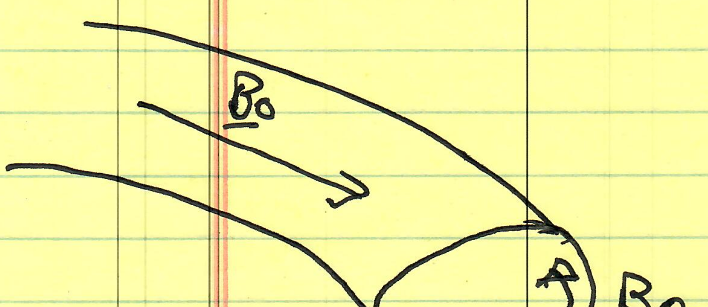
Physics 218c

Lecture 2c : Transport Heuristics Part c: A Simple Perspective on Turbulent Transport in Tokamaks → Scalings, etc.

Here → seek some simple applications of mixing length ideas to tokamak transport

→ address basic scaling questions.



turbulence:

- quasi-2D cells
  $k_\perp \gg k_\parallel$
- $R_o \ll 1$
  $\updownarrow$
  Rossby #

$R_o \sim V_\perp / l_\perp \Omega_i$

- so $\omega < \Omega_0$

$\tilde{\underline{v}}_\perp = \frac{c}{B}\underline{E} \times \hat{z}$ $\Rightarrow$ $E \times B$ advection

→ typically, cells localized at: ("pinned" turbulence)

$\underline{k} \cdot \underline{B} = 0$ surfaces

$q = m/n$

$q = r B_T / R B_\theta = \frac{m}{n}$ [redacted]

$q \downarrow$ pitch lines

$\downarrow$ mode pitch

Why: $\frac{\underline{k} \cdot \underline{B}}{|B|} = k_\parallel$

$r = m/n$

→ defines surface, mode rational surface

large $k_\parallel$ → $\left\{ \begin{array}{l} \text{Landau Damping} \\ \text{Bending Energy} \\ \vdots \end{array} \right\}$

→ turbulence:

∇T_i, ∇T_e, ∇n

driven.

Drift / Drift-Alfven Turbulence

→ akin to $\left\{ \begin{array}{l} \text{Rayleigh-Benard} \\ \text{Thermal Rossby Wave} \end{array} \right\}$

convection with $g \to \left\{ \begin{array}{l} \nabla B \\ \text{curvature} \end{array} \right.$ drift.

⇒ $\boxed{\text{Buoyancy}}$

key here is that

$$g_{eff} = g_{eff}(\eta)$$

↓

position along fieldline

N.B. $\boxed{\begin{array}{l} \text{Buoyancy is} \\ \text{critical to} \\ \text{tokamak turbulence} \end{array}}$

→ Dimensionless #'s

– $Re = \tilde{V} L / \nu$ irrelevant

⇒ meaningless, as dissipation is not viscous

$$Re \sim |\underline{V} \cdot \nabla \underline{V} / \nu \nabla^2 V|$$

– $(Re)_{eff}$ not so large ......

— Kubo # $\underline{\underline{is}}$ relevant. — but not a
(stochastic) control parameter

$$Ku \sim \frac{\text{scattering length}}{\text{correlation length}}$$

$$\sim \delta x / \Delta$$

perturbations correlation time

$$\sim \int \tilde{v}\, dt \Big/ \Delta \sim \frac{\tilde{v}\, \tau_c}{\Delta}$$

correlation length

$Ku < 1 \Rightarrow$ many kicks in Δ
diffusive / random (easy)

$Ku > 1 \Rightarrow$ strong kick in Δ
coherent (hard)

$Ku \sim 1$ is, also, typical of
saturated turbulence
(cross-over regime)

$$\tilde{v} \sim \frac{\Delta}{\tau_c}$$

Most MFE turbulence has $Ku \lesssim 1$.

Opinion: [redacted] Deeper study of $Ku \geq 1$ is needed.

$\rightarrow$ $Ku \leq 1$ $\Rightarrow$ turbulence is not "strong"

$\sim$ more akin to wave turbulence than high Re Fluid Turbulence.

Wave Turbulence:

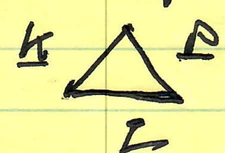
$$\partial_t a \sim c\,a\,a + \dots$$

$\rightarrow$ quadratically nonlinear

$$\partial_t \mathcal{E} \sim \partial_t (|a|^2) \sim c \langle a a a \rangle$$

$\Rightarrow$ energy transfer ("cascade")

$\sim \langle a a a \rangle$ $\Rightarrow$ triad interactions



$\underline{k} = \underline{p} + \underline{q}$

$\rightarrow$ key physics is triad coherence time $\rightarrow$ time for (coherent) energy transfer

(Fermi Golden Rule)

$$\tau_{corr} \simeq \pi\, \delta(\omega_{\underline{k}} - \omega_{\underline{p}} - \omega_{\underline{q}}) \rightarrow \left\{ \begin{array}{l} \text{Wave} \\ \text{Regime} \end{array} \right.$$

$$\sim 1/\Delta\omega$$

$\hookrightarrow$ bandwidth

vs.

$$\tau_{corr} \sim \ell / v(\ell) \qquad \Rightarrow K41$$

eddy turn-over.

wave $\leftrightarrow$ weak : Many $\underline{\text{kicks}}$ in energy evolution

$$\frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} \tau_c < 1$$

Hydro $\leftrightarrow$ strong $\qquad \frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial t} \tau_c \rightarrow 1$

N.B. [Drift wave - Zonal Flow interaction can be cast in framework of wave turbulence.

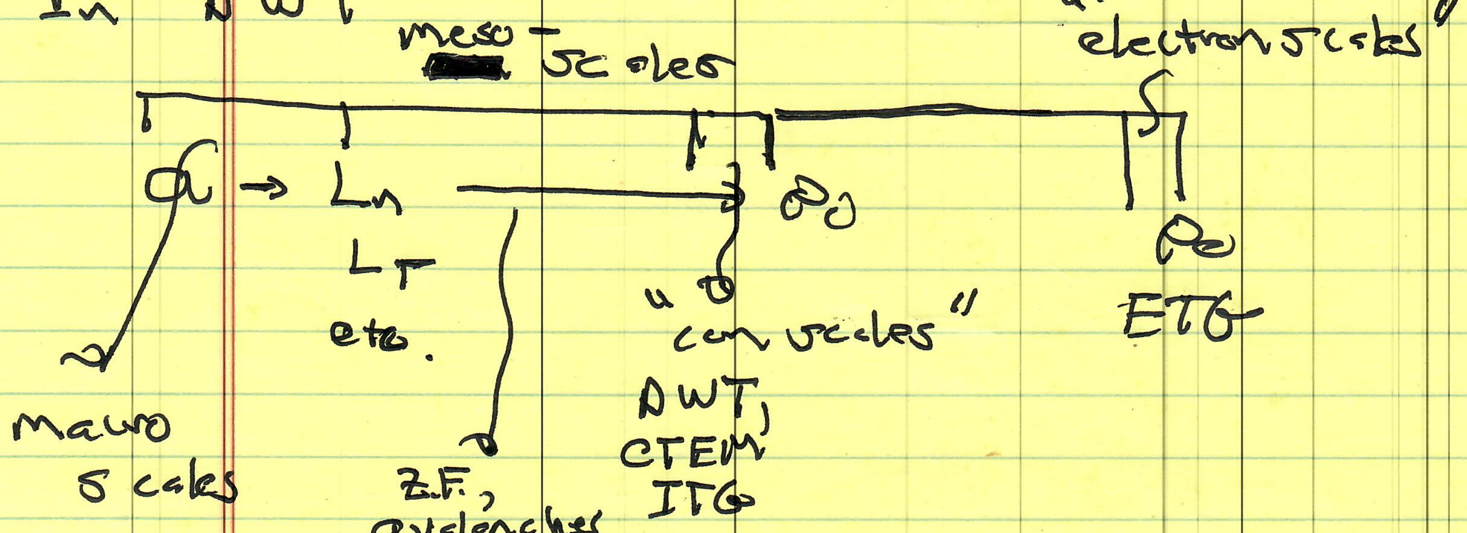
→ <u>But</u> dynamic range of tokamak drift wave turbulence is <u>large</u>, even absent high Re.

In K41, Dynamic Range $\sim \dfrac{l_0}{l_d}$ → excitation, l_d → dissipn

$\sim Re^{3/4}$

(show this!)

In DWT



so $l_0 / l_{min} \sim a/\rho_e \sim \sqrt{\dfrac{M_i}{m_e}}\,\dfrac{1}{\rho_*}$

BIG

→ This brings us to a key issue →

ρ_*

→ Unlike pipe, two dynamical scales for drift wave turbulence

key contrast

$\rho_i \equiv$ gyro-radius, non-dissipative inner scale

$a, L_T \ldots \equiv$ cross-section macro, outer scale

$\rho_* = \rho_i / L_T \ll 1$ $\sim 10^{-3}$, and going down for ITER.

⇓

key smallness parameter of magnetic confinement.

ρ_* scaling is of great interest vis-a-vis benefits of size scaling → Is bigger better?

(ITER would say 'yes'!)

→ so, mixing length l_{mix} on range $[\rho_i, L_T] \rightarrow [\rho_i, \rho_i/\rho_*]$

or $$\boxed{l_{mix} \sim L_T (\rho_*^{\alpha})}$$

α tbd.

No corresponding scale to ρ_i exists for pde.

→ Then diffusivity → τ_E $\quad 1/\tau_E \sim \frac{D}{L_\perp^2}$

$D \sim \tilde{V}\, l_{mix}$ $\quad$ analogous $u_t\, x$

do not confuse!

for drift waves, expect 'wave breaking' to occur $\tilde{V} \sim V_d \sim V_*$

diamagnetic velocity

(i.e. fluctuating velocity ~ characteristic wave velocity)

Why? – coming ...

$V_d \sim \rho_* c_s$

so $$\boxed{D \sim \rho_* c_s\, l_{mix}}$$

$\Rightarrow$

$$D \sim \rho_* c_s L_T \rho_*^\alpha$$

$$\sim \rho_s c_s \rho_*^\alpha$$

$$\sim \underbrace{D_B}_{\text{Bohm Diffusion} \sim T/B} \rho_*^\alpha$$

$\tau_E \sim a^2$

$\alpha = 0 \Rightarrow$ Bohm $\Rightarrow$ bigger <u>not</u> better

$\alpha = 1 \Rightarrow D \sim D_{GB} \sim D_B \rho_* \sim \frac{\rho_s}{L_T}(\rho_s c_s)$

$\Rightarrow$ bigger <u>is</u> better

$\tau_E \sim a^3$

correlation with eddy scales?

$0 < \alpha < 1 \Rightarrow$ "Broken" Gyro-Bohm

(with α closer to 1 ....)

is symptomatic of reality.

c.f. (McKee., et.al. 2006 $\rightarrow$)

Typical :- $l_{mix} \sim (\rho_i L_T)^{1/2}$ $\Rightarrow$ possible, ad-hoc!

So pessimist → $\alpha = 0$ (pipe analogy)

$$D \sim D_B$$

optimist → $\alpha = 1$

$$D \sim D_{GB}$$

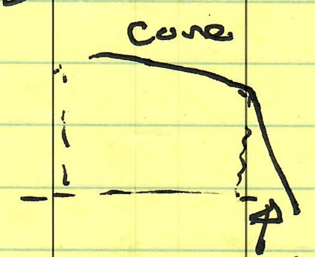
$\boxed{\text{realist} \rightarrow \text{what physics regulates } \alpha \text{ ?}}$

* ⇒ { - curvature / ballooning
- $E \times B$, magnetic shear
etc.

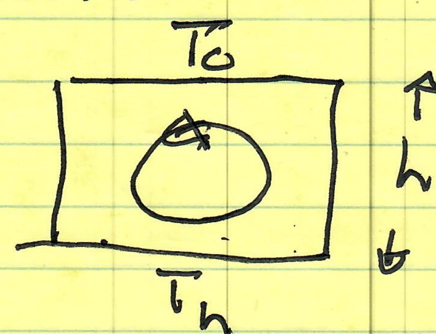
N.B.: - ρ_* scaling complicated by

multi-zone structure of confinement physics

i.e.



- on RBC
(see Notes 2b)

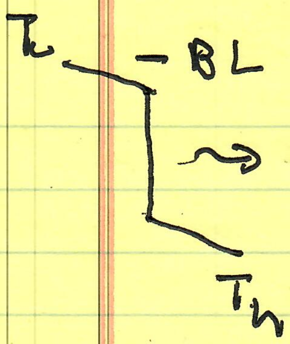


Analogous question is

Nu vs Ra

Nusselt #   Rayleigh #

$$Nu = \frac{\langle \tilde{v} \tilde{T} \rangle}{\chi_0 \Delta T / h}$$



$$Ra \sim \frac{h^3 g \alpha \Delta T}{\chi_0 \nu_0}$$

<u>all</u> $T_h - T_c$ held in Boundary Layer
(~ marginal)

∴ "Thermal short circuit" in core ⇒ "bigger box doesn't help ⟺ Bohm"

$\boxed{\text{see Ahlers, Siggia Reviews}}$

then $\langle \tilde{v} \tilde{T} \rangle$ indep. box size.

$$\langle \tilde{v} \tilde{T} \rangle = Nu \frac{\chi_0 \Delta T}{h}$$ must be indep. h.

$Ra \sim h^3$ so $\boxed{Nu \sim (Ra)^{1/3}}$ classic scaling

$\boxed{Nu \sim (Ra)^{1/3} \text{ is \underline{well} supported by simulation, experiment.}}$

N.B. Convection tracks 'Bohm' picture.

→ More Theory, please!

Generically, (cf. Taylor & McNamara)

$$\underline{\underline{D}}_{\perp} = \int_0^\infty d\tau \, \langle \tilde{\underline{V}}(0) \tilde{\underline{V}}(-\tau) \rangle \quad \text{(Kubo)}$$

time history → scalar.

Consider ions (quasi-2D, ambi-polarity tbd)

$$\underline{V}_{\perp} \simeq \frac{c}{B_0} \sum_{\underline{k}} \underline{E}_{\perp \underline{k}} e^{i \underline{k} \cdot \underline{r}}$$

so

$$D_{\perp} \cong \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 e^{-i \underline{k} \cdot \underline{r}_0} e^{i \underline{k} \cdot \underline{r}(-\tau)}$$

$$\underline{r}(-\tau) = \underline{r}_0 + \delta \underline{r}(-\tau)$$

excursion due scattering → stochastic.

[n.b. no time transform, "modes", etc.

⟨ ⟩ ≡ averages over ensemble of

$$D_1 \cong \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 \langle e^{i\underline{k}\cdot\delta\underline{r}(-\tau)} \rangle$$

Now, cumulant expansion:

$$\langle e^{i\underline{k}\cdot\delta\underline{r}(-\tau)} \rangle \cong \left\langle 1 + i\underline{k}\cdot\delta\underline{r}(-\tau) - \frac{(\underline{k}\cdot\delta\underline{r})^2}{2} + \cdots \right\rangle$$

$$\cong \left\langle 1 - \frac{k^2 \delta r^2}{2} \right\rangle$$

$$\cong 1 - k^2 D_1 \tau \rightarrow e^{-k_\perp^2 D \tau}.$$

so

$$D_\perp \cong \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-k_\perp^2 D_1 \tau}$$

$$= \sum_{\underline{k}} |\tilde{V}_k|^2 / k_\perp^2 D_1$$

D_1 obtained recursively

N.B. → D_1 controls τ_c &

→ conservation of n $\left(\partial_t \int d\underline{x}\, n = 0 \right)$

⇒ slow τ_c at large scale.

then

$$D \cong \left(\sum_k |\tilde{V}_k|^2 / k_\perp^2 \right)^{1/2}$$

spectral structure matters!

n.b. in 2D, symmetric spectrum

$$D \cong \left(\sum_k |\tilde{V}_k|^2 / k_\perp^2 \right)^{1/2} = \left(\int_0^\infty dk_\perp \, |\tilde{V}_k|^2 / k_\perp \right)^{1/2}$$

spectrum structure

$$\tilde{V}_k = \frac{c}{B} \tilde{\underline{E}}_\perp \times \hat{z}$$

$\rightarrow$ infrared scaling important!

so, back to - scaling wise:

$\ell_{mix} \leftrightarrow k_\perp^{-1}$

$$D \cong \tilde{V} \ell_{mix}$$

then, $\tilde{v}_k \cong \frac{c}{B} i k_\theta \tilde{\Phi}_k \rightarrow k_\theta \rho_s \, c_s \, |e| \tilde{\Phi}_k / T$

How estimate potential ?

For drift wave turbulence:

$$v_{Thi} < \frac{\omega}{k_\parallel} < v_{The}$$

a/s/ ion-acoustic mode

and $\tilde{n}_i = \tilde{n}_e$ $\quad l \gg \lambda_D$

so

$$\frac{\tilde{n}}{n} \sim \frac{|e|\hat{\phi}}{T_e}$$

⇒

$$\tilde{v} \cong k_\theta \rho_s c_s \frac{\tilde{n}}{n}$$

∴

$$\boxed{D \cong D_B k_\theta \frac{\tilde{n}}{n} l_{mix}}$$

⇒ basic scaling.

Now, in MFE, "Mixing Length Theory / Estimate" refers to how determine/estimate $\tilde{n}/n$ (BBK '65)

MLT:

$$\partial_t \tilde{n} + \underbrace{\tilde{v} \cdot \nabla \tilde{n}}_{} \cong -\tilde{v} \cdot \nabla \langle n \rangle$$

could be T, etc.

saturation if scattering/fwr mixing balances relaxation

$$\cancel{\tilde{v}}\, k_r \tilde{n} \cong -\cancel{\tilde{v}} \frac{\partial \langle n \rangle}{\partial r}$$

$$\tilde{n}/n \cong 1/k_r L_n$$

$\Rightarrow$ ~ upper bound

$$\ell_c^2\, \cancel{D} \langle \tilde{n}^2 \rangle \sim \cancel{D} \left(\frac{\partial \langle n \rangle}{\partial r} \right)^2$$

$$D \cong D_B \frac{k_\theta}{k_r L_n} \ell_{mix}$$

and further:

- approximate isotropy

$$\Rightarrow \boxed{D \sim D_B \frac{\ell_{mix}}{L_\perp}}$$

$$\partial_t \langle \tilde{n}^2 \rangle + T \langle \tilde{n}^2 \rangle \sim -\nabla n\, \Gamma_n$$

$$\cong D (\nabla \langle n \rangle)^2$$

$$\frac{\langle \tilde{n}^2 \rangle}{n_0^2} \sim \tilde{\tau}_c D \left(\frac{\nabla \langle n \rangle}{\langle n \rangle} \right)^2$$

" " if $l_{max} \sim L_\perp \rho_*^{\alpha}$

$$\boxed{D \sim D_B \, \rho_*^{\alpha}}$$

$$\begin{cases} l_{max} \sim \rho_s \rightarrow D_{GB} \\ l_{max} \sim L_\perp \rightarrow D_B \end{cases}$$

etc.

N.B — scaling related magnitude, not necessarily equal.

— inferred divergence can complicate story.

— [redacted] D_B interesting in that
- $l_{max} \sim l_{macro}$
- D_B is intensive (no macro-scale dependence)

— low k behavior of spectrum can introduce $L_\perp$ dependence → "non-locality".

→ How about a bit more Theory?
— How does shearing change things?

Recall:

$$D_{\underline{k}} = \int_0^\infty d\tau \left\langle \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 e^{-i\underline{k}\cdot\underline{r}_0} e^{i\underline{k}\cdot\underline{r}(-\tau)} \right\rangle$$

$$= \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \left\langle e^{i\underline{k}\cdot\delta\underline{r}(-\tau)} \right\rangle$$

$$V_y = V_{E\times B}$$

$$\delta\underline{r} = -V_y \tau + \delta r_s(-\tau)$$

and as can always transform away constant $\underline{V}$,

$$\delta r = -\underbrace{V_y'}_{\text{shear}} x \tau + \delta r_0(-\tau)$$

$$V_y = V_{y_0} + x V_y' \hat{y}$$

but realize x scattered

$$\delta r \cong -V_y' \int^{\tau} d\tau' \, \delta x(-\tau') + \delta r_s(-\tau)$$

$$D_\perp = \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \Big\langle \exp\Big(-i k_y V_y' \int^{\tau} d\tau' \, \delta x(-\tau')\Big) \ast e^{+i \underline{k} \cdot \delta \underline{r}(-\tau)} \Big\rangle$$

$$= \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \Big\langle \exp\Big(-i k_y V_y' \underbrace{\int^{\tau} d\tau' \, \delta x(-\tau')}_{!}\Big) \ast e^{i \underline{k} \cdot \delta \underline{r}(-\tau)} \Big\rangle$$

N.B. shear couples $\left\{ \begin{array}{l} \text{streaming} \\ \text{scattering} \end{array} \right.$

$\Rightarrow$ enhanced decorrelation

so

$$D_\perp = \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_{\underline{k}}|^2 \exp\Big[-\frac{k_y^2}{3} V_y'^2 D_r \tau^3 \ast e^{-k_\perp^2 D_\perp \tau}\Big]$$

$$1/\tau_c^3 \equiv \frac{k_y^2 V_y' D_n}{3}$$

$$1/\tau_\perp \equiv k_r^2 D_r$$

two time history effect

$$D \equiv \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 \exp\left[-\left(\tau/\tau_c\right)^3 - \tau/\tau_\perp\right]$$

- neglected $\tilde{V}_x \tilde{V}_y$ cross correlation
- $k_r^2 D_r$
- for dominant decorrelation:

$$\left(k_y^2 V_y'^2 D_n\right)^{1/3} \text{ vs } k_r^2 D_n$$

$$k_y^2 V_y'^2 \cancel{D_r} \text{ vs } (k_r^2)^3 D_r^{\cancel{3}\,2}$$

$$\frac{k_y^2 V_y'^2}{k_r^2} > (k_r^2 D)^2$$

$1/k_r^2 \sim \Delta r^2 \sim l_{mix}^2 \rightsquigarrow$ radial scale

$$\Rightarrow \quad k_y^2 V_y'^2 \Delta_r^2 > (k_r^2 D)^2$$

$$\boxed{k_y V_y' \Delta_r > k_r^2 D}$$

shearing rate vs decorrelation rate criterion BDT '90

then

$$D \cong \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-(\tau/\tau_c)^3}$$

$$\cong \sum_k |\tilde{V}_k|^2 \tau_c$$

indep of l_{mix}, radial via time scale.

$$\cong \sum_k |\tilde{V}_k|^2 \tau_\perp \underbrace{\frac{\tau_c}{\tau_\perp}}_{<1}$$

i.e. all else same, $D/D_0 \downarrow$ by $\tau_c/\tau_\perp$ ratio!

And, at estimate level

$$D \cong \sum_{k} |\tilde{v}_{k}|^2 \frac{1}{\left(k_0^2 V'^2 D\right)^{1/3}}$$

$$D^{4/3} \cong \sum_{k} |\tilde{v}_{k}|^2 \Big/ \left(k_0^2 V'^2\right)^{1/3}$$

$$\Rightarrow \quad D \cong \left[\sum_{k} |\tilde{v}_{k}|^2 \Big/ \left(k_0^2 V'^2\right)^{1/3}\right]^{3/4}$$

- $\sim \left[V'\right]^{-1/2}$ decay
- some sensitivity low k_0 & $\sim (k_0)^{-1/2}$

$\sim \langle \tilde{v}^2 \rangle^{3/4}$, not $\langle \tilde{v}^2 \rangle^{1/2}$.

$\Rightarrow$ stronger velocity fluctuation sensitivity.

T.B.C., but next: Models!