Adiabatic Theory and Wave Action Density

⇒ Quasiparticle Picture

c.f. : Whitham, relevant chapter posted

Wave Adiabatic Theory / Wave Kinetics

- frequently encounter continuum problems with slowly varying parameters ⇒ adiabatic theory needed

⇒

- $\boxed{\text{wave kinetic equation}}$ (consequence of Liouville Thm.)

$$\partial_t N + (\underline{v}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \partial_{\underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \partial_{\underline{k}} N$$

$$= C(N)$$; obvious analogy to Boltzmann Eqn.

$N \equiv \mathcal{E} / \omega_{\underline{k}}$ ≡ wave action density (could guess by analogy) / wave quanta density

$\mathcal{E}$: wave energy density

$$\mathcal{E} = \frac{\partial}{\partial \omega} (\omega \epsilon_{\underline{k}}) \Big|_{\omega_{\underline{k}}} \frac{|E_{\underline{k}}|^2}{8\pi}$$, for e.s. waves

$N \Leftrightarrow e$

characteristics:

$$\frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} \hat{k} + \underline{V} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V})$$

refraction by shear

refraction by parametric variation

- need:

$$\omega < \frac{1}{\lambda} \frac{d\lambda}{dt} \qquad \lambda \equiv \text{parameter}$$

→ $\boxed{\text{space and time scale separation}}$

$$\frac{1}{N} (\underline{v}_{gr} \cdot \nabla N) \ll \omega \quad \Rightarrow \quad \underline{\kappa} \cdot \underline{v}_{gr} < \omega$$

Transport Eqn - $\underline{QM}$

$$\frac{\partial n}{\partial t} + v_{gr}\cdot\nabla n - \frac{\partial \omega}{\partial \underline{x}}\cdot\nabla_{\underline{k}} n = C(n)$$

$$\frac{\partial n}{\partial t} + \frac{\partial \hbar\omega}{\partial \hbar \underline{k}}\cdot\nabla n - \frac{\partial \hbar\omega}{\partial x}\cdot\nabla_{\hbar k} n = C(n)$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \frac{\partial \epsilon}{\partial \underline{p}}\cdot\nabla n - \frac{\partial \epsilon}{\partial x}\cdot\nabla_{p} n = C(n)}$$

valid for:

$$\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial x} < 1/\lambda_{DB}$$

$$\lambda_{DB} = \hbar/p.$$

C(N) → interactions with comparable scale.
ignore here.

Examples:

- beach
- linear theory of Langmuir turbulence
  i.e. when will phonon grow?
- QL theory of Langmuir turbulence
  i.e. determine evolution of plasma energy → net impact?!
- drift waves and sheared flow.

→ transport equation, super-fluids $N = \frac{\varepsilon}{\omega}$

→ dynamics?!

Fundamentals of wave kinetics

→ where does conservation of action emerge from?

→ answer: $\boxed{\text{phase symmetry (of wave train) underlies wave kinetics}}$

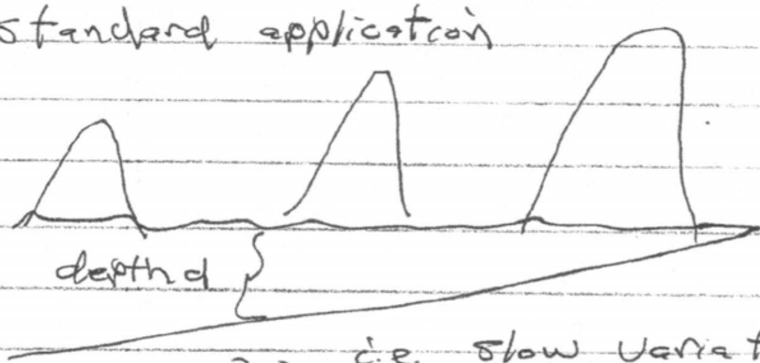
→ How show, beyond approach by analogy?

→ approach via variational principle.

c.f. Whitham: "Linear and Nonlinear Waves"
Chapt. 14.

⇒ standard application

⇒



beach ⟷

waves in shallow water

⇒ i.e. slow variation

$$\frac{1}{d}\frac{d}{dx}d(x) \ll k$$

- influx of wave energy
- depth $H(x,y)$ decreases

⇒ wave amplification, breaking.

- Derivation

Consider a system, [like ideal $\overset{\text{fluid,}}{\wedge}$ MHD,] acoustics which can be described in terms of displacement $\underline{\Sigma}$;

i.e. $\underline{\Sigma} = \mathrm{re}\left\{ A e^{i\overset{\text{phase}}{\phi}} + A^* e^{-i\phi} \right\}$

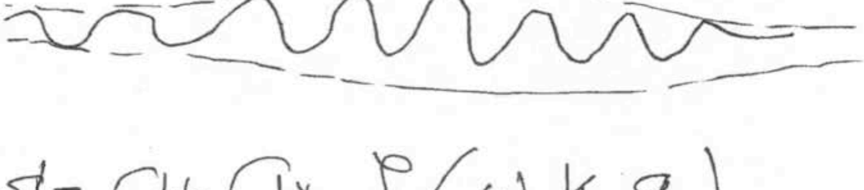
i.e. acoustics!

{ displacement can be viewed as excitation level

then wave equation arises from:

$$\delta S = \delta \int dt \int d\underline{x}\ \mathcal{L}(\Sigma)$$

- Envision a wave train, with slowly varying amplitude so eikonal approach optimal i.e. fast variation in phase, ala WKB:



$$S = \int dt \int d\underline{x}\ \mathcal{L}(\omega, \underline{k}, \underset{\text{amplitude}}{a})$$

$$\begin{cases} \underline{k} = \underline{\nabla}\phi \\ \omega = -\partial_t\phi \end{cases}$$

$$= \int dt \int d\underline{x}\ \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

- neglect all corrections to eikonal theory. (no higher order WKB).

- here L corresponds to period-averaged Lagrangian
- ϕ undetermined to const → phase symmetry!

∴ to vary:

$$\left.\begin{array}{l} \delta S/\delta a = 0 \\ \delta S/\delta \phi = 0 \end{array}\right\} \rightarrow \text{2 eqns}$$

Now, in linear theory:

[$G(k,\omega) \Leftrightarrow G$]

- $\mathcal{L} = G(\omega, k)\, a^2$

$\left\{ \begin{array}{l} G(\omega, k) = 0 \\ \omega^2 = k^2 c_s^2 \end{array} \right.$ disp'n

i.e. for continuum, i.e. acoustics, as in wave section:

$$\mathcal{L} = \frac{1}{2}\rho\, \dot{\underline{\xi}}^2 - \frac{1}{2}\rho \left[\underline{\underline{D}}(k, x, t) \right]^2 \underline{\xi}^2$$

concrete form of Lagrangian

↳ $\left\{ \begin{array}{l} \text{eikonal form of} \\ \text{stiffness matrix} \\ (\rightarrow \text{potential energy}) \end{array} \right.$

$$\Rightarrow \quad \underline{\xi} \cdot \underline{\underline{D}} \cdot \underline{\xi}$$

$D(k, \omega, \theta)$, as for linear waves

if: $\underline{\xi} = \underline{A} e^{i\phi} + A^* e^{-i\phi}$

$$\underline{\underline{i}} \quad G(\omega, \underline{k}) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2\right]$$

Now, 1) $\delta S/\delta a = 0$

$$\Rightarrow \quad G(\omega, k) = 0 \quad \rightarrow \left\{\begin{array}{l}\text{dispn}\\ \text{relation}\end{array}\right.$$

but

$$G(\omega, k) = \rho\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\underline{\nabla}\phi, \underline{x}, t)\right]^2$$

$$= \rho\omega^2 - \underbrace{D^2}_{\text{stiffness fctn.}}$$

$\Rightarrow$ dispn. relation

2) $\delta S/\delta\phi = 0$

$$\delta S = \int dt \int d^3x \left\{ \frac{\partial\mathcal{L}}{\partial(-\phi_t)}\,\delta(-\phi_t) + \frac{\partial\mathcal{L}}{\partial(\phi_{\underline{x}})}\,\delta(\phi_{\underline{x}}) \right\}$$

end pts fixed, i b p

$$= \int dx \int d^3x \left\{ \partial_t\left(\frac{\partial\mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial(\underline{\nabla}\phi)}\right)\right\}\delta\phi$$

$\delta S = 0 \Rightarrow$

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(-\phi_t)}\right) - \underline{\nabla}\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0$$

conservation eqn.

Now, have: $\quad G(k, \omega) = 0 \qquad$ (dispn. reln.)

$$\partial_t\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$dG = 0 \Rightarrow \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}}\cdot d\underline{k} = 0$$

$$\therefore \quad \underline{v}_{gr} = \frac{d\omega}{d\underline{k}} = \frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega} \qquad \text{(akin } \omega)$$

$$\therefore \quad \boxed{\partial_t\left((\partial G/\partial \omega)\, a^2\right) + \underline{\nabla}\cdot\left[\frac{-\partial G/\partial \underline{k}}{\partial G/\partial \omega}\,\frac{\partial G}{\partial \omega}\, a^2\right] = 0}$$

and so $\quad N \equiv \underbrace{\frac{\partial G}{\partial \omega}\, a^2}$

$$\boxed{\frac{\partial N}{\partial t} + \underline{\nabla}\cdot\left(\underline{v}_{gr} N\right) = 0} \qquad \text{Cons. eqn.}$$

though: $\quad G = \rho\omega^2 - \Lambda^2 \qquad$ (N not yet action)

$$\partial G/\partial \omega = 2\rho\omega = 2\rho\omega$$

$$\frac{\partial G}{\partial \omega} a^2 \rightarrow \mathcal{E}/\omega \;,$$

i.e. $\omega \frac{\partial G}{\partial \omega} a^2 = 2\rho \frac{\omega^2}{\omega} a^2 = \frac{2(KE)}{\omega} = \mathcal{E}/\omega \rightarrow$ action density

Also note energy conserved $\Leftrightarrow$ G invariant to time translations $(\mathcal{L} = G a^2)$.

Thus, Noether's thm $\Rightarrow$ there exists an energy conservation eqn.

Can expect form (see previous)

$$\partial_t \mathcal{E} + \underline{\nabla} \cdot \underline{S} = 0$$

Question: What is N? relation to $\mathcal{E}$, $\underline{S}$?

<u>an argument</u>

Now, have:

$$\mathcal{L} = G(k, \omega) a^2$$

$$\partial \mathcal{L} / \partial a = 0 \Rightarrow G(k, \omega) = 0$$

$$\partial \mathcal{L} / \partial \phi = 0 \Rightarrow \partial_t \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

and also realize:

$$\underline{\nabla} \times \underline{k} = 0 \,, \qquad \underline{k} = \underline{\nabla} \phi$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{i.e.} \quad \partial_t \underline{\nabla} \phi = -\underline{\nabla} (-\partial_t \phi)$$

$$= -\underline{\nabla} \omega$$

expect: $\dfrac{\partial \mathcal{L}}{\partial \omega} = N$

so $\quad \omega \, \partial \mathcal{L} / \partial \omega = \mathcal{E}$.

Now, assert:

$$\partial_t \left(\omega \frac{\partial \mathcal{L}}{\partial \omega} \right) + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

as energy eqn.

To check:

→ add zero $\quad (\text{i.e. } \mathcal{L} = \underbrace{G a^2}_{=0})$

$$\partial_t \left(\omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right) + \underline{\nabla} \cdot \left[-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right] = 0$$

then

$$(\partial_t \omega) \mathcal{L}_\omega + \omega \partial_t \mathcal{L}_\omega - \partial_t \mathcal{L} + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

but

$$\partial_t \mathcal{L}_\omega = \underline{\nabla} \cdot (\mathcal{L}_{\underline{k}})$$ phase variation eqn

$$(\mathcal{L}_\omega)(\partial_t \omega) + \cancel{\omega [\underline{\nabla} \cdot \mathcal{L}_{\underline{k}}]} - \cancel{\omega (\underline{\nabla} \cdot \mathcal{L}_{\underline{k}})}$$
$$- (\partial \mathcal{L} / \partial \underline{k}) \cdot \underline{\nabla} \omega - \partial_t \mathcal{L} = 0$$

$$\partial_t \underline{k} = -\underline{\nabla} \omega$$

$$(\partial_t \omega)(\mathcal{L}_\omega) + (\partial_t \underline{k}) \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}} \right) - \frac{\partial \mathcal{L}}{\partial t} = 0$$

is an identity.

so $$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L} \right\} + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

but $\mathcal{L} = 0$

so

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \underline{\nabla} \cdot \left(\underline{V_{gr}}\, \omega \frac{\partial \mathcal{L}}{\partial \omega} \right) = 0$$

$\Rightarrow$ energy equation.

$$\mathcal{E} = \omega \frac{\partial \mathcal{L}}{\partial \omega} = \text{wave energy density}$$

so then

$$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega = N$$

→ wave action density

→ by construction, an adiabatic invariant for wave packet.

But $G(\omega, k) = 0 \Rightarrow \mathcal{L} = 0$

$\therefore$

$$\partial_t \left\{ \omega \frac{\partial \mathcal{L}}{\partial \omega} \right\} + \nabla \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}} \right) = 0$$

Poynting Form

so

$$\mathcal{E} \equiv \omega \frac{\partial \mathcal{L}}{\partial \omega} \rightarrow \left\{ \begin{array}{l} \text{wave} \\ \text{energy density} \end{array} \right.$$

so $$\frac{\partial \mathcal{L}}{\partial \omega} = \mathcal{E}/\omega \rightarrow \left\{ \begin{array}{l} \text{wave} \\ \text{action density} \end{array} \right. \text{ \& }$$

$$= N(\underline{k}, \underline{x}, t) \rightarrow \text{adiabatic invariant for wave packet.}$$

so have:

$$\boxed{\partial_t (N) + \nabla \cdot (\underline{v}_{gr} N) = 0}$$

wave - kinetics

To demonstrate equivalence,

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \nabla N - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_{\underline{k}} N = 0$$

and Liouville Thm:

$$\partial_t N + \nabla \cdot (\underline{v}_{gr} N) + \nabla_{\underline{k}} \cdot \left(-\partial \omega / \partial \underline{x} \, N \right) = 0$$

$\int d\underline{k}$, and assume narrow spread in $\underline{k}$
(i.e. wave packet) $\Rightarrow$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left[\underline{V}_{gr} N \right] = 0$$

Observe:

$\rightarrow$ Vlasov-like equation in eikonal phase space $(\underline{x}, \underline{k})$

$$\frac{\partial N}{\partial t} + \underline{V}_{gr} \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and

$\rightarrow$ continuity-type equation in $\underline{x}$-space, for packet

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot (\underline{V}_{gr} N) = 0$$

Also observe:

- seeming issue re:

$$\frac{\partial k}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \quad \text{vs} \quad \frac{dk}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

$(\omega = D)$

Now $\partial_t \underline{k} = -\partial\omega/\partial\underline{x}$ is (Eulerian)
(partial) relation in $\underline{x}, t$

$d\underline{k}/dt = -\partial\omega/\partial\underline{x}$ is (Lagrangian)
(total) relation, following packet)

(here $\omega = D(\underline{k}, \underline{x}, t)$, as $\sigma = 0$)

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr}\cdot\nabla\underline{k}$$

$$= -\frac{\partial\omega}{\partial\underline{x}} + \underbrace{\frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\underline{k}}{\partial\underline{x}}}_{irrot}$$

∴

$$\frac{\partial\underline{k}}{\partial t} = -\frac{\partial D}{\partial\underline{x}}$$ agree!

→ Now, can convert from N to E!

i.e. $N = \mathcal{E}/\omega$

$$\left.\frac{dN}{dt}\right|_{rays} = \frac{d}{dt}\left(\mathcal{E}/\omega\right) = 0$$

$$\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\bigg|_{rays} - \frac{1}{\omega^2}\,\mathcal{E}\,\frac{d\omega}{dt}\bigg|_{rays} = 0$$

Now $\displaystyle \frac{d\omega}{dt} = \partial_t\omega + \frac{\partial\omega}{\partial\underline{x}}\cdot\frac{d\underline{x}}{dt} + \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{d\underline{k}}{dt}$

From eikonal eqns:

$$= \partial_t\omega + \frac{\partial\omega}{\partial\underline{x}}\cdot\frac{\partial\omega}{\partial\underline{k}} - \frac{\partial\omega}{\partial\underline{k}}\cdot\frac{\partial\omega}{\partial\underline{x}}$$

so if $\partial_t\omega = 0$ energy conserved

$\therefore$ $\displaystyle \frac{dN}{dt} = 0 \Rightarrow \frac{d\mathcal{E}}{dt} = 0$.

so $\displaystyle \partial_t\mathcal{E} + \underline{V}_{gr}\cdot\underline{\nabla}\mathcal{E} - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\mathcal{E} = 0$

and exploiting Liouville Thm, etc $\Rightarrow$

$$\boxed{\frac{d\mathcal{E}}{dt} = \partial_t\mathcal{E} + \nabla\cdot\left[\underline{V}_{gr}\,\mathcal{E}\right] = 0}$$

conserved energy density

So, for conservative case i.e. $\partial_t \omega = 0$

$$\boxed{\partial_t \mathcal{E} + \nabla \cdot [\underline{V}_{gr} \mathcal{E}] = 0}$$

If stationary, $\partial_t \mathcal{E} = 0$

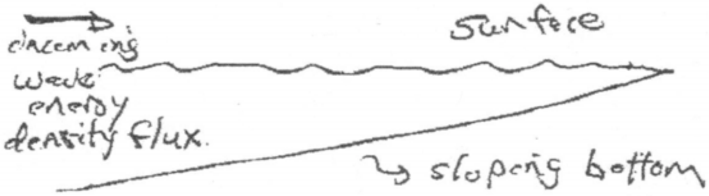
$$\Rightarrow \boxed{\underline{\nabla} \cdot [\underline{V}_{gr} \mathcal{E}] = 0}$$

incompressible wave energy flux!

$\Rightarrow V_{gr}$ drops $\Rightarrow$
$\mathcal{E} \uparrow \Rightarrow$ blocking, breaking

③ The beach....

Consider:



$H = H(x)$

Now, in shallow water
$(\lambda > H)$

(continuity)

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(vh) = 0 \\[2ex] \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -g\dfrac{\partial h}{\partial x} \end{cases}$$

slope

(mom.)

shallow water eqns.

$\dfrac{\partial h}{\partial x} \rightarrow$ replaces pressure

$v = \cancel{v_0} + \tilde{v}$, $h = H + \tilde{h}$

$$\Rightarrow \quad -i\omega\tilde{h} + ikH\tilde{v} = 0$$
$$-i\omega\tilde{v} = -ikg\tilde{h}$$

$\therefore \Rightarrow \omega^2 = k^2 gH$ is dispersion relation
$gH \leftrightarrow c_s^2$

$\Rightarrow$ analogy with acoustics is obvious.

$h \leftrightarrow \rho \qquad c_s^2 = gH$
$v \leftrightarrow v \qquad$ etc.

~~energy~~ ⇒

$$\frac{\partial \tilde{v}}{\partial t} = -g\frac{\partial \tilde{h}}{\partial x} \qquad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H\frac{\partial \tilde{v}}{\partial x} \qquad (2)$$

$$\Rightarrow \quad (1) \times \tilde{v} + (2) \times \left(g\,\frac{\tilde{h}}{H}\right)$$

$$\therefore \quad \frac{\partial}{\partial t}\tilde{v}^2 = -g\,\tilde{v}\,\frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H}\frac{\partial}{\partial t}\tilde{h}^2 = -g\frac{\cancel{H}}{\cancel{H}}\,\tilde{h}\,\frac{\partial \tilde{v}}{\partial x}$$

Wave energy density flux

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H}\right) + \frac{\partial}{\partial \underline{x}}\cdot\left(g\,\tilde{h}\,\underline{\tilde{v}}\right) = 0$$

is energy theorem

$$\Rightarrow \quad \mathcal{E} = \frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H}$$ is wave energy density.

$$\omega/k = (gH)^{1/2}$$ is wave phase velocity

so .... as no explicit time dependence:

$$\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla}\cdot\left(v_{gr}\,\mathcal{E}\right) = 0$$

$\Rightarrow \quad V_g(x)\,\mathcal{E}(x) = V_\infty\, \mathcal{E}_\infty = \underline{I}$

$\underline{I}$ ↓ incoming wave flux

$V_g = \sqrt{gH(x)}$ ↳ shallow water waves have zero dispersion

$\therefore \quad \sqrt{g(H(x))}\ \mathcal{E}(x) = \underline{I}$

as $x \to$ shore $V_g \downarrow$ so wave energy ~~[illegible]~~ must increase.

Now $\mathcal{E}(x) = \dfrac{\overline{\tilde{v}^2}}{2} + \dfrac{g\overline{\tilde{h}^2}}{2H} \simeq \dfrac{g\tilde{h}^2}{\cancel{2}H}$

$\sqrt{gH(x)}\ \dfrac{g\tilde{h}^2}{\cancel{2}H(x)} = I$

$\dfrac{\tilde{h}^2}{H(x)^2} = \dfrac{\cancel{2}I}{(\sqrt{g})^3}\left(\sqrt{H(x)}\right)^{-3}$

i.e. $\boxed{(\tilde{h}/H)^2 \sim (\text{const})\, I \big/ (H(x))^{3/2}}$

i.e. $\tilde{h}/H \to 1 \Leftrightarrow$ breaking $\Leftarrow$ as $H(x)$ drops.

N.B.:

→ if know bottom profile, can deduce displacement profile, and approximate breaking point.

→ 2D bottom contours ⇒ wave refraction

$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial \underline{x}} = -kg\left(\frac{\partial}{\partial \underline{x}} H(x,y)\right)$$

i.e. wavefronts tend to align with bottom contours approaching shore.