Physics 218c

Lecture 2b : Transport Heuristics Part b : Convection

→ Recall essence of mixing length 'theory' or model for pipe flow

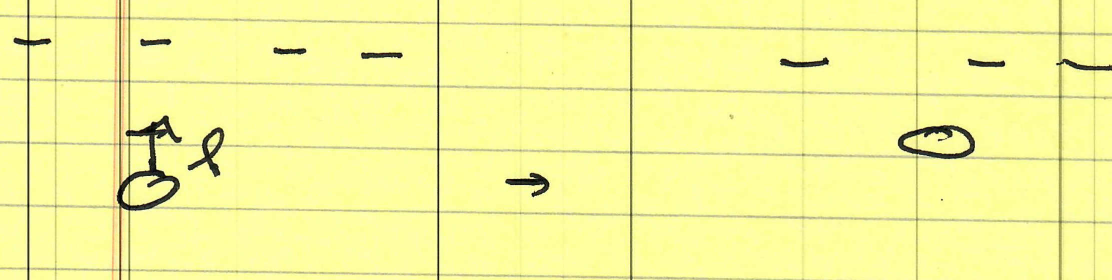
$$\underset{\substack{\downarrow \\ \text{turbulent viscosity}}}{\nu_T} \cong \overset{\substack{\text{friction velocity} \\ \uparrow}}{v_*}\, l_{mix} \simeq v_* x$$

friction velocity ↔ macroscopic distance

$\underbrace{l_{mix} \simeq x}$: mixing length ~ distance from wall

and rate v_* / l_{mix}

→ Mixing Length Theory – MLT – tied to notion of a "parcel"



lifetime $\tau \sim \dfrac{l}{v}$

↕ mean free time

→ 

Essence: what sets the mixing length?

- boundary (i.e. pipe ⇒ self-similarity on range $[x_*, a]$)
- boundary layer – convection (scale over which gradient held)
- ν, χ etc. ⇒ diffusion (ambient)
- shear ⇒ parcel lifetime σ ↑ ⟶
- nonlinear / emergent scale ⟺ staircase etc.

i.e. $$1/l_{mix}^2 = \frac{1}{l_0^2} + \frac{1}{l_{oz}^2} \Rightarrow \text{BLY '98}$$

$\Updownarrow$

emergent scales ⟺ rate balance of nonlinear with linear ↓ drive dependent

Ozmidov scale ⟶ $\frac{V(l)}{l} \sim \omega_{BV}$

- mixing length theory can be non-local c.f. Spiegel, Improved MLT (posted)

→ A little philosophy:

- mixing length theory is ancient

c.f. 50's → - Erika Bohm-Vitense (stars)
Ed Spiegel
(pre-simulation) - B.B. Kadomtsev (1965) (Plasma)

and thus is often criticized

- M.L.T. keeps coming back ...

c.f. - staircases
- water-ice interfaces (scarring)

common element: Hard, non-obvious answer
→ simple insight needed

and

- M.L.T. is a model, and so:

"All models are wrong,
some models are useful."

- George Box (statistician).

MLT frequently is useful....

⇒ "Mixing length theory always works – <u>if</u> you know the mixing length"

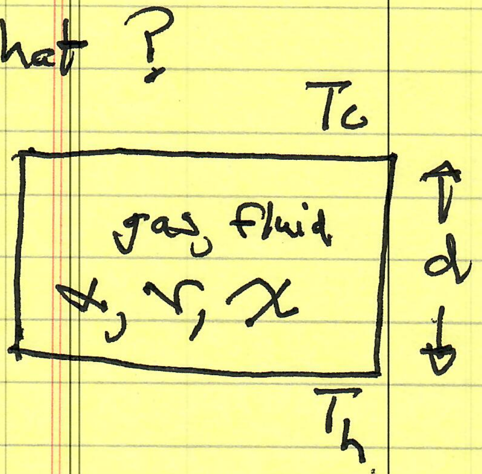
— P.D.

⇒ key physics issue in MLT is identification of critical <u>scales</u>.

⇒ in case of pipe flow, boundary + universality ⇒ scale y.

This brings us to a more interesting case: (Rayleigh–Benard) Convection

⇒ what?



$$\Delta T = (T_h - T_c) - \Delta T_{c_{rit}}$$

$$\frac{\Delta T}{d} \neq \text{local gradient}$$

Control parameter

$$Ra \equiv \frac{g \alpha \Delta T d^3}{\nu \chi}$$

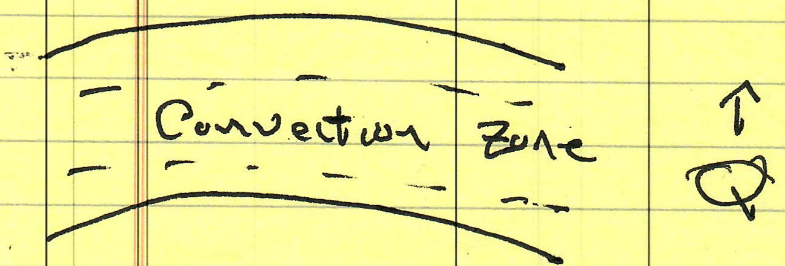
⇒ Rayleigh #

$Ra \equiv$ strength drive $\rightarrow \Delta T$

also

$P = \nu/\chi \rightarrow$ Prandtl #
(ratio dissipations)

also: <u>Sun</u>



key difference: boundary conditions

$\rightarrow$ box $\rightarrow$ no slip / free   it matters!

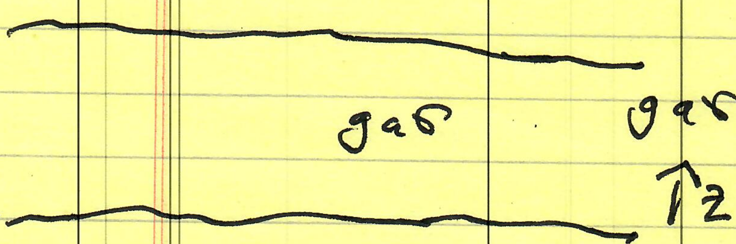
$\rightarrow$ sun $\rightarrow$?

$\rightarrow$ Why: RBC is naturally described by parcels / mixing length theory

$\rightarrow$ Plasma? : "RBC is analogous to ITG mode turbulence"

→ many similarities, and one can understand something intuitively

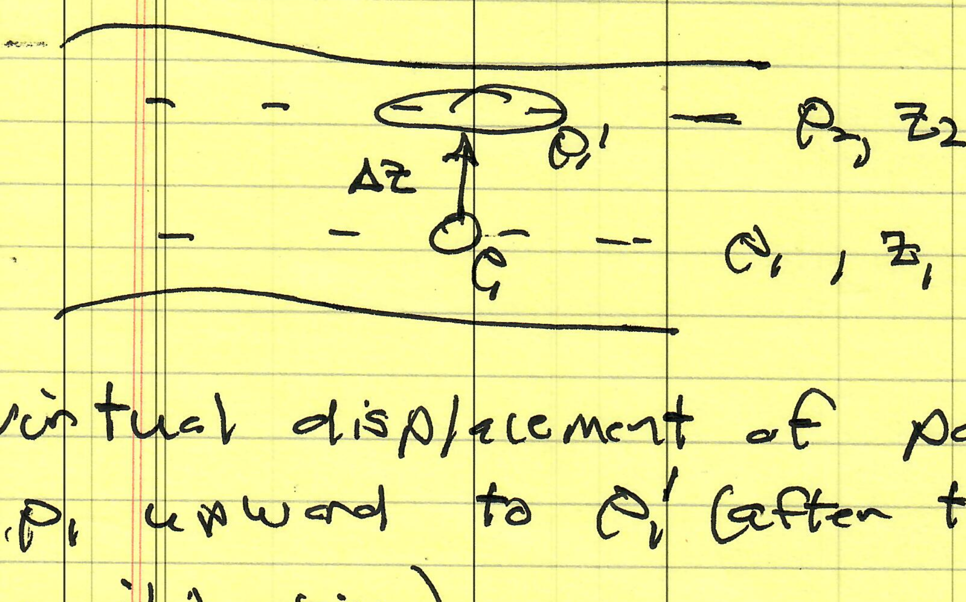
→ Basic Physics: Schwarzschild criterion (ala' parcels) (Review)



$$\frac{dp}{dz} = -\rho g \qquad \frac{\partial \rho}{\partial z} < 0 \qquad \frac{\partial p}{\partial z} < 0$$

$P\rho^{-\gamma} \simeq$ const   ideal gas

so



virtual displacement of parcel ρ_1 upward to ρ_1' (after thermodynamic equilibration) so $\rho_1' < \rho_2$ → parcel lighter than surroundings, rises → unstable

$\rho_1' > \rho_2$ → parcel heavier than surrounding, sinks → stable

Now,

$$\rho_2 = \rho_1 + \frac{\partial \rho}{\partial z} \Delta z$$

For ρ' → density of perturbed parcel ρ_1

Point:

- blob/parcel ρ_1 equilibrates pressure with surroundings ⇒ ρ_1'

- why? $\frac{\Delta z}{c_s} \ll \tau_{rise}$

i.e. rise time is long, slow ⇒ "incompressible"

then $P_1 \rho_1^{-\gamma} = P_1' \rho_1'^{-\gamma}$

but $p_1' = p_2$ $\Rightarrow$ pressure of parcel equilibrates with surroundings

$d\rho = 0$
↳ incompressible

$$p_2 = \left(p_1 + \frac{dp}{dz}\Delta z\right)$$

so

$$p_1 \rho_1^{-\gamma} = \left(p_1 + \frac{dp_1}{dz}\Delta z\right) \rho_1'^{-\gamma}$$

$$\left(\rho_1'/\rho_1\right)^{\gamma} = \left(1 + \frac{\Delta z}{p}\frac{dp_1}{dz}\right)$$

so

$$\rho_1' = \left(1 + \frac{1}{\gamma}\frac{\Delta z}{p_1}\frac{dp_1}{dz}\right)\rho_1$$

perturbed parcel

$$\rho_2 = \left(1 + \frac{1}{\rho_1}\Delta z\frac{d\rho_1}{dz}\right)\rho_1$$

surroundings

so $\rho_1' < \rho_2 \iff \frac{1}{\gamma}\frac{\Delta z}{p_1}\frac{dp_1}{dz} < \frac{\Delta z}{\rho_1}\frac{d\rho_1}{dz}$

parcel lighter than surroundings $\rightarrow$ buoyant so → Schwarzschild Criterion for Buoyancy Instability

$$\boxed{\frac{1}{\gamma}\frac{1}{P_1}\frac{dP_1}{dz} < \frac{1}{\rho_1}\frac{d\rho_1}{dz}}$$

or, as both gradients negative

$$\left|\frac{1}{\gamma}\frac{1}{P_1}\frac{dP_1}{dz}\right| > \left|\frac{1}{\rho_1}\frac{d\rho_1}{dz}\right|$$

Now entropy $S = (\text{const}) \ln\left(P\rho^{-\gamma}\right)$

[redacted] $$\frac{dS}{dz} = (\,)\left[\frac{1}{P}\frac{dP}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}\right]$$

so buoyancy instability $\Longleftrightarrow$ $\underbrace{\frac{dS}{dz}}_{\text{entropy gradient}} < 0$

(free energy available)

$$\frac{dS}{dz} \begin{matrix} < \\ = \\ > \end{matrix} 0 \quad \left.\begin{matrix}\text{superadiabatically} \\ \text{adiabatically} \\ \text{subadiabatically}\end{matrix}\right\} \text{stratified}$$

super → (ideal) unstable

adiabatic → " marginal

sub → " stable

Now $\frac{dS}{dz} < 0 \Rightarrow \frac{1}{P}\frac{dP}{dz} < \frac{\gamma}{\rho}\frac{d\rho}{dz}$

$P = k_B\, \rho T$

$$\boxed{\frac{1}{T}\frac{dT}{dz} < \frac{(\gamma-1)}{\rho}\frac{d\rho}{dz}}$$

[$\gamma - 1 \rightarrow$ eqn. of state → specifies how steep ∇T must be relative to density]

n.b.: Schwarzschild "is, in essence an η_{crit}" condition

i.e. $\frac{d\ln T}{d\ln\rho} \equiv \frac{d\ln T}{d\ln n} >$ "η_{crit}"
$\downarrow$
$\gamma - 1$

Basic Physics of ITG mode is same, with much more intricate "η_{crit}". For fluid ITG, "η_{crit}" indeed set by E-o-S.

Scales: $\tilde{s} = \left[\frac{\tilde{T}}{T} - (\gamma-1)\frac{\tilde{\rho}}{\rho_0}\right]$

$$\delta p = 0 \Rightarrow \frac{\tilde{\rho}}{\rho_0} = \frac{-\tilde{T}}{T_0}$$

so $\tilde{s} = \gamma \frac{\tilde{T}}{T_0}$ ⇒ entropy perturbation tied to temperature perturbation alone.

$$\text{so} \quad \gamma \frac{\partial \tilde{T}/T_0}{\partial t} = -\tilde{v}_z \frac{ds_0}{dz} = -\tilde{v}_z\left(\frac{1}{T_0}\frac{dT_0}{dz} - \frac{1}{T_0}\frac{dT_{ad}}{dz}\right)$$

$$\partial_t \tilde{v}_z = \underset{\delta p \to 0}{-\cancel{\frac{\partial_z \tilde{p}}{\rho_0}}} \quad g \underset{-\tilde{T}/T_0}{\underset{\downarrow}{\frac{\tilde{\rho}}{\rho_0}}} \tilde{z}$$

Thus time scale for buoyancy instability:

$$\boxed{1/\tau_b^2 \sim \frac{g}{\gamma}\frac{ds_0}{dz}}$$

buoyancy time scale

N.B. (Almost) no equations!

Now for parcel of scale l:

$1/\tau_\nu \cong \nu/\ell^2$ ⇒ time scale to diffuse parcel momentum to surroundings.

viscous dissipation rate

$1/\tau_\chi \cong \chi/\ell^2$ ⇒ time scale for heat conduction from parcel to surroundings ⇒ parcel cools, looses buoyancy

thermal diffusion rate

$(\tau_\chi \tau_\nu)^{-1} > 1/\tau_b^2$ ⇒ diffusion will smear out parcel heat

⇒ need

$$\frac{\tau_\nu \tau_\chi}{\tau_b^2} \sim \frac{g}{\gamma} \frac{\partial \langle s \rangle}{\partial z} \ell^4 / \nu\chi > \# \underset{O(1)}{\sim} \Rightarrow \text{b.c.'s}$$

⇒

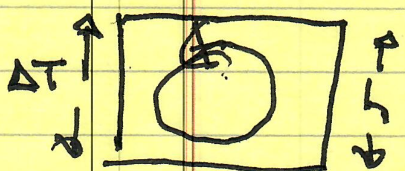
$$\boxed{Ra = \frac{g}{\gamma} \frac{\partial s}{\partial z} \ell^4 / \nu\chi}$$

→ Rayleigh #

i.e. need sufficient free energy to overcome dissipation

$\Leftrightarrow$ $Ra > Ra_{crit}$

→ For fluid in box:



$\delta\rho = -\alpha\,\delta T$

↓

coeff termal expansion

$$\boxed{Ra = g\,\Delta T\,\alpha\,h^3/\nu\chi}$$

$Ra > Ra_{crit}$

$\Rightarrow \dfrac{\Delta T}{d} > \left.\dfrac{\Delta T}{d}\right|_{crit}$

- see
  - notes
  - Chandrasekhar

  } detailed linear theory.

- in astro situation $P_r \ll 1$ (radiative conduction).

  ν somewhat feeble.

→ Transport

At high Ra, interested in flux of heat and momentum.

⇒ $\langle \tilde{V}_z \tilde{\Theta} \rangle$ / $\langle \tilde{V}_z \tilde{T} \rangle$ $\quad\Theta$ → potential temperature

$\langle \underline{V}\ \underline{V} \rangle$ $\quad$ Reynolds stress

How formulate theory ? $\quad$ Mixing Length (cf Gough, Spiegel)

→ n.b. here $\tilde{V}_z$ is spontaneous, due instability, not driven, a la' pipe flow

→ Compute heat flux.

$$\partial_t \frac{\tilde{T}}{T_0} + \tilde{\underline{V}} \cdot \underline{\nabla} \frac{\tilde{T}}{T_0} = - \tilde{V}_z \frac{d}{dz}\left(\langle T \rangle - T_{ad}\right)$$

high Ra. $\qquad \tilde{\underline{V}} \cdot \underline{\nabla} \sim \frac{\tilde{\underline{V}}}{\ell_{mix}}$

$$\Rightarrow \frac{\tilde{T}}{T_0} \simeq - \underset{\substack{\uparrow \\ \text{displ.} \\ \text{fluid particle}}}{\xi} \frac{d}{dz} \frac{\left(\langle T \rangle - T_{ad}\right)}{T_0}$$

then $\underset{\substack{\downarrow \\ \text{displacement}}}{\xi} \sim \underset{\substack{\searrow \\ \text{mixing length}}}{l_{mix}}$

$$\frac{\tilde{T}}{T_0} \simeq - l_{mix}$$

and for $\underline{v}$

$$\rho(\partial_t \tilde{\underline{v}} + \widetilde{\underline{u} \cdot \nabla \underline{v}}) = -\cancel{\nabla \tilde{P}} - g \tilde{\rho}$$

$$\frac{d}{dz}(v_z^2) = + g \frac{\tilde{T}}{T_0} \qquad \text{steady state}$$

$\Rightarrow$

$$v_z^2 \simeq l_{mix}\, g \frac{\tilde{T}}{T_0}$$

$$\simeq l_{mix}^2\, g \left| \frac{d}{dz}\left(\frac{\langle T \rangle - T_{ad}}{T_0}\right)\right|$$

$$\underbrace{\tilde{v}_z}_{\text{mixing velocity}} \cong l_{mix} \left[g \left| \frac{d}{dz}\left(\frac{\langle T \rangle - T_{ad}}{T_0}\right)\right| \right]^{1/2}$$

Then

$$Q = \langle \tilde{V}_z \tilde{T} \rangle$$

$$\cong - l_{mix}^2 \left[g \left| \frac{d}{dz} \left(\frac{\langle T \rangle - T_{ad}}{T_0} \right) \right| \right]^{1/2} \frac{d}{dz} \left(\langle T \rangle - T_{ad} \right)$$

$$\cong - \underbrace{\chi_T}_{\text{turbulent heat diffusivity}} \frac{d}{dz} \left(\langle T \rangle - T_{ad} \right)$$

$$\chi_T \cong l_{mix}^2 \left[g \left| \frac{d}{dz} \left(\frac{\langle T \rangle - T_{ad}}{T_0} \right) \right| \right]^{1/2}$$

$$\tilde{V} \cdot \nabla \tilde{T} \rightarrow -\chi_T \nabla^2 \sim \frac{\chi_T}{\ell^2}$$

χ_T as needed
ν_T for marginality

One could also compute Reynolds stress:

$$\langle \tilde{V}^2 \rangle \cong l_{mix}^2 \, g \left| \frac{d}{dz} \left(\frac{\langle T \rangle - T_{ad}}{T_0} \right) \right|$$

$$\underline{V} \cdot \underline{\nabla} \, \underline{V} \rightarrow \nu_T \nabla^2 \sim \frac{\nu_T}{\ell^2}$$

N.B.: $\rightarrow$ $\chi_T / \langle \tilde{V}^2 \rangle$ ratio indep. of l_{mix}^2

$\rightarrow$ ratios more suitable for theory work than absolutes

→ Now, what is the mixing length?

– Sun, $l_{mix} \sim H_p$

Pressure scale ht

long

$$\frac{dP}{dz} = -\rho g$$

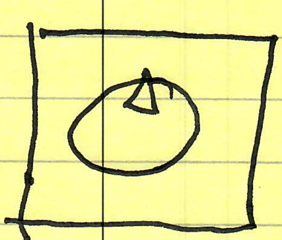
$$\frac{P}{H_p} \sim \rho g \qquad \frac{\cancel{\rho}\, c_s^2}{H_p} \sim \cancel{\rho}\, g$$

$$H_p \sim c_s^2/g.$$

⇒ integral form / non-local conductivity

see Spiegel, Improved MLT

→ Box

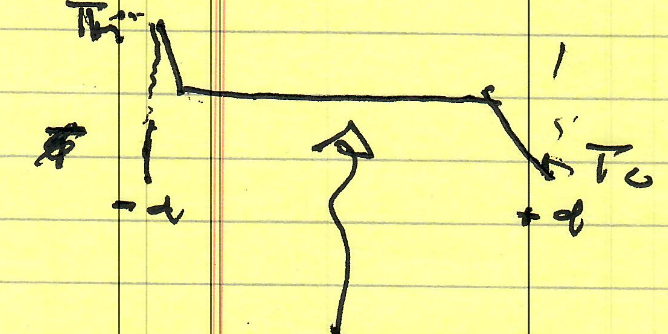


d ΔT $\frac{\Delta T}{d} \neq$ local gradient

Experiments (Malkus, 1954 et. seq

Simulations Sreeni. 2020)

a)



core $\nabla T \sim \nabla T_{ad}$.

all ∇T "held" in 2 BL's

at top, bottom.

$\Rightarrow$ Suggests a "thermal short circuit"

in core of flow. All transport

controlled by B.L.

Now, how is score kept?

Scalings: $Nu\,(Ra, Pr)$

More generally: $Nu\,(Ra, Pr)$

$Re\,(Ra, Pr)$

$Nu \rightarrow$ Nusselt #

$$Nu = \frac{\text{Flux}}{\text{Conducted Flux}}$$

$$= \frac{\langle \tilde{V}_z \tilde{T} \rangle - \chi \Delta T/d}{\chi \Delta T/d}$$

$$\cong \frac{\langle \tilde{V}_z \tilde{T} \rangle}{\chi \Delta T/d} \quad \left\{ \begin{array}{l} \text{for} \\ Ra \gg Ra_{crit} \end{array} \right.$$

$$Nu \sim \frac{\chi_{Turb}}{\chi_{neo}}$$ for ITG.
(loose analogy)

$Re \sim VL/\nu$ for large scale flow in box.

$Nu\,(Ra, Pr)$
$Re\,(Ra, Pr)$ → key scalings.

$Re\,(Ra, Pr) \leftrightarrow$ ZF.

→ Heat transport scalings:

Classic

⇒ transport } held to BL
Gradient }

∴ no gain in confinement from larger box size ($\sim$ Bohm)

⇒

$$\langle \tilde{V}_z \tilde{T} \rangle \sim \frac{\chi \Delta T}{d} Nu(Ra)$$

$$\sim \frac{\chi \Delta T}{d} \Big(\underbrace{\frac{d^3 \Delta T g \alpha}{\chi \nu}}_{Ra} \Big)^{\gamma}$$

⇒ $\gamma \sim 1/3$

$$\boxed{Nu \sim Ra^{1/3}}$$

⇒ $Nu \sim Ra^{1/3}$ is de-facto counterpart of Bohm scaling

i.e. "bigger is not better".

→ Kraichnan, Spiegel 1962.

→ simulations 2020 (Sreeni) support

→ for boundary layer thickness,

c.f. Malkus 1954, 1956

⇒ marginal stability arguments.

"Ultimate" Regime (Spiegel '63)

$$Nu \sim (Ra)^{1/2} \Rightarrow \text{ultimate scaling}$$

Now,
$$Nu \sim \frac{\chi_T}{\chi}$$

$$\sim \frac{l_{mix}^2 \left[g \frac{d}{dz} \frac{\langle T \rangle}{T_0} \right]^{1/2}}{\chi}$$

$l_{mix}^2 \sim d^2$

$l_{mix} \sim r^{(0)}$

"ideal" scaling at high Ra, l_{mix} index viscosity

$$\therefore Nu \sim \left[Ra \right]^{1/2}$$

→ So far, "Ultimate" regime not observed

→ $Nu \sim Ra^{1/3}$ scaling robust → simulation and experiment.

→ For more:

- Spiegel papers
- Siggia Review
- Lohse, Grossman
- Howard, Krishnamurti

⋮

N.B. Rather little on flux-driven convective turbulence...

Chapman, Proctor '1982,

Key: $Re = Re(Q, r)$