Physics 218c

Lecture II a. – Transport Heuristics

a) → Transport → Mixing → Profile

- Pipe Flow
- Stellar Convection

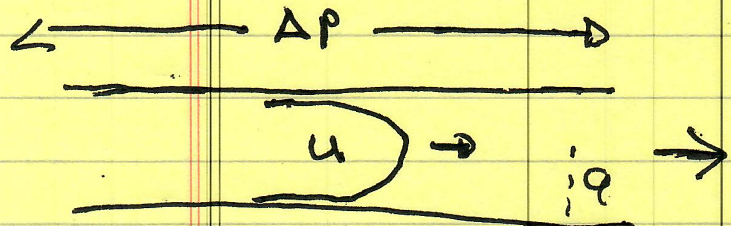
b) → Scalings – MFE

- Flux, D
- ρ_* – the critical ratio
- D → Gyro-Bohm, Bohm and between
- Shearing Effects

→ Transport ↔ Mixing ⇒ Profile

a.) Pipe Flow – Turbulent (Navier-Stokes) (cf. Landau & Lifshitz)

→ inhomogeneous, bounded system



$\frac{\Delta P}{\ell}$ → drive, pressure drop per length.

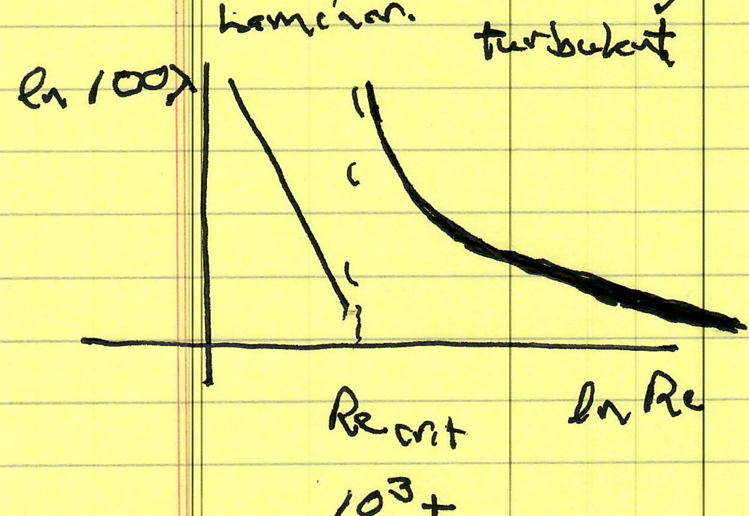
no slip
$V_t = 0$, wall

$$\left(\partial_t \underline{v} + \underline{v}\cdot\nabla\underline{v} = -\frac{\nabla P}{\rho} + \nu\nabla^2\underline{v} \right)$$

keeping score:

$$\lambda = 2a\,\Delta P/\ell \Big/ \tfrac{1}{2}\rho U^2$$ → resistance factor

(akin $\tau_E \sim W/P_{in}$) (shear-driven)



$$Re = \frac{2Ua}{\nu}$$

better $1/\lambda$ vs Re
or KE vs. ΔP

more rapidly than in turbulent flow.

Figure 32 shows a logarithmic graph of λ as a function of R. The steep straight line corresponds to laminar flow (formula (43.6)), and the less steep curve (which is almost a straight line also) to turbulent flow. The transition from the first line to the second occurs, as the Reynolds number increases, at the point where the flow becomes turbulent; this may occur for various Reynolds numbers, depending on the actual conditions (the intensity of the perturbations). The resistance coefficient increases abruptly at the transition point.

$$\lambda \sim \frac{\Delta p}{\rho u^2/2}$$

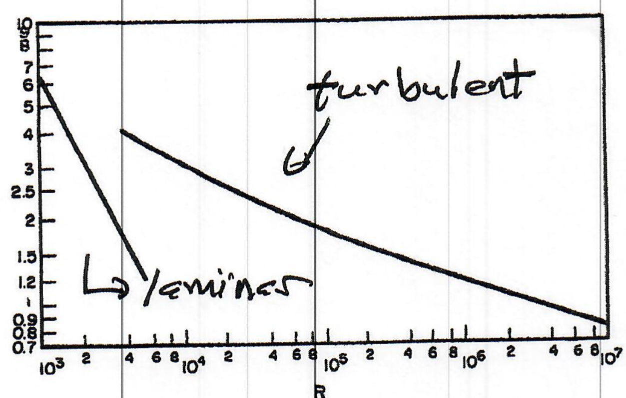


FIG. 32 Re

→ turbulent resistance curve ⇒

momentum confinement scaling.

→ also → ⓐ universal boundary layer structure

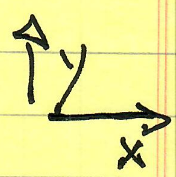
core → flat/plug

→ log law ↔ empirical

Prandtl '25, '32.

"universal" ⇒ rescale different flows of diff'nt size, etc, Re (so long as turbulent)

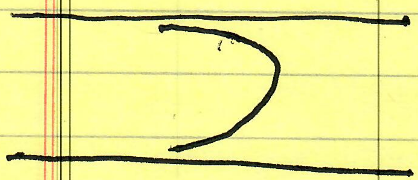
→ same shape $U(y)$ profile



↔ counterpart of K41, etc.

⇒ counterpart of "profile consistency", "resiliency", "stiffness" etc.

What is going on here?



no slip b.c.

← $\Delta p/\ell$ →

- drag on flow → λ → due momentum flux to wall
- turbulent transport/mixing of momentum

⇒

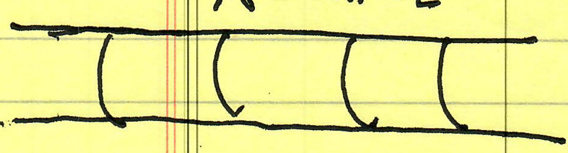
- wall stress must balance pressure drop.

so

$$\tau_{wall} = \rho u_*^2$$

$u_* \equiv$ typical turbulent velocity

$A = 2\pi a L$



← ℓ →

Δp

Force on Wall ~ $(L \sim \ell)$

$\rho u_*^2 A_{wall}$

$\sim \rho u_*^2 2\pi a \ell$

and Force on Fluid (per ℓ) $\sim$ (Pressure drop)(X A)
$\sim \Delta p \, \pi a^2$

$\Rightarrow$ balance $\Rightarrow$

$$\rho u_*^2 (2\pi a \ell) = \Delta p (\pi a^2)$$

$$\boxed{u_* = \left(\Delta p / 2\rho\right)^{1/2} \left(\frac{a}{\ell}\right)^{1/2}}$$

~ "friction velocity" — characteristic velocity

~ "typical velocity" of turbulence in (turbulent) pipe flow

N.B. — viscous vs turbulent stress? → tbd

— u_* ~ isotropic $\langle v_y v_x \rangle \to u_*^2$

Reynolds stress.

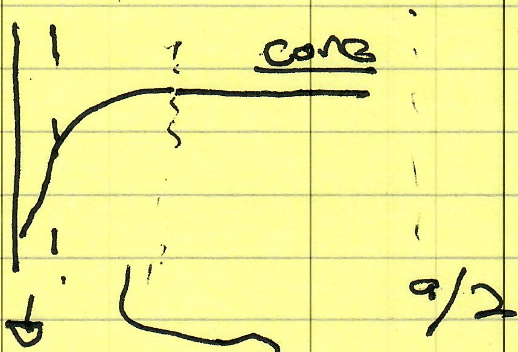
— Laminar
$$-\nu \nabla^2 V_x = -\frac{\partial_x P}{\rho}$$
— Poiseuille
— parabolic.

→ THE Question

- What is the profile $u(y)$?

~ practical question — chin MFE

- Dimensional Reasoning



Pipe flow profile: (obvious analogy with K41)

l_{vs} viscous sublayer — inertial sub-layer > - universal

Can note:

Length scales: a → radius

$$l_{vs} \sim \frac{\nu}{u_*}$$ → viscous sublayer scale

in small scale region, close to wall,

$$\Pi_{y,x} = \cancel{\rho}\, u_*^2 = -\nu \frac{\partial u}{\partial y} \cancel{\rho}$$

viscous transport

$$u(y) = \frac{\cancel{\rho}\, u_*^2}{\nu} y \quad \rightarrow \begin{cases} \text{linear profile} \\ \text{viscous sublayer} \end{cases}$$

$$= \frac{u_*^2}{\nu} y$$

$y \equiv$ distance from wall.

But what of inertial sublayer?

$l_{vs} < y < a$

Only parameters: $\underbrace{u_*, \rho}_{\tau_w}$, $\underbrace{y}_{\text{distance from wall}}$

Key: inertial sublayer is scale invariant!

So, if seek velocity gradient

$$\frac{\partial u}{\partial y} = ?$$

Only possibilities are u_*

$y \rightarrow$ distance from wall

$$\frac{\partial u}{\partial y} = \frac{u_*}{y}$$

$$\boxed{\text{Log Profile}}$$

$\Rightarrow u = u_* \ln(y)$

$\rightarrow = u_* \ln(y/l_v) + U_{b*}$
(measured from $y=0$)

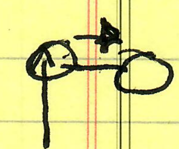
to match to viscous sublayer at l_v.
Constant K (Von Karman) enters.

- Heuristic Reasoning

Consider turbulent mixing as a momentum transport process, akin to kinetic theory of gases.

Flux driven transport !

$$\tau_w = \rho u_*^2 = \rho \langle \tilde{v}_y \tilde{v}_x \rangle$$



transport via distribution of slugs/parcels of momentum.

$\rightarrow$ conserved locally.

so $$\tilde{v}_x(y) = U_x(y-\ell) - U_x(y)$$

parcel scattered

$$\cong$$ [redacted] $$-\ell \frac{\partial U_x}{\partial y}$$

$\tilde{v}_y \sim u_*$

so $$\tau_w = -\rho \langle u_* \ell \rangle \frac{\partial U_x}{\partial y}$$

momentum diffusivity "eddy/turbulent viscosity"

What is ℓ → mixing length (analogous l_{mfp})

<u>Scale invariance</u> ⇒

mixing length restricted only by distance to nearest boundary (i.e. no scale)

So → mixing length restricted only by distance to (nearest) wall

∴ $\ell \sim y$

$$\tau_w \cong -\rho u_* y \frac{\partial U_x}{\partial y}$$

and $$-\rho u_* y \frac{\partial U_x}{\partial y} = \rho u_*^2$$

$$U_x = u_* \ln y + C$$ (measured from $y=0$)

(with const)

$$\boxed{U_x \to \frac{u_*}{K} \ln(y/\ell_{vs})}$$

inertial Layer profile → "Law of the Wall"

<u>Profile</u> → turbulent pipe flow

⇒ Welcome to Prandtl → the "mother" of all mixing length theory.

Some comments:

→ as in K41, clear phenomenology critical to model.

- data!

→ many ongoing:

- studies of transport, turbulence physics:

streaks, vortices, staircases, anomalous scaling, staircase ....

but

→ Log law works pretty well!!
(can fit λ vs Re plot)

→ why single u_*? → value of turbulent velocity?

$\tilde{v}$ from mixing of mean gradient

$$\tilde{v} \sim l \frac{\partial u}{\partial y} \sim l \frac{u_*}{y} \sim y \frac{u_*}{y}$$

→ Scale separation ?

– Better Question !

ie for diffusive model, should have

$l_{mix} / L_{macro} < 1$ → i.e. validity Chapman Enskog.

here $l_{mix} \sim y$

$L_{macro}^{-1} \sim \frac{1}{u} \frac{\partial u}{\partial y}$

$\sim \frac{1}{u_*(\ln y)} \frac{u_*}{y}$

so $l_{mix}/L_{macro} \sim 1/\ln y \lesssim 1$

→ Marginal ..

– note y/y scaling

but

– it works! ..

c.f. Spiegel '63 (posted) for more.

→ What of turbulent dissipation ?

Consider N-S Eqn:

$$\partial_t \underline{v} + \underline{\tilde{v}}\cdot\underline{\nabla}\,\underline{\tilde{v}} + \langle V_x\rangle \partial_x \underline{\tilde{v}} + \tilde{v}_y\, \partial_y \langle V_x\rangle = -\underline{\nabla} p + \nu \nabla^2 \underline{v}$$

$\underline{\tilde{v}}\cdot$ and avg:

$$\partial_t \langle \tilde{v}^2\rangle + \langle \overbrace{\cancel{\underline{\tilde{v}}\cdot\underline{\nabla}\frac{\underline{v}^2}{2}}}^{\underline{\nabla}\cdot\underline{v}=0} \rangle + \langle V_x\rangle \langle \overbrace{\cancel{\underline{\tilde{v}}\cdot\partial_x \underline{\tilde{v}}}}^{\text{odd}} \rangle$$

$$+ \langle V_y V_x\rangle \partial_y \langle V_x\rangle = -\langle \underbrace{\cancel{\underline{v}\cdot\underline{\nabla} p}}_{\text{cbp}} \rangle - \nu \langle (\underline{\nabla}\underline{\tilde{v}})^2\rangle$$

so

$$\partial_t \mathcal{E} = \underbrace{-\nu\langle(\underline{\nabla}\tilde{v})^2\rangle}_{\text{viscous dissipation}} \underbrace{- \langle \tilde{v}_y \tilde{v}_x\rangle \frac{\partial\langle V_x\rangle}{\partial y}}_{\text{Reynolds work}}$$

Reynolds work (input of energy to fluctns from mean flow)

obviously > 0.

define:

$$\epsilon = \langle \tilde{v}_y v_x \rangle \frac{\partial U_x}{\partial y}$$

$\Downarrow$

turbulent dissipation rate

Can use MLT:

$$\langle \tilde{v}_y \tilde{v}_x \rangle = u_* x \frac{\partial u}{\partial y}$$

$\therefore$

$$\epsilon = (u_* x) \frac{\partial u}{\partial y} = \nu_T \left(\frac{\partial u}{\partial y} \right)^2$$

$\Downarrow$

"rate of heating" of fluctns by turb. viscous relaxation

ultimately:

$$\int \nu \langle (\nabla \tilde{v})^2 \rangle = \int \nu_T \left(\frac{\partial U_x}{\partial y} \right)^2$$

[in steady state.

so

$$\epsilon = (u_* y)\left(\frac{u_*}{y}\right)^2$$

$$= u_*^3 / y \quad \checkmark \qquad \Delta$$

[dissipation rate largest near wall (∇u_x)]

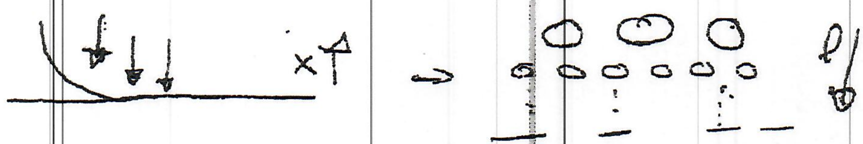
ϵ finite as $\nu \to 0$

Proof ? – Pipe counterpart of 4/5 Law ?!

N.B. Mixing Length Theory, while not rigorous, is

- useful → especially in complicated problems.
  ↓
  staircases, flow over ice
- non-trivial

c.f. Spiegel '63 (p-stat)



→ Now, interesting to tabulate comparison between Pipe Flow and K41 Problem

Pipe Flow (Prandtl)	K41 (Kolmogorov)
scales: $a, x, \nu/u_*$ invariance: $x \to$ real space	ℓ_0, ℓ_n, ℓ_d $\ell \to$ scale space
inertial sublayer viscous sublayer	inertial range dissipation range
lance: $u_*^2 = \nu_T \frac{\partial u}{\partial x}$	$\epsilon = \frac{v(\ell)^2}{\tau(\ell)}$
namics: eddy viscosity $\nu_T = u_* x$	turn-over rate $1/\tau(\ell) = \frac{v(\ell)}{\ell}$
ult: $u = \frac{u_*}{\kappa} \ln(x)$	$v(\ell) = \epsilon^{1/3} \ell^{1/3}$
universal profile	universal spectral scaling
sipation: $\nu = \nu_T$ $x_0 = \nu/u_*$	$v(\ell)/\ell = \nu/\ell^2$ $\ell_d = \nu^{3/4}/\epsilon^{1/4}$.