Physics 218C

Lecture 4C: ⇒

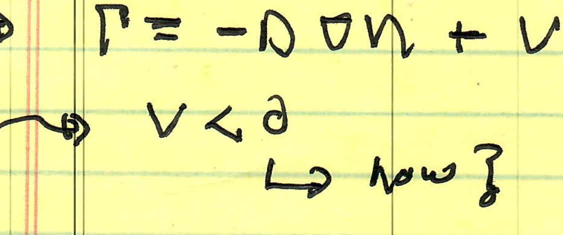
Forming the Density Profile II - Universality ?!

⇒ Pinch - Review, Homogenization and TEP

Recall:

- extensive discussion of ITG / η_i drift wave turbulence, and variants, with background phenomenology
- Inward convection

⇒ How to form a density profile ?



How ?

→ $\Gamma = -D\nabla n + Vn$

→ $V < 0$ (n.b. Fickian picture)

↳ how ?

Aside:

- density profile formation is one of the key questions in tokamak self-org.

- Key Questions (remainder):
  - density peaking ✓
  - ✓ zonal flows vs avalanches } and GB breaking
  - intrinsic rotation
  - L-H transition / ITB formation
  - Edge Relaxation – ELMs, QH, SOL
  - density limit

~ cont'd

For $\nabla < 0$, prototype: Ion Mixing Mode

⇒ realization of Keller–Segel Chemotaxis Mechanism, with $\nabla C \to \nabla T$

$$\Rightarrow \frac{\tilde{n}_k}{n_\omega} = \frac{|e|\hat{\phi}_k}{T\omega}\left[1 - i\frac{d_k}{\omega}, + i\frac{d_k}{\omega_2}\right]$$

$\delta_{k,\omega}^{1} \to \nabla n \to \partial$

outward/down gradient flow

(→0 in I.M.M.)

$\delta_{k,\omega}^{2} \to \nabla T_e$

$\oplus$

$\langle \tilde{v}_r \tilde{n} \rangle_{\delta_2} < 0$ inward

→ produced by ITG mode.

i.e.

$$\Gamma_n = \sum_k \frac{|\tilde{V}_{r_{k,\omega}}|^2}{\chi_e k_\parallel^2} \underbrace{\nabla T_e}_{<0} < 0$$

- developed analogy with Keller-Segel

- demonstrated:

$\frac{dS}{dt} > 0$ → net entropy production (Ludwig B. happy)

as $\chi_\parallel k_\parallel^2 \gg \gamma_{k\omega}, 1/\tau_{c,n}$ so $\frac{\omega}{\chi_\parallel k_\parallel^2} < 1$

( ) $\sim 1/nT$

( ) $\left.\frac{dS}{dt}\right|_{\nabla T, \nabla T} = \chi_T \left(\frac{\nabla T_i}{T}\right)^2$

$\gg$

$$\frac{dS}{dt} = -\underbrace{\chi_{cross}}_{O\left(\omega/\chi k_{ri}\right)} \frac{\nabla n}{n} \frac{\nabla T_i}{T}$$

Onsager Matrix

entropy production

$$\frac{dS}{dt} = \text{[redacted]} - \left(\frac{\nabla T}{T}, \frac{\nabla n}{n}\right) \begin{pmatrix} \overset{>0,\ \text{big}}{\boxed{\chi_T}} & -\chi_{cr} \\ -\chi_{cr} & \underset{>0}{D_n} \end{pmatrix} \begin{pmatrix} \frac{\nabla T}{T} \\ \frac{\nabla n}{n} \end{pmatrix}$$

entropy production

Thermo Forces

$\Rightarrow$ pinch consistent with entropy production.

$\Rightarrow$ Some questions:

- Ion mixing mode $\rightarrow$ collisional electrons, limited to edge (albeit useful)

- structurally similar approaches for core, exist but

  → very mode/parameter etc. dependent ⟨ ITG, CTEM η_i, η_e ..... ?

  → yet profile peaking is universal.

How ? → can we find a robust, universal mechanism ?

⇒ Theory of "Turbulent Equipartition Pinch" (c.f. Isichenko, P.D.; Naulin, Garbet)

What is it ?

- homogenization (hence "turbulent equipartition")

  → 2D fluid, PV. (Why 2D ?)

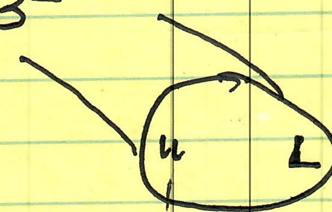
- due $\underline{\nabla} \cdot \underline{V}_{E \times B} \neq 0$ due $\underline{B}$ inhomogeneity (toroidal geometry)

⇒ its n/B which is homogenized/ mixed! (f/B, for kinetics)

Then

$$\nabla(n/B) = 0 \Rightarrow \frac{\nabla n}{B} + -\frac{\nabla B\, n}{B^2}$$

$$\Rightarrow \frac{\nabla n}{n} = \frac{\nabla B}{B} \cong -\frac{1}{R}$$



⇒ – a robust, universal mechanism

$\nabla \cdot V_{E\times B} \neq 0$!

– geometric!

but – the catch: modest peaking;

i.e.

$$\frac{\nabla T}{T} \sim \frac{1}{L_T} \gtrsim \frac{1}{a}$$

$$\frac{\nabla n}{n} = -\frac{1}{R}$$

so T comparatively more peaked than n.

future $\Rightarrow$ i.e ITER

Most scenarios based on modest peaking / flattish n profiles

$\Rightarrow$ typical of H-mode.

Contrast IOC / Supershot (TFTR)

To develop:

ⓐ – theory of (PV, scalar) homogenization $\updownarrow$ mixing { Fundamental, L – entropy }

(also barriers)

ⓑ – application to convective mixing

ⓒ – rmks on general theory

ⓓ – density profiles $\Rightarrow$ status.

ⓐ Homogenization { Prandtl, Batchelor, also Rhines, Young }

recall $\underline{\omega}$ evolution for (2D) { $\underline{\nabla}\cdot\underline{V} = 0$, closed eddy }

$$\frac{\partial \omega}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\omega - \underline{\nabla}\cdot\nu\,\underline{\nabla}\,\omega = \cancel{\underline{\omega}\cdot\underline{\nabla}V}$$

ν: impt

$\cancel{\ }$: 2D.

equally relevant for c, A, q ...

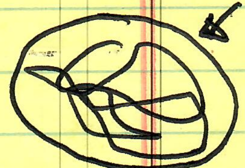
A: scalar potential (flux expulsion)

c: passive concentration

q: PV

$\underline{v} = \underline{\nabla}\phi \times \hat{z}$

What happens ?



closed boundary streamline $\to$?

i.c.

$t \to \infty$

$$\underline{\nabla}\phi \times \hat{z} \cdot \nabla \omega = \underline{\nabla} \cdot (\nu \underline{\nabla} \omega)$$

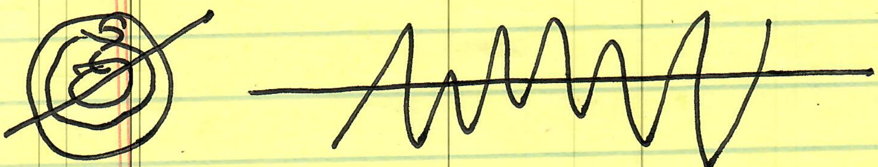
$\nu \to 0$ $\quad \nabla\phi \times \hat{z} \cdot \nabla \omega = 0$

so arbitrary $\omega = \omega(\phi)$ is solution.

$\Rightarrow$ can develop arbitrary fine scale

$$\omega(\phi) \to \begin{cases} \text{closed streamlines} \\ \text{perfect memory} \end{cases}$$

⇒ fine scale structure develops, as no inter-streamline mixing ($\nu \to 0$)



no smoothing of sharp gradients.

Now: → obviously unphysical

⇒ small but finite mixing

⇒ different result ⇒ globally !

n.b. $\boxed{\text{"Not all solutions of the Navier-Stokes equations are realized in nature" — Landau + Lifshitz}}$

Point is that for $\nu \neq 0$ $\left\{ \begin{array}{l} Re \\ Pe \end{array} \right. \gg 1$

instead of arbitrarily fine scale structure,

have $W(\phi) \to$ const as $t \to \infty$

Like wake, a small ammount of ν makes a *global* difference.

i.e. singular perturbation!

<u>Theorem and Proof</u>

Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by <u>closed streamline</u> C_0. Then, <u>if</u> <u>diffusive dissipation</u>:

$$\text{i.e.} \quad \partial_t \omega + \nabla\phi \times \hat{z} \cdot \underline{\nabla}\, \omega = \underline{\nabla} \cdot (\nu \underline{\nabla} \omega)$$

<u>Then</u> $\omega \to$ uniform (state of mixing or homogenization) as $t \to \infty$.

[N.B. No comment on "how long?"]

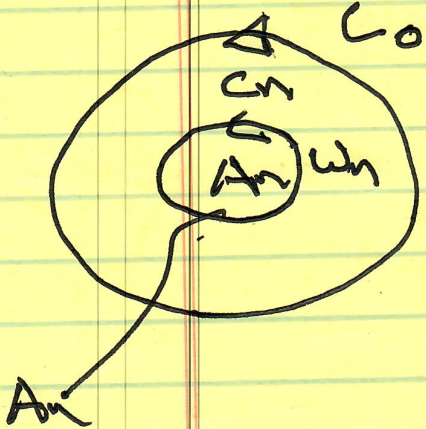
<u>Proof</u>: $t \to \infty$, $\partial_t \omega \to 0$

$$\nabla\phi \times \hat{z} \cdot \nabla\, \omega = \underline{\nabla} \cdot \underline{\nu} \cdot \underline{\nabla}\omega$$

(for stationarity) n.b. $t \to \infty$ "before" $\nu \to 0$ finite ν!

- choose arbitrary closed C_n within C_0

C_n is a streamline



n.b.

- simply connected region $\rightarrow$ no holes
- stationarity $\rightarrow$ ω constant along streamline
- C_0 $\rightarrow$ specify on bndry

$\therefore$ $\omega \Rightarrow \omega_0$ on bndry (b.c.)

$\omega \rightarrow \omega_n$ on C_n.

if A_n is area enclosed by C_n,

$$\int_{A_n} d^2x \; \underline{v} \cdot \underline{\nabla} \omega = \int_{A_n} d^2x \; \underline{\nabla} \cdot (v \underline{\nabla} \omega)$$

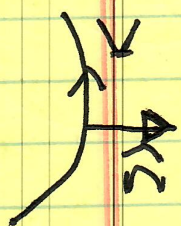
but

$$\int_{A_n} d^2x \; \underline{v} \cdot \underline{\nabla} \omega = \int_{A_n} d^2x \; \underline{\nabla} \cdot [\underline{v} \omega]$$

by Gauss's law:

$$= \int_{C_n} d\ell \; \underset{\substack{\uparrow \\ \text{normal to } C_n}}{\hat{n}} \cdot (\underline{V}\omega)$$

$$= 0 \; , \quad \text{as} \quad \hat{n} \cdot \underline{V} = 0$$



$\hat{n} \cdot \underline{V} = 0$ as $\underline{V}$ along streamline and $\hat{n}$ orthogonal

so

$$0 = \int_{A_n} d^2x \; \underline{\nabla} \cdot (\nu \, \underline{\nabla} \omega)$$

$$= \int_{C_n} d\ell \; \nu \, \hat{n} \cdot \underline{\nabla} \omega$$

but $\omega = \omega(\phi_n)$ in steady state,

$$\omega_{C_n} = \omega(\phi_n)$$

so

$$0 = \int_{C_n} d\ell \; r \; \hat{n}\cdot\nabla\phi_n \frac{\delta \omega}{\delta \phi_n}$$

Now $$0 = r\frac{\delta \omega}{\delta \phi_n} \int_{C_n} d\ell \; \hat{n}\cdot\nabla\phi_n$$

$$\Gamma = \int d\underline{\ell}\cdot\underline{V} \Rightarrow \text{circulation}$$

$$= -\int d\ell \; \nabla\phi\cdot\hat{n}$$

$\Rightarrow$

$$0 = r\frac{\delta \omega}{\delta \phi_n}\Gamma_n$$

$\Rightarrow$ $\delta\omega/\delta\phi_n = 0$

but ϕ_n arbitrary: $\boxed{\delta\omega/\delta\phi = 0 \quad \text{all } \phi}$

no variation from streamline - to - streamline

$\Rightarrow$ $\boxed{\omega \text{ homogenized}}$ i.e. $\omega \to$ const.

Comments

⇒ rate ? → discussed below

⇒ Poor Man's (Physicist) argument

$$\frac{\partial \omega}{\partial t} + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\,\underline{\omega} = \underline{\nabla} \nu \cdot \underline{\nabla}\, \omega$$

$$\frac{\partial \langle \omega \rangle}{\partial t} + \frac{\partial}{\partial r} \langle \tilde{v}_r \omega \rangle = \partial_r \nu\, \partial_r \omega$$

$$\tilde{v}_r \Big|_{0,\, c_0} = 0 \qquad v_r = 0 \;\rightarrow\; \text{closed streamline}$$

$$\frac{\partial}{\partial t} \int \frac{\langle \omega \rangle^2}{2} = \int \langle \tilde{v}_r \omega \rangle\, \partial_r \langle \omega \rangle$$

$$= -\int \nu \left(\partial_r \langle \omega \rangle \right)^2$$

$$\partial_t \int \frac{\langle \omega \rangle^2}{2} = -\int \nu \left(\partial_r \langle \omega \rangle \right)^2$$

then $\partial_t \dfrac{\langle \omega \rangle^2}{2} \rightarrow 0$ ⇒ $\partial_r \langle \omega \rangle = 0$

$\nu \neq 0$ homogenized state !

if $\langle \tilde{v}_r \tilde{\omega} \rangle \neq 0$

assume $\langle \tilde{v}_r \tilde{\omega} \rangle = -D_T \partial_r \langle \omega \rangle$

turbulent D_T

$$\partial_t \int \frac{\langle \omega \rangle^2}{2} = -\int d^2x \left[D_T + \underline{\nu} \right] (\partial_r \langle \omega \rangle)^2$$

akin Q L Plateau argument:

$$\frac{\partial \langle f \rangle}{\partial t} = \partial_v D \partial_v \langle f \rangle$$

$$\frac{\partial}{\partial t} \int \frac{\langle f \rangle^2}{2} = - \int D \left(\partial_v \langle f \rangle \right)^2$$

for stationarity

$$D \neq 0 \Rightarrow \underbrace{\frac{\partial \langle f \rangle}{\partial v} = 0}$$ → plateau.

(or $D \to 0$).

→ Possible to have $v = v(\phi)$.
n.b. Conservative structure crucial!
⇒ layered states (homogenized bands)

⇒ e.g. CHNS (Fan, P.D. et. al.)
↓
targets

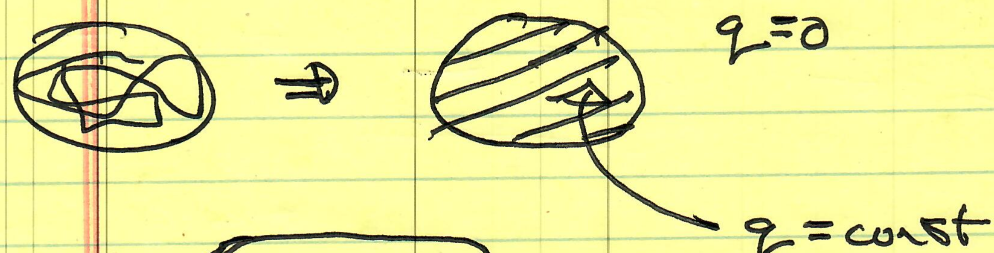
but ultimate end state is homogenized, globally.

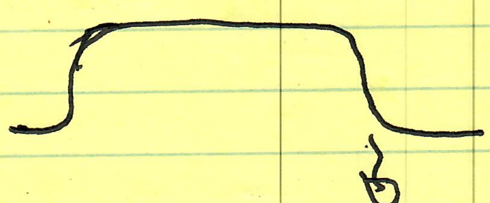
→ Key assumptions:

- closed, bounding streamline
- simply connected domain
- isolated structure

n.b. spreading rate?

→ Homogenization → barriers



⇒ 

∇q at C_0 ⇒ boundary vorticity gradient

$\nabla \omega = \nabla V'$ ⇒ strong gradient in shear

⇒ barrier.

$\omega_{Rossby} \sim k_\theta \nabla q / k^2$

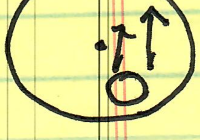
$$D \cong \sum_k \langle \tilde{v}^2 \rangle_k \tau_{c_k} = \sum_k \langle \tilde{v}^2 \rangle_k \Delta\omega_k^{-1}$$

$$\rightarrow \sum_k \langle \tilde{v}^2_k \rangle \frac{\Delta\omega_k}{\left[\underset{\substack{\uparrow \\ \neq 0 \text{ only on barrier.}}}{\omega_{R_k}^2} + \Delta\omega_k^2\right]}$$

mixing thru barrier $\downarrow$ where $\omega_R \gg \Delta\omega_k$ at fixed amplitude.

"Rossby wave elasticity"
"self-sharpening" $\Big\}$ Dritschel and McIntyre.

→ Rate



- synergy of shear + r
- $r\theta = y$

$$\frac{dy}{dt} = V_y(r)$$

so $$\frac{d}{dt}\delta y = \frac{\partial V_y}{\partial r}\delta r$$

$$\delta y = \int dt \left(\frac{\partial V_{oy}}{\partial r}\right) \delta r$$

$$\langle \delta y^2 \rangle \cong \left(\frac{\partial V_{oy}}{\partial r}\right)^2 \underbrace{\langle \delta r^2 \rangle\, t^3}$$

$$\cong \left(\frac{\partial V_{oy}}{\partial r}\right)^2 \frac{D_r}{3} t^3 \qquad \text{(properly)}$$

$$= \left(\frac{\partial V_{oy}}{\partial r}\right)^2 \frac{\nu}{3} t^3 \quad , \quad \text{here}$$

$$1/\tau_{mix}^3 = \frac{1}{L_y^2} \left(\frac{\partial V_{oy}}{\partial r}\right)^2 \frac{\nu}{3}$$

ref scale $\sim$ typical k_y

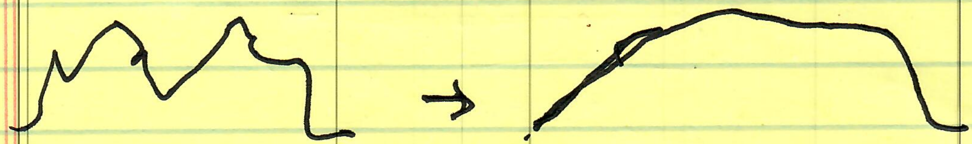
so

$$\boxed{1/\tau_{mix} \cong \left(\frac{1}{L_y^2} \left(\frac{\partial V_{oy}}{\partial r}\right)^2 \frac{\nu}{3}\right)^{1/3}}$$

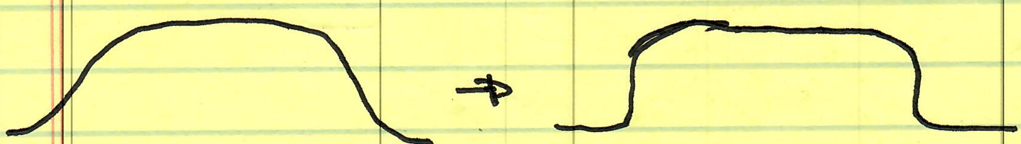
$$\boxed{\tau_{mix} \sim Re^{1/3}\, \tau_{Rot.}}$$

so — 'fast' mixing to smooth state

on $\tau_{mix} \sim Re^{1/3}\,\tau_{Rot}$



— 'slow' mixing on diffusive time scales



$\tau \sim Re\,\tau_{Rot}$

— can have:

- exact streamlines, v, ω
- coarse grained streamlines, v_T, $\langle\omega\rangle$.

(b) Convective Mixing

Consider now:

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n \underline{V}) = 0$$

$$\underline{V} = \underline{V}_E = -\frac{c}{B(\underline{x})} \underline{\nabla}\phi \times \hat{z}$$

→ inhomogeneous B, i.e. tokamak ($\underline{B} = B_0(R)$)

Point:
E×B flow in tokamak is compressible, due geometry.

⇒

$$\frac{\partial n}{\partial t} - \frac{c}{B_0} \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} n + n\, \underline{\nabla} \cdot \left[-\frac{c}{B} \underline{\nabla}\phi \times \hat{z} \right] = 0$$

$\underline{B} = B(x)\, \hat{z}$, for simplicity

recombine to:

$$\frac{\partial n}{\partial t} - c\underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\left(\frac{n}{B}\right) = 0$$

coarse grain

Now, $\frac{n}{B}$ undergoes incompressible advection $\Rightarrow$ [$\frac{n}{B}$ Frozen-in contrast MHD]

$\boxed{n/B \text{ homogenized}}$

$$\frac{\partial n}{\partial t} + \underline{\tilde{V}}' \cdot \underline{\nabla}\left(\frac{n}{B}\right) = 0$$

for mean field evolution

so

$$\frac{\partial \langle n \rangle}{\partial t} + \underline{\nabla} \cdot \left\langle \tilde{V} \widetilde{\left(\frac{n}{B}\right)} \right\rangle = 0$$

B static

then $\widetilde{\left(\frac{n}{B}\right)}_{\underline{k}} = -\tilde{V}^*_{\underline{k}} R_{\underline{k}} \nabla \left\langle \frac{n}{B} \right\rangle$

response fctn (contains details)

∴ have mean field eqn:

$$\partial_t \langle n \rangle + \nabla \cdot \Gamma_n = 0$$

$$\Gamma_n = -\sum_{\underline{k}} |\tilde{V}_{x k}|^2 R_{\underline{k}} \nabla \left(\langle n/B \rangle \right)$$

$$\partial_t \langle n \rangle = \partial_r D \, \partial_r \left\langle \frac{n}{B} \right\rangle$$

$$= \partial_r D \, \partial_r \left[\frac{\langle n \rangle}{B} \right]$$

at B fixed.

then, as before

$$\partial_t \int \langle n \rangle^2 \, dx = - \int D \left(\partial_x \langle n \rangle \right) \partial_x \left[\frac{\langle n \rangle}{B} \right]$$

stationarity of entropy production for:

- $D = 0$ (no turbulence) $\rightarrow$ trivial (but near marginal)

- $\partial_x \langle n \rangle = 0$ $\rightarrow$ trivial and undesirable

or

$$\rightarrow \boxed{\partial_x \left[\frac{\langle n \rangle}{B} \right] = 0}$$

$\rightarrow$ TEP state

- then

$$\frac{\partial_x \langle n \rangle}{\langle n \rangle} = -\frac{\partial_x B}{B}$$

$$\cong -1/R$$

$\Rightarrow$ "canonical", or fully mixed or TEP profiles

- nothing but homogenization, with realization that its n/B which is homogenized!

- result is independent (modulo time scale) of details of QL closure

(d) Status

- Thermoelectric → stronger but very model sensitive

- TEP → ⓒ universal, but weak
  ↳ because geometric!

$$\frac{\nabla \cdot n}{n} = -1/R$$

- what of ⓒ marginal states (non-diffusive flux).

- what of stellarator (1/R ?)

i.e. what's in Respons?

(3)

Two comments:

- obviously, B is not thermodynamic variable. How interpret role of B in profile relaxation? (Garbet '05)

⇒ answer: stability threshold.

No flux if $|\nabla(\text{Thermo})/\text{Thermo}| >$ $1/R$.

- What of kinetics? (I, G, D '97)

⇒ Answer: Relevant to ~2D dynamics ⇒ trapped particles carry TEP.