}} \frac{\mathbf{k}}{(\mathbf{k}^2 + 2r)\left[(\mathbf{k} - \mathbf{l})^2 + 2r\right]} = \frac{1}{2}\int_{\mathbf{k}} \frac{\mathbf{l}}{(\mathbf{k}^2 + 2r)\left[(\mathbf{k} - \mathbf{l})^2 + 2r\right]} \tag{4.63}$$

which can be proven by introducing a new integration variable $\mathbf{k}' = -\mathbf{k} + \mathbf{l}$ in the left-hand side. In this way, we obtain for the second diagram



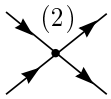
$$= i\lambda\alpha^2 \left(I_d^2 - J_d\right), \tag{4.64}$$

where J_d is the integral

$$J_d = 2r \int_{\mathbf{k},\mathbf{l}} \frac{1}{\mathbf{k}^2 + 2r} \frac{1}{\mathbf{l}^2 + 2r} \frac{1}{(\mathbf{k} - \mathbf{l})^2 + 2r} \tag{4.65}$$

which remains finite in the limit $d \to 2$. Finally, the last diagram in Fig. 4.8 is given by (4.55) with $-i\lambda$ replaced with the counter term (4.59).

When the results are added, the sum of the three diagrams in Fig. 4.8 is shown to diverge again when the space dimension is given its physical value $(d = 2)$. The theory can be rendered finite to this order by introducing the two-loop counter term



$$= -16i\,\Omega_2^2 \frac{1}{\epsilon^2} \lambda \left(4\lambda^2 - \alpha^2\right). \tag{4.66}$$

Surprisingly, the counter term does not contain a simple pole in ϵ. Since the beta function is determined by the residue of the simple pole, it follows that the one-loop result (4.60) is unchanged at the two-loop level.

The calculation of the beta function was extended to third order in the loop expansion by Freedman, Lozano, and Rius [218] using differential regularization, and to fourth order by this author [219] applying the more conventional methods explained in this section. We found that the third and fourth order diagrams diverge like $1/\epsilon^3$ and $1/\epsilon^4$, respectively. In particular, the simple poles in ϵ dropped out so that the one-loop beta function is unaffected by three- and four-loop corrections. It is tempting to conjecture that this feature persists to all orders in perturbation theory, implying that—just as in the neutral system which corresponds to taking the limit $\theta \to \infty$— the one-loop beta function (4.60) is exact.

The beta function is schematically represented in Fig. 4.9. It yields an IR fixed point ${\lambda^*}^2 = \frac{1}{4}\alpha^2$ determined by $\theta = \alpha^{-1}$, or, equivalently, by the filling factor. More precisely, the strength of the repulsive coupling at the fixed point $\lambda^* = \pi(2l + 1)$ increases with the number $2l + 1\,(= 1, 3, 5, \ldots)$ of flux quanta bound to the electron. The presence of the fixed point shows that the CSGL theory undergoes a 2nd-order phase transition when the chemical potential of the composite particles tends to zero. Since the self-energy Σ is identically zero, the nontrivial fixed point has nevertheless Gaussian exponents, $\nu = \frac{1}{2},\ z = 2,\ \eta = 0$. It should be noted that only the location of the fixed point depends on θ, the critical exponents—which in contrast to the strength of the coupling at the fixed point are independent of the regularization and renormalization scheme—are universal and independent of the filling factor. This "triviality" is

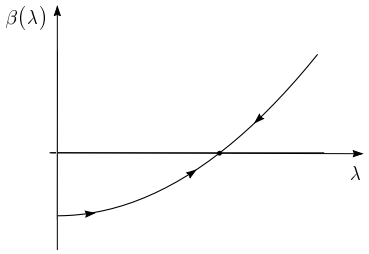


Figure 4.9: Schematic representation of the β-function.

in accord with the experimentally observed universality of the FQHE. A dependence of the critical exponents on θ could from the theoretical point of view hardly be made compatible with the hierarchy construction [220] which implies a cascade of phase transitions. From this viewpoint the present results are most satisfying: the CSGL theory is shown to encode a new type of field-induced 2nd-order quantum phase transition that is simple enough not to obscure the observed universality of the FQHE.

Most of the properties found in the nonrelativistic theory are not shared by the relativistic version of the CSGL theory. On the contrary, by calculating the effective potential, Pryadko and Zhang [215] argued that the relativistic theory undergoes a fluctuation-induced 1st-order transition—similar to the one in massless scalar electrodynamics in 3+1 spacetime dimensions first studied by Coleman and Weinberg [221]. From the standpoint of dimensional analysis the discrepancy in critical behavior is not surprising. Because the effective spacetime dimensionality in the nonrelativistic theory is increased by one, the scaling dimension of a given operator generally differs in both theories.

As a side remark we mention that the theory (4.39) with zero chemical potential has also been successfully applied to the problem of anyon scattering [216]. From it, the two-particle scattering amplitude, which was known from the work of Aharonov and Bohm [222, 211] on the scattering of a charged particle from a magnetic flux tube, could be calculated. The result of Aharonov-Bohm could be reproduced provided the coupling constant was given the specific value $\lambda^* = \frac{1}{2}\alpha$ corresponding to the IR fixed point. It is sometimes presented as if taking this value is a mere choice. However, the scattering amplitudes considered here pertain to low-energy, small-momentum phenomena, and these are governed by the IR fixed point. In other words, in describing these processes, there is no arbitrariness at all; one must fix the value of the self-coupling λ to correspond to the IR fixed point.

Because of the necessity to introduce a dimensional parameter to avoid infrared divergences and to be able to carry out the renormalization-group program, the scale invariance of the classical theory (4.39) with $\mu = 0$, first discussed in Ref. [223], is broken at nonfixed points. Only at the fixed point the theory is scale invariant. From the discussion above, it follows that the low-energy, small-momentum behavior of the system is also scale invariant since it is governed by the fixed point.

The same remarks apply to the thermodynamic properties of an anyon gas which can also be computed from the Lagrangian (4.39), as was done in Ref. [224]. These, too, are low-energy, small-momentum properties governed by the IR fixed point so that

one must set $\lambda^* = \frac{1}{2}\alpha$ when calculating them.

The results obtained so far are gratifying. The CSGL theory, which captures the essential properties of a quantum Hall liquid, was shown to be capable of also describing the Hall-liquid-to-insulator transitions—at least in the absence of impurities. However, impurities plays an essential part in the FQHE, so a realistic description of the critical behavior of a quantum Hall liquid should account for impurities. Including (quenched) impurities in the superfluid state of repulsively interacting bosons was shown to lead to the phenomenon of localization which is of paramount importance to the FQHE. Since upon invoking a statistical gauge field this superfluid state represents a quantum Hall liquid, this is in principle a good starting point to study the quantum critical behavior of the CSGL model in the presence of impurities. There is, however, a technical problem. We have worked in a double epsilon expansion thereby leaving physical spacetime. Unfortunately, the Chern-Simons term governing the statistical gauge field of the theory is defined only in $2 + 1$ dimensions. This makes a study of the critical behavior of the random CSGL theory in a double epsilon expansion impossible, and a different method, possibly nonperturbative in character, is required.

Luckily, using general scaling arguments and information contained in the effective theories considered in this report one is able to acquire some additional understanding of the behavior of various quantum phase transitions studied in experiment, although precise estimates for critical exponents other than the dynamic one cannot be made on this basis.

4.5 Scaling Theory

The traditional scaling theory of classical 2nd-order phase transitions, first put forward by Widom [225], is easily extended to include the time dimension [183] because relation (4.1) implies the presence of only one independent diverging scale. Let $\delta = K - K_{\rm c}$, with K the parameter that drives the phase transition, measure the distance from the critical coupling $K_{\rm c}$. A physical observable at the absolute zero of temperature $O(k_0, |\mathbf{k}|, K)$ can in the critical region close to the transition be written as

$$O(k_0, k, K) = \xi^{d_O} \mathcal{O}(k_0 \xi_t, |\mathbf{k}|\xi), \qquad (T = 0), \tag{4.67}$$

where d_O is the dimension of the observable O. The right-hand side does not depend explicitly on K; only implicitly through ξ and ξ_t. The closer one approaches the critical coupling $K_{\rm c}$, the larger the correlation length and time become.

Since a physical system is always at some finite temperature, we have to investigate how the scaling law (4.67) changes when the temperature becomes nonzero. The easiest way to include temperature in a quantum field theory is to go over to imaginary time $\tau = it$, with τ restricted to the interval $0 \leq \tau \leq \beta$. The temporal dimension becomes thus of finite extend. The critical behavior of a phase transition at finite temperature is, as we remarked before, still controlled by the quantum critical point provided $\xi_t < \beta$. If this condition is fulfilled, the system does not see the finite extend of the time dimension. Instead of the zero-temperature scaling (4.67), we now have the finite-size scaling

$$O(k_0, |\mathbf{k}|, K, T) = \beta^{d_O/z} \mathcal{O}(k_0 \beta, |\mathbf{k}|\beta^{1/z}, \beta/\xi_t), \qquad (T \neq 0). \tag{4.68}$$

The distance to the quantum critical point is measured by the ratio $\beta/\xi_t \sim |\delta|^{z\nu}/T$.

Let us continue to derive hyperscaling relations. To this end, we consider the two terms in the effective theory (1.177) quadratic in the Goldstone field φ with m effectively set to 1 and write it in the most general form [226]:

$$\mathcal{L}^{(2)}_{\rm eff} = -\tfrac{1}{2}\bar{\rho}_{\rm s}(\nabla\varphi)^2 + \tfrac{1}{2}\bar{n}^2\kappa(\partial_0\varphi)^2. \tag{4.69}$$

The coefficient $\rho_{\rm s}$ is the superfluid mass density which in the presence of, for example, impurities does not equal $m\bar{n}$—even at the absolute zero of temperature. The other coefficient,

$$\bar{n}^2\kappa = \frac{\partial\bar{n}}{\partial\mu} = \lim_{|\mathbf{k}|\to 0}\Pi_{00}(0,\mathbf{k}), \tag{4.70}$$

with Π_{00} the (0 0)-component of the full polarization tensor (1.179), involves the full compressibility and particle number density. This is because the chemical potential is represented in the effective theory by $\mu = -\partial_0\varphi$ and

$$\frac{\partial^2\mathcal{L}_{\rm eff}}{\partial\mu^2} = \bar{n}^2\kappa. \tag{4.71}$$

Equation (4.69) leads to the general expression of the sound velocity

$$c^2 = \frac{\bar{\rho}_{\rm s}}{\bar{n}^2\kappa} \tag{4.72}$$

at the absolute zero of temperature.

Let us next assume that the chemical potential is the control parameter, so that $\delta \propto \mu - \mu_{\rm c}$ denotes the distance from the phase transition, and $\xi \sim |\delta|^{-\nu}$. Now, on the one hand, the singular part of the free energy density $f_{\rm sing}$ arises from the low-energy, long-wavelength fluctuations of the Goldstone field. (Here, we adopted the common practice of using the symbol f for the density Ω/V and of referring to it as the free energy density.) The ensemble averages give

$$\langle(\nabla\varphi)^2\rangle \sim \xi^{-2}, \quad \langle(\partial_0\varphi)^2\rangle \sim \xi_t^{-2} \sim \xi^{-2z}. \tag{4.73}$$

On the other hand, dimensional analysis shows that the singular part of the free energy density scales near the transition as

$$f_{\rm sing} \sim \xi^{-(d+z)}. \tag{4.74}$$

Combining these hyperscaling arguments, we arrive at the following conclusions:

$$\rho_{\rm s} \sim \xi^{-(d+z-2)}, \quad \bar{n}^2\kappa \sim \xi^{-(d-z)} \sim |\delta|^{(d-z)\nu}. \tag{4.75}$$

The first conclusion is consistent with the universal jump (3.12) predicted by Nelson and Kosterlitz [132] which corresponds to taking $z = 0$ and $d = 2$. Since $\xi \sim |\delta|^{-\nu}$, $f_{\rm sing}$ can also be directly differentiated with respect to the chemical potential to yield for the the singular part of the compressibility

$$\bar{n}^2\kappa_{\rm sing} \sim |\delta|^{(d+z)\nu-2}. \tag{4.76}$$

Fisher and Fisher [226] continued to argue that there are two alternatives. Either $\kappa \sim \kappa_{\rm sing}$, implying $z\nu = 1$; or the full compressibility κ is constant, implying $z = d$. The former is consistent with the Gaussian values $\nu = \frac{1}{2},\ z = 2$ found by Uzunov [189] for the pure case in $d < 2$. The latter is believed to apply to repulsively interacting bosons in a random media. These remarkable simple arguments thus predict the exact value $z = d$ for the dynamic exponent in this case.

The above hyperscaling arguments have been extended by Fisher, Grinstein, and Girvin [188] to include the $1/|\mathbf{x}|$-Coulomb potential. The quadratic terms in the effective theory (1.201) may be cast in the general form

$$\mathcal{L}^{(2)}_{\rm eff} = \frac{1}{2}\left(\rho_{\rm s}\mathbf{k}^2 - \frac{|\mathbf{k}|^{d-1}}{\hat{e}^2}k_0^2\right)|\varphi(k_0,\mathbf{k})|^2, \tag{4.77}$$

where $\hat{e}$ is the renormalized charge. From (1.201) we find that to lowest order:

$$\hat{e}^2 = 2^{d-1}\pi^{(d-1)/2}\Gamma\left[\tfrac{1}{2}(d-1)\right]e_0^2. \tag{4.78}$$

The renormalized charge is connected to the (0 0)-component of the full polarization tensor (1.107) via

$$\hat{e}^2 = \lim_{|\mathbf{k}|\to 0}\frac{|\mathbf{k}|^{d-1}}{\Pi_{00}(0,\mathbf{k})}. \tag{4.79}$$

A simple hyperscaling argument like the ones given in the preceding paragraph shows that near the transition, the renormalized charge scales as

$$\hat{e}^2 \sim \xi^{1-z}. \tag{4.80}$$

They then argued that in the presence of random impurities this charge is expected to be finite at the transition so that $z = 1$. This again is an exact results which replaces the value $z = d$ of the neutral system.

The quantum phase transitions we are considering take place in charged systems and are mainly probed by conductivity σ or resistivity ρ measurements. To see how the conductivity σ relates to the superfluid mass density $\rho_{\rm s}$, we minimally couple the effective theory (4.69) to an electromagnetic gauge field. The only relevant term for this purpose is the first one in (4.69) with $\nabla\varphi$ replaced by $\nabla\varphi - e\mathbf{A}$, where we allow the superfluid mass density to vary in space and time. The term in the action quadratic in $\mathbf{A}$ then becomes in the Fourier representation

$$S_\sigma = -\tfrac{1}{2}e^2\int_{k_0,\mathbf{k}}\mathbf{A}(-k_0,-\mathbf{k})\rho_{\rm s}(k_0,\mathbf{k})\mathbf{A}(k_0,\mathbf{k}). \tag{4.81}$$

The electromagnetic current,

$$\mathbf{j}(k_0,\mathbf{k}) = \frac{\delta S_\sigma}{\delta\mathbf{A}(-k_0,-\mathbf{k})} \tag{4.82}$$

obtained from this action can be written as

$$\mathbf{j}(k_0,\mathbf{k}) = \sigma(k_0,\mathbf{k})\mathbf{E}(k_0,\mathbf{k}) \tag{4.83}$$

with the conductivity

$$\sigma(k) = -ie^2 \frac{\rho_{\rm s}(k)}{k_0} \tag{4.84}$$

essentially given by the superfluid mass density divided by k_0, where it should be remembered that the mass m is effectively set to 1 here. By virtue of the scaling relation (4.75), it follows that σ scales as [227]

$$\sigma \sim \xi^{-(d-2)}. \tag{4.85}$$

In other words, the scaling dimension of the conductivity and therefore that of the resistivity is zero in two space dimensions.

Let us now consider a quantum phase transition triggered by changing the applied magnetic field, i.e., $\delta \propto H - H_{\rm c}$. The critical field scales with ξ as $H_{\rm c} \sim \Phi_0/\xi^2$. In fact, this expresses a more fundamental result, namely that the scaling dimension $d_{\mathbf{A}}$ of $\mathbf{A}$ is one,

$$d_{\mathbf{A}} = 1, \tag{4.86}$$

so that $|\mathbf{A}| \sim \xi^{-1}$. From this it in turn follows that $E \sim \xi_t^{-1}\xi^{-1} \sim \xi^{-(z+1)}$, and that the scaling dimension d_{A_0} of A_0 is z,

$$d_{A_0} = z, \tag{4.87}$$

so that $A_0 \sim \xi_t^{-1} \sim \xi^{-z}$. For the DC conductivities in the presence of an external electric field we have on account of the general finite-size scaling form (4.68) with $k_0 = |\mathbf{k}| = 0$:

$$\sigma(H, T, E) = \varsigma(\delta^{\nu z}/T, \delta^{\nu(z+1)}/E). \tag{4.88}$$

This equation shows that conductivity measurements close to the quantum critical point collapse onto a single curve when plotted as function of the dimensionless combinations $\delta^{\nu z}/T$ and $\delta^{\nu(z+1)}/E$. The best collapse of the data determines the values of νz and $\nu(z+1)$. In other words, the temperature and electric-field dependence determine the critical exponents ν and z independently.

4.6 Experiments

4.6.1 Superconductor-To-Insulator Transition

The first experiments we wish to discuss are those performed by Hebard and Paalanen on superconducting films in the presence of random impurities [192, 139]. It has been predicted by Fisher [228] that with increasing applied magnetic field such systems undergo a zero-temperature transition into an insulating state. (For a critical review of the experimental data available in 1993, see Ref. [229].)

Let us restrict ourselves for the moment to the $T\Delta$-plane of the phase diagram by setting the applied magnetic field H to zero. For given disorder strength Δ, the system then undergoes a Kosterlitz-Thouless transition induced by the unbinding of magnetic vortex pairs at a temperature $T_{\rm KT}$ well below the bulk transition temperature (see Sec. 3.2). The Kosterlitz-Thouless temperature is gradually suppressed to zero when the

disorder strength approaches criticality $\hat{\Delta} \to \hat{\Delta}_{\rm c}$. The transition temperature scales with the correlation length $\xi \sim |\hat{\Delta}_{\rm c} - \hat{\Delta}|^{-\nu}$ as $T_{\rm KT} \sim \xi^{-z}$.

In the $H\Delta$-plane, i.e., at $T = 0$, the situation is as follows. For given disorder strength, there is now at some critical value $H_{\rm c}$ of the applied magnetic field a phase transition from a superconducting state of pinned vortices and condensed Cooper pairs to an insulating state of pinned Cooper pairs and condensed vortices. The condensation of vortices disorder the ordered state as happens in classical, finite temperature superfluid- and superconductor-to-normal phase transitions [17]. When the disorder strength approaches criticality again, $H_{\rm c}$ is gradually suppressed to zero. The critical field scales with ξ as $H_{\rm c} \sim \Phi_0/\xi^2$. Together, the scaling results for $T_{\rm KT}$ and $H_{\rm c}$ imply that [228]

$$H_{\rm c} \sim T_{\rm KT}^{2/z}. \tag{4.89}$$

This relation, linking the critical field to the Kosterlitz-Thouless temperature, provides a direct way to measure the dynamic exponent z at the $H = 0$, $T = 0$ transition. This has been done first by Hebard and Paalanen [192, 139]. Their experimental determination of $T_{\rm KT}$ and $H_{\rm c}$ for five different films with varying amounts of impurities confirmed the relation (4.89) with $2/z = 2.04 \pm 0.09$. The zero-temperature critical fields were obtained by plotting $\mathrm{d}\rho_{xx}/\mathrm{d}T|_H$ versus H at the lowest accessible temperature and interpolating to the the field where the slope is zero. The resulting value $z = 0.98 \pm .04$ is in accordance with Fisher's prediction [228], $z = 1$, for a random system with a $1/|\mathbf{x}|$-Coulomb potential.

Hebard and Paalanen [192] also investigated the field-induced zero-temperature transition. The control parameter is here $\delta \propto H - H_{\rm c}$. When plotted as function of $|H - H_{\rm c}|/T^{1/\nu_H z_H}$ they saw their resistivity data collapsing onto two branches; an upper branch tending to infinity for the insulating state, and a lower branch bending down for the superconducting state. The unknown product $\nu_H z_H$ is experimentally determined by looking for which value the best scaling behavior is obtained. Further experiments carried out by Yazdani and Kapitulnik [230] studying the electric-field dependence of the resistivity also determined the product $\nu_H(z_H + 1)$. The two independent measurements together fix the critical exponents ν_H and z_H separately. From their best data, Yazdani and Kapitulnik extracted the values [230]

$$z_H = 1.0 \pm 0.1, \quad \nu_H = 1.36 \pm 0.05. \tag{4.90}$$

4.6.2 Quantum-Hall Systems

We continue to discuss the field-induced quantum phase transitions in quantum Hall systems. Since an excellent discussion recently appeared in the literature [182], we shall be brief, referring the reader to that review for a more thorough discussion and additional references.

One can image transitions from one Hall liquid to another Hall liquid with a different (integer or fractional) filling factor, or to the insulating state. Experiments seem to suggest that all the quantum-Hall transitions are in the same universality class. The transitions are probed by measuring the conductivities σ_{xx} and σ_{xy}. The scaling of the width of the transition regime with temperature as predicted by Eq. (4.88) has been

corroborated by DC experiments on various transitions between integer quantum-Hall states which were all found to yield the value $1/\nu z = 0.42 \pm 0.04$ [231]. Also the scaling $\delta \sim E^{1/\nu(z+1)}$ has been corroborated by experiment which yielded the value $\nu(z+1) \approx 4.6$ [232]. Together with the previous result obtained from the temperature scaling this gives

$$z \approx 1, \quad \nu \approx 2.3. \tag{4.91}$$

The value of the dynamic exponent strongly suggests that it is a result of the presence of the $1/|\mathbf{x}|$-Coulomb potential. The correlation length exponent ν is seen to be large.

4.6.3 $2d$ Electron Systems

Recently, silicon MOSFET's at extremely low electron number densities has been studied [202, 203, 204, 205]. Earlier experiments at higher densities seemed to confirm the general believe, based on the work by Abrahams *et al.* [227], that such two-dimensional electron systems do not undergo a quantum phase transition. In that influential paper, it was demonstrated that even weak disorder is sufficient to localize the electrons at the absolute zero of temperature thus excluding conducting behavior. Electron-electron interactions were however not included. As we saw in Sec. 1.6, the $1/|\mathbf{x}|$-Coulomb interaction becomes important at low densities and the analysis of Abrahams *et al.* [227] no longer applies.

The recent experiments have revealed a zero-temperature conductor-to-insulator transition triggered by a change in the charge carrier density $\bar{n}$. That is, the distance to the critical point is in these systems measured by $\delta \propto \bar{n} - \bar{n}_{\rm c}$. Like in the quantum-Hall systems, these transitions are probed by measuring the resistivity. It scales with temperature near the transition according to the scaling form (4.88) with H set to zero. For $\bar{n} < \bar{n}_{\rm c}$, where the Coulomb interaction is dominant and fluctuations in the charge carrier density are suppressed, the electron system is insulating. On increasing the density, these fluctuations intensify and at the critical value $\bar{n}_{\rm c}$, the system reverts to a conducting phase. By plotting their conductivity data as function of $T/\delta^{\nu z}$ with $\nu z = 1.6 \pm 0.1$, Popovíc, Fowler, and Washburn [205] saw it collapse onto two branches; the upper branch for the conducting side of the transition, and the lower one for the insulating side. A similar collapse with a slightly different value $1/\nu z = 0.83 \pm 0.08$ was found in Ref. [203], where also the collapse of the data when plotted as function of $\delta/E^{1/(z+1)\nu}$ was obtained. The best collapse resulted for $1/(z+1)\nu = 0.37 \pm 0.01$, leading to

$$z = 0.8 \pm 0.1, \quad \nu = 1.5 \pm 0.1. \tag{4.92}$$

The value for the dynamic exponent is close to the expected value $z = 1$ for a charged system with a $1/|\mathbf{x}|$-Coulomb interaction, while that of ν is surprisingly close to the value (4.90) found for the superconductor-to-insulator transition.

A further experimental result for these two-dimensional electron systems worth mentioning is the suppression of the conducting phase by an applied magnetic field found by Simonian, Kravchenko, and Sarachik [204]. They applied the field *parallel* to the plane of the electrons instead of perpendicular as is done in quantum-Hall measurements. In this way, the field presumably couples only to the spin of the electrons and the complications arising from orbital effects do not arise. At a fixed temperature,

a rapid initial raise in the resistivity was found with increasing field. Above a value of about 20 kOe, the resistivity saturates. It was pointed out that both the behavior in a magnetic filed, as well as in zero field strongly resembles that near the superconductor-to-insulator transition discussed above, suggesting that the conducting phase might in fact be superconducting.

4.6.4 Conclusions

We have seen that general scaling arguments combined with the effective theories, can be employed to understand the scaling behavior observed in various quantum phase transitions. Most of the experiments seem to confirm the expected value $z = 1$ for a random system with a $1/|\mathbf{x}|$-Coulomb interaction. The number of different universality classes present is yet not known. Even if the conductor-to-insulator transition observed in silicon MOSFET's at low electron number densities turns out to be in the same universality class as the superconductor-to-insulator transition, there are still the field-induced transitions in quantum-Hall systems, which have a larger correlation-length exponent.

The paradigm provided by a repulsively interacting Bose gas, seems to be a good starting point to describe the various systems. However, high-precision estimates calculated from this theory with impurities and a $1/|\mathbf{x}|$-Coulomb interaction included are presently lacking.

Bibliography

[1] V. N. Popov, *Functional Integrals and Collective Excitations* (Cambridge University Press, Cambridge, 1987).

[2] Y. Takahashi, Fortschr. Phys. **36**, 63 (1988).

[3] M. Greiter, F. Wilczek, and E. Witten, Mod. Phys. Lett. B **3**, 903 (1989).

[4] C. Eckart, Phys. Rev. **54**, 920 (1938); W. Yourgrau and S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory* (Pitman, London, 1968).

[5] R. Kronig and A. Thellung, Physica **18**, 749 (1952).

[6] A. M. J. Schakel, Int J. Mod. Phys. B **10**, 999 (1996).

[7] See, for example, B. K. Shivamoggi, *Theoretical Fluid Dynamics* (Martinus Nijhoff, Dordrecht, 1985), p. 191, and references therein. (Unfortunately, the relevant equation contains a misprint in that the combination $\Phi_{zz} - 2/a^2\Phi_x\Phi_y$ in front of Φ_{xy} is left out.)

[8] See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 505.

[9] P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin-Cummings, Menlo Park, 1984).

[10] F. London, *Superfluids* (Wiley, New York, 1954), Vol. II.

[11] R. P. Feynman, *Statistical Mechanics* (Addison Weseley, New York, 1972), Sec. 10.12.

[12] E. Fick, *Einführing in die Grundlagen der Quantentheorie* (Akademische Verlagsgesellschaft, Frankfurt am Main, 1968).

[13] W. J. Ellon, M. Matters, U. Geigenmüller, and J. E. Mooij, Nature **371**, 594 (1994).

[14] H. von Helmholtz, Journal für die reine und angewandte Mathematik **55**, 25 (1858).

[15] W. Thomson, Trans. Roy. Soc. Edingb. **25**, 217 (1869).

[16] A. Clebsch, Journal für die reine und angewandte Mathematik **56**, 1 (1859); H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1942), p. 248.

[17] H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989), Vol. I.

[18] H. Kleinert, Int. J. Mod. Phys. A **7**, 4693 (1992).

[19] H. Kleinert, in *Formation and Interactions of Topological Defects*, Proceedings NATO ASI, Cambridge, 1994, edited by A. C. Davis and R. Brandenburger (Plenum, New York, 1995).

[20] E. B. Sonin, Phys. Rev. B **55**, 485 (1997) and references therein.

[21] L. P. Pitaevskii, Sov. Phys. JETP **8**, 888 (1959).

[22] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

[23] N. N. Bogoliubov, V. V. Tolmachev, and D. V. Sirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, New York, 1959).

[24] J. Bardeen, L. N. Cooper, J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[25] S. Weinberg, Prog. Theor. Phys. Suppl. **86**, 43 (1986).

[26] C. M. Fraser, Z. Phys. C **28**, 101 (1985); I. J. R. Aitchison and C. M. Fraser, Phys. Rev. D **31**, 2605 (1985).

[27] A. J. Leggett in *Modern Trends in the Theory of Condensed Matter*, edited by A. Pekalski and J. Przystawa (Springer-Verlag, Berlin, 1980), p. 13.

[28] R. D. Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem* (McGraw-Hill, New York, 1976).

[29] R. Haussmann, Z. Phys. B **91**, 291 (1993).

[30] M. Drechsler and W. Zwerger, Ann. Phys. (Germany) **1**, 15 (1992).

[31] C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).

[32] M. Marini, F. Pistolesi, and G. C. Strinati, e-print cond-mat/9703160 (1997).

[33] U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. **30**, 6419 (1989).

[34] A. M. J. Schakel, Mod. Phys. Lett. B **4**, 927 (1990).

[35] I. J. Aitchison, P. Ao, D. J. Thouless, and X.-M. Zhu, Phys. Rev. B **51**, 6531 (1995).

[36] M. Stone, Int. J. Mod. Phys B, **9**, 1359 (1995).

[37] V. M. Loktev and S. G. Sharapov, e-print cond-mat/9706285 (1997).

[38] J. Gavoret and P. Nozières, Ann. Phys. (N.Y.), **28**, 349 (1964).

[39] H. Kleinert, Fortschr. Phys. **26**, 565 (1978).

[40] P. N. Brusov and V. N. Popov, Sov. Phys. JETP **53**, 804 (1981); Phys. Lett. A **87**, 472 (1982).

[41] M. Randeria, J. -M. Duan, and L. -Y. Shieh, Phys. Rev. B **41**, 327 (1990).

[42] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentence-Hall, Englewood Cliffs, 1963).

[43] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor and Francis, London, 1990).

[44] Ch. Bruder and D. Vollhardt, Phys. Rev. B **34**, 131 (1986).

[45] A. M. J. Schakel and F. A. Bais, J. Phys.: Condens. Matter **1**, 1743 (1989).

[46] G. E. Volovik, *Exotic Properties of Superfluid* ^{3}*He* (World Scientific, Singapore, 1992).

[47] D. L. Stein and M. C. Cross, Phys. Rev. Lett. **42**, 504 (1979).

[48] For a discussion of various representations of the superfluid ^{3}He order parameter see H. W. Capel and A. M. J. Schakel, Physica A **160**, 409 (1989).

[49] A. M. J. Schakel, Mod. Phys. Lett. B **5**, 833 (1991).

[50] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).

[51] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).

[52] E. P. Gross, Nuovo Cimento **20**, 454 (1961); L. P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).

[53] H. M. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995).

[54] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[55] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).

[56] S. T. Beliaev, Sov. Phys. JETP **7**, 289 (1958).

[57] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).

[58] J. Bernstein and S. Dodelson, Phys. Rev. Lett. **66**, 683 (1991); K. M. Benson, J. Bernstein, and S. Dodelson, Phys. Rev. D **44**, 2480 (1991).

[59] H. Leutwyler, Phys. Rev. D **49**, 3033 (1994).

[60] A. M. J. Schakel, Int. J. Mod. Phys. B **8**, 2021 (1994).

[61] J. F. Donoghue, Phys. Rev. D **50**, 3847 (1994); e-print gr-qc/9512024 (1995).

[62] T. Y. Cao and S. S. Schweber, Synthese **97** (1993); in *Renormalization*, edited by L. M. Brown (Springer-Verlag, Berlin, 1993).

[63] N. M. Hugenholtz, Rep. Prog. Phys. **28**, 201 (1965).

[64] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hullet, Phys. Rev. Lett. **75**, 1687 (1995).

[65] P. B. Weichman, Phys. Rev. B **38**, 8739 (1988).

[66] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[67] U. C. Täuber and D. R. Nelson, Phys. Rep. **289**, 157 (1997).

[68] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).

[69] P. Nozières and D. Pines, *The Theory of Quantum Liquids* (Addison-Wesley, New York, 1990), Vol. II.

[70] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983), Ch. 6.

[71] J. I. Kapusta, *Finite-Temperature Field Theory*, (Cambridge University Press, Cambridge, 1989), Ch. 7.

[72] P. C. Hohenberg and P.C. Martin, Ann. Phys. (N.Y.), **34**, 291 (1964).

[73] L. S. Brown, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1992).

[74] L. Tisza, Nature **141**, 913 (1938); J. Phys. Radium **1**, 165, 350 (1940); Phys. Rev. **72**, 838 (1947).

[75] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950); reprinted in *L. D. Landau, Collected Papers* (Pergamon, London, 1965), p. 546.

[76] L. P. Gorkov, Sov. Phys. JETP **9**, 1364 (1959).

[77] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[78] P. M. Chaikan and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).

[79] A. A. Abrikosov, Sov. Phys. JETP **5**, 1174 (1957).

[80] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).

[81] X. G. Wen and A. Zee, Nucl. Phys. B**326**, 619 (1989).

[82] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison Wesley, New York, 1991).

[83] I. E. Dzyaloshinski, A. M. Polyakov, and P. B. Wiegmann, Phys. Lett. A **127**, 112 (1988).

[84] F. D. M. Haldane, Phys. Rev. Lett. **61**, 1029 (1988); E. Fradkin and M. Stone, Phys. Rev. B **38**, 7215 (1988).

[85] M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. **164**, 218 (1988).

[86] H. So, Prog. Theor. Phys. **74**, 585 (1985).

[87] K. Ishikawa and T. Matsuyama, Z. Phys. C **33**, 41 (1986); T. Matsuyama, Prog. Theor. Phys. **77**, 711 (1987).

[88] G. E. Volovik, Sov. Phys. JETP **67**, 1804 (1989);

[89] A. Coste and M. Lüscher, Nucl. Phys. B **323**, 631 (1989); Z. Hlousek, D. Sénéchal, and S. H. H. Tye, Phys. Rev. D **41**, 3773 (1990).

[90] G. E. Volovik, A. Solovyov, and V. M. Yakovenko, JETP Lett, **49**, 65 (1989); G. E. Volovik and V. M. Yakovenko, J. Phys.: Condens. Matter **1**, 5263 (1989).

[91] F. Wilczek and A. Zee. Phys. Rev. Lett. **41**, 2250 (1983).

[92] A. A. Belavin and A. M. Polyakov, JETP Lett, **22**, 245 (1975).

[93] N. D. Mermin, Rev. Mod. Phys. **51**, 591 (1979).

[94] H.-R. Trebin, Adv. Phys. **31**, 195 (1982).

[95] V. P. Mineev, in *Soviet Scientific Reviews*, Section A: Physics Reviews, Vol. 2, edited by I. M. Khalatnikov (Harwood, New York, 1980), p. 173.

[96] G. Toulouse and M. Klèman, J. Phys. (Paris) **37**, L149 (1976).

[97] M. J. Bowick, D. Karabali, and L. C. R. Wijewardhana, Nucl. Phys. B **271**, 417 (1986).

[98] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics* (World Scientific, Singapore, 1995).

[99] K. Huang, Y. Koike, and J. Polonyi, Int. J. Mod. Phys. A **6**, 1267 (1991).

[100] I. Ya. Arefyeva and S. I. Azakov, Nucl. Phys. B **162**, 298 (1980).

[101] A. Kovner and B. Rosenstein, Nucl. Phys. B **346**, 576 (1990).

[102] K. Bitar and E. Manousakis, Phys. Rev. B **43**, 2615 (1991).

[103] A. M. J. Schakel, in *Proceedings of the Körber Symposium on Superfluid ^{3}He in Rotation*, Helsinki, 1991, edited by M. M. Salomaa, Physica B **178**, 280 (1992).

[104] P. W. Anderson and G. Toulouse, Phys. Rev. Lett. **38**, 508 (1977); V. R. Chechetkin, Sov. Phys. JETP **44**, 766 (1976).

[105] M. M. Salomaa and G. E. Volovik, Rev. Mod. Phys. **59**, 533 (1987).

[106] For reviews see, for example, V. J. Emery in *Highly Conducting One-Dimensional Solids*, edited by J. T. Devrese, R. P. Evrard, and V. E. van Doren (Plenum, New York,1979); I. Affleck in *Fields, Strings and Critical Phenomena*, proceedings of Les Houches summerschool, Session XLIX, 1988, edited by E. Brézin and J. Zinn-Justin (Elsevier, Amsterdam, 1989).

[107] H. Bethe, Z. Phys. **71**, 205 (1931).

[108] R. E. Peierls, *Quantum Theorie of Solids* (Oxford University Press, Oxford, 1955), Sec. 5.3.

[109] H. Fröhlich, Proc. Roy. Soc., London, Sec. A **233**, 296 (1954).

[110] For a review on charge density waves see, for example, B. Horovitz, in *Modern Problems in Condensed Matter Physics*, Vol. 17 edited by S. E. Trullinger, V. E. Zakharov, and V. L. Pokrovsky (Elsevier, Amsterdam, 1986).

[111] A. M. J. Schakel, Mod. Phys. Lett. B **6**, 637 (1992).

[112] A. M. Polyakov, *Gauge Fields and Strings* (Harwood, New York, 1987).

[113] T. Jaroszewics, Phys. Lett. B **146**, 337 (1984).

[114] F. D. M. Haldane, J. Appl. Phys. **57**, 3359 (1985); I. Affleck, Nucl. Phys. B **257**, 397 (1985); for a review see I. Affleck, J. Phys.: Condens. Matter **1**, 3047 (1989).

[115] R. Shankar and N. Read, Nucl Phys. B **336**, 457 (1990); I. Affleck, Phys. Rev. Lett. **66**, 2429 (1991).

[116] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[117] N. Dorey and N. E. Mavromatos, Phys. Lett. B **250**, 107 (1990); Phys. Rev. B **44**, 5286 (1991).

[118] A. Kovner and B. Rosenstein, Phys. Rev. B **42**, 4748 (1990); Mod Phys. Lett. A **5**, 2661 (1990).

[119] A. M. J. Schakel, Phys. Rev. D **44**, 1198 (1991).

[120] G. Gat, A. Kovner, A. Krasnitz, B. Rosenstein, and A. M. J. Schakel, in *Proceedings of the Conference on Physics in* $2D$*, Neuchatel, 1991*, Helv. Phys. Acta **65**, 367 (1992).

[121] For reviews, see *Solid State Physics*, Springer Tracts in Modern Physics, Vol. 78, edited by G. Hohler (Springer-Verlag, Berlin, 1976); *Narrow Gap Semiconductors Physics and Applications*, Lecture Notes in Physics, Vol. 133, edited by W. Zawadzki (Springer-Verlag, Berlin, 1980); *Physics of Narrow Gap Semiconductors*, *ibid.*, Vol. 153, edited by E. Gornik, *et al.* (Springer-Verlag, Berlin, 1982).

[122] G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).

[123] A. M. J. Schakel, Phys. Rev. D **43**, 1428 (1991).

[124] A. M. J. Schakel and G. W. Semenoff, Phys. Rev. Lett. **66**, 2653 (1991).

[125] Y. H. Chen and F. Wilczek, Int. J. Mod. Phys. B **3**, 117 (1989).

[126] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. **140**, 372 (1982); R. Jackiw in *Current Algebra and Anomalies*, edited by S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten (World Scientific, Singapore, 1985).

[127] A. Neagu and A. M. J. Schakel, Phys. Rev. D **48**, 1785 (1993).

[128] L. S. Forte, *Rev. Mod. Phys.* **64**, 193 (1992).

[129] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252 (1941).

[130] V. L. Berezinskii, Sov. Phys. JETP **34**, 610 (1972).

[131] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).

[132] D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).

[133] S. Schenker, in *Recent Advances in Field Theory and Statistical Mechanics*, proceedings of Les Houches summerschool, Session XXXIX, 1982, edited by J. B. Zuber and R. Stora (Elsevier, Amsterdam, 1984).

[134] F. A. Schaposnik, Phys. Rev. D **18**, 1183 (1978).

[135] J. Pearl, in *Low Temperature Physics—LT9*, edited by J. G. Danut, D. O. Edwards, F. J. Milford, and M. Yagub (Plenum, New York, 1965).

[136] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, New York, 1966).

[137] M. R. Beasley, J. E. Mooij, and T. P. Orlando, Phys. Rev. Lett. **42**, 1165 (1979).

[138] M.-C. Cha, M. P. A. Fisher, S. M. Girvin, M. Wallin, and A. P. Young, Phys. Rev. B **44**, 6883 (1991); S. M. Girvin, M. Wallin, M.-C. Cha, M. P. A. Fisher, and A. P. Young, Prog. Theor. Phys. Suppl. **107**, 135 (1992).

[139] A. F. Hebard and M. A. Palaanen, Helv. Phys. Act. **65**, 197 (1992).

[140] A. Das and A. Karev, Phys. Rev. D **36**, 2591 (1987).

[141] R. Rajaraman, in *Anomalies, Phases, Defects, ...*, edited by M. Bregola, G. Marmo, and G. Morandi (Bibliopolis, Napoli, 1990).

[142] A. V. Smilga, Phys. Lett. B **278**, 371 (1992).

[143] N. K. Nielsen and B. Schroer, Nucl. Phys. B **120**, 62 (1977); Phys. Lett. B **66**, 373 (1977).

[144] N. K. Nielsen and B. Schroer, Nucl. Phys. B **127**, 493 (1977).

[145] R. P. Feynman. Phys. Rev. **80**, 440 (1950).

[146] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[147] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948).

[148] M. Kiometzis and A. M. J. Schakel, Int. J. Mod. Phys. **B** 7, 4271 (1993).

[149] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1996), Vol. II.

[150] S. Elitzur, Phys. Rev. D **12**, 3978 (1975).

[151] D. Sen, Phys. Rev. D **41**, 1227 (1990).

[152] H. Kleinert and A. M. J Schakel, in *Proceedings of the Xth International Conference "Problems of Quantum Field Theory"*, edited by D. V. Shirkov, D. Kazakov, and A. Vladimirov (JINR, Dubna, 1996).

[153] P. Dimopoulos, K. Farakos, and G. Koutoumbas, e-print hep-lat/9703004 (1997).

[154] H. Kleinert, Lett. Nuovo Cimento **35**, 405 (1982); Phys. Lett. B **128**, 69 (1983).

[155] B. I. Halperin, T. C. Lubensky and S. Ma, Phys. Rev. Lett. **32**, 292 (1974).

[156] G. Parisi, Lectures presented at the Cargèse Summer School 1973 (unpublished); J. Stat. Phys. **23**, 49 (1980).

[157] G. Parisi, *Statistical Field Theory* (Addison-Wesley, New-York, 1988).

[158] J. S. Kang, Phys. Rev. D **10**, 3455 (1974).

[159] I. F. Herbut and Z. Tešanović, Phys. Rev. Lett. **76**, 4588 (1996).

[160] S. Hikami and E. Brézin, J. Phys. A **11**, 1141 (1978).

[161] S. Hikami, Prog. Theo. Phy. **62**, 226 (1979).

[162] A. N. Vasil'ev and M. Yu. Nalimov, Teor. Mat. Phys. **56**, 643 (1983).

[163] A. Kovner, P. Kurzepa, and B. Rosenstein, Mod. Phys. Lett. **A** 8, 1343 (1993).

[164] T. Banks, B. Meyerson, and J. Kogut, Nucl. Phys. B **129**, 493 (1977).

[165] M. Peskin, Ann. Phys. **113**, 122 (1978).

[166] P. R. Thomas and M. Stone, Nucl. Phys. B **144**, 513 (1978).

[167] R. Savit, Rev. Mod. Phys. **52**, 453 (1989).

[168] W. Helfrich and W. Müller, in *Continuum Models of Discrete Systems* (Waterloo University Press, Waterloo, 1980), p. 753.

[169] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981).

[170] A. Kovner, B. Rosenstein, and D. Eliezer, Mod. Phys. Lett. A **5**, 2733 (1990); Nucl Phys. B **350**, 325 (1991); A. Kovner and B. Rosenstein, Phys. Rev. Lett. **67**, 1490 (1991).

[171] P. A. M. Dirac, Phys. Rev. **74**, 817 (1948).

[172] Y. Nambu, Phys. Rev. D **10**, 4262 (1974).

[173] E. C. Marino, Phys. Rev. D **38**, 3194 (1988); see also E. C. Marino, G. C. Marques, R. O. Ramos, and J. S. Ruiz, Phys. Rev. D **45**, 3690 (1992); E. C. Marino, Ann. Phys. (NY) **224**, 225 (1993).

[174] K. Bardakci and S. Samuel, Phys. Rev. D **18**, 2849 (1978).

[175] H. Kawai, Prog. Theor. Phys. **65**, 351 (1981).

[176] M. Kiometzis, H. Kleinert, and A. M. J. Schakel, Phys. Rev. Lett. **73**, 1975 (1994); Fortschr. Phys. **43**, 697 (1995).

[177] N. Overend, M. A. Howson, and I. D. Lawrie, Phys. Rev. Lett. **72**, 3238 (1994).

[178] Z.-H. Lin, G. C. Spalding, A. M. Goldman, B. P. Bayman, and O. T. Valls, Europhys. Lett. **32**, 573 (1995).

[179] I. F. Herbut, J. Phys. A: Math. Gen. **29**, 423 (1996).

[180] K. Kajantie, M. Karalainen, M. Laine, and J. Peisa, e-print hep-lat/97040562 (1997).

[181] P. Olsson and S. Teitel, Phys, Rev. Lett. **80**, 1964 (1998).

[182] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Rev. Mod. Phys. **69**, 315 (1997).

[183] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, London, 1976).

[184] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).

[185] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[186] I. Affleck, Phys. Rev. B **41**, 6697 (1990).

[187] T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **76**, 2571 (1996).

[188] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. **64**, 587 (1990).

[189] D. I. Uzunov, Phys. Lett. A **87**, 11 (1981); *Introduction to the Theory of Critical Phenomena* (World Scientific, Singapore, 1993).

[190] O. Bergman, Phys. Rev. D **46**, 5474 (1992).

[191] G. G. Batrouni, R. T. Scalettar, and G. T. Zimanyi, Phys. Rev. Lett. **65**, 1765 (1990); G. G. Batrouni and R. T. Scalettar, Phys. Rev. B **46**, 9051 (1992).

[192] A. F. Hebard and M. A. Palaanen, Phys. Rev. Lett. **65**, 927 (1990).

[193] E. R. Korutcheva and D. I. Uzunov, Phys. Lett. A **106**, 175 (1984).

[194] P. B. Weichman and K. Kim, Phys. Rev. B **40**, 813 (1989).

[195] S. N. Dorogovtsev, Phys. Lett. A **76**, 169 (1980).

[196] G. Grinstein and A. Luther, Phys. Rev. B **13**, 1329 (1976).

[197] T. C. Lubensky, Phys. Rev. B **11**, 3573 (1975).

[198] For a review see, for example, J. A. Hertz, Physica Scripta T**10**, 1 (1985).

[199] K. Huang and H.-F. Meng, Phys. Rev. Lett. **69**, 644 (1992).

[200] S. Giorgini, L. Pitaevskii, and S. Stringari, Phys. Rev. B **49**, 12938 (1994).

[201] A. M. J. Schakel, Phys. Lett. A, **224**, 287 (1997).

[202] S. V. Kravchenko, G. V. Kravchenko, J. E. Furneaux, V. M. Pudalov, and M. D'Iorio, Phys. Rev. B **50**, 8039 (1994); S. V. Kravchenko, W. E. Mason, G. E. Bowker, J. E. Furneaux, V. M. Pudalov, and M. D'Iorio, Phys, Rev. **51**, 7038 (1995).

[203] S. V. Kravchenko, D. Simonian, M. P. Sarachik, W. E. Mason, and J. E. Furneaux, Phys. Rev. Lett. **77**, 4938 (1996).

[204] D. Simonian, S. V. Kravchenko, and M. P. Sarachik, e-print cond-mat/9704071 (1997).

[205] D. Popovíc, A. B. Fowler, and S. Washburn, e-print cond-mat/9704249 (1997).

[206] T. Giamarchi and H. J. Schulz, Europhys. Lett. **3**, 1287 (1987); Phys. Rev. B **37**, 325 (1988).

[207] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[208] S. M. Girvin, in *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, Berlin, 1986), Ch. 10; S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987).

[209] S.-C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[210] S.-C. Zhang, Int. J. Mod. Phys. B **6**, 25 (1992).

[211] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).

[212] D. P. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).

[213] S. Kivelson, D.-H. Lee, and S.-C. Zhang, Phys. Rev. B **46**, 2223 (1992); D.-H. Lee, Z. Wang, and S. Kivelson, Phys. Rev. Lett. **70**, 4130 (1993).

[214] X.-G. Wen and Y.-S. Wu, Phys. Rev. Lett. **70**, 1501 (1993).

[215] L. Pryadko and S.-C. Zhang, Phys. Rev. Lett. **73**, 3282 (1994).

[216] G. Lozano, Phys. Lett. B **283**, 70 (1992); O. Bergman and G. Lozano, Ann. Phys. (N.Y.) **229**, 416 (1994).

[217] P. Ramond, *Field Theory, A Modern Primer* (Benjamin-Cummings, Reading, 1981).

[218] D. Z. Freedman, G. Lozano, and N. Rius, Phys. Rev. D **49**, 1054 (1994).

[219] A. M. J. Schakel, Nucl. Phys. B [FS] **453**, 705 (1995).

[220] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[221] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).

[222] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[223] R. Jackiw and S. Y. Pi, Phys. Rev. D **42**, 3500 (1990).

[224] M. A. Valle Basagoiti, Phys. Lett. B **306**, 307 (1993); R. Emparan and M. A. Valle Basagoiti, Mod. Phys. Lett. A **8** 3291 (1993).

[225] B. Widom, J. Chem. Phys. **43**, 3892 (1965); **43**, 3898 (1965).

[226] D. S. Fisher and M. P. A. Fisher, Phys. Lett. **61**, 1847 (1988).

[227] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 637 (1989).

[228] M. P. A. Fisher, Phys. Rev. Lett. **65**, 923 (1990).

[229] T. Liu and A. M. Goldman, Mod. Phys. Lett. B **8**, 277 (1994).

[230] A. Yazdani and A. Kapitulnik, Phys. Rev. Lett. **74**, 3037 (1995).

[231] H. P. Wei, D. C. Tsui, M. A. Paalanen, and A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1294 (1988).

[232] H. P. Wei, L. W. Engel, and D. C. Tsui, Phys. Rev. B **50**, 14609 (1994).