Physics 218C: Lecture notes (5-6)

Dynamical Models

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1. Overview

In plasma physics, there are a zillion models. Since most of them are just a few key models with variations (so that people can put their names on the covers), we'd better first know the key models well. As Pat said, once we understand a few cats, we understand them all, even including "big cats" like lions, tigers, leopards, and cheetahs. Normally, we have two approaches to study these key models, Top→Down and Bottom→Up. Both approaches have advantages and disadvantages. A brief description for these two approaches and their comparison are listed in table 1.

Approach	Hierarchy	Advantage	Disadvantage
Top→Down	Vlasov-Boltzmann → Gyro/Drift Kinetics → Fluids/Gyro Fluid → Reduced Fluid (i.e. Hasegawa-Mima)	More systematic	Laborious
Bottom→Up	Time/space scale → Reduction Principle/Idea → Reduced Model (Hasegawa-Mima) → Less reduced Model (4 fields, 6 fields)	More insight	Less systematic

Table 1 "Top→Down" and "Bottom→Up" approach

Historically, both top→down (gyro kinetics: Frieman, Rutherford' late 60s; Chen, Frieman' early 80s; MHD: Rosenbluth, Kadomtsev-Pogutse' 60s; et al.) and bottom→up (Hasegawa, Kadomtsev, Sagdeev, et al.) evolved simultaneously. So a synergism of both approaches is useful. Lecture 5 and lecture 6 focus on simple drift wave models from a bottom-up perspective.

In these two lectures, the basic setup is:

- A strong $\boldsymbol{B_0}$, which means a strongly magnetized plasma. Because of this strong field, the turbulence is anisotropic ($l_\parallel \gg l_\perp$ or $k_\parallel \ll k_\perp$).
- Time scales are larger than ion-cyclotron, i.e., $\omega < \Omega_i$ (ion cyclotron frequency).
- Quasi-neutrality is always guaranteed ($l > \lambda_D$).
- Consider electrostatic case first, in which $\boldsymbol{v}_\perp = c\boldsymbol{E} \times \hat{\boldsymbol{z}}/B$.

2. An introduction to Potential Vorticity

Traditionally, we can derive Hasegawa-Mima equation from ion continuity equation (with Boltzmann electron). But now we seek a new, more insightful way to derive it by using the conservation of **potential vorticity** (**PV**).

2.1. A simple definition of PV and motivation

We plan to derive Hasegawa-Mima equation from the point of view of Potential vorticity conservation. Then one might ask what PV is? A quick answer is that PV is a conserved "effective charge" density or a generalized vorticity. Here the charge means total charge, which contains both guiding center charge and polarization charge.

But why do we care about PV? There are several reasons. First of all, PV is conserved so that it is easy to keep track of its exchange between mean field and fluctuations. In addition, unlike phase space density f, another conserved quantity, PV is a macroscopic quantity, which makes it useful. What's more, one of the key points of Hasegawa-Mima's paper is that drift wave turbulence is in some sense like geophysical fluid turbulence and geophysical fluid dynamics (GFD) makes heavy use of PV. So for our purpose, it is essential to study PV.

2.2. The expression for PV

We will start from fluid at first, then we will generalize this idea to plasma. There is a common point in both cases: the Rossby number is small, i.e., $Ro \ll 1$. For fluid, the Rossby number is defined as $Ro \sim v_\perp / L_\perp \Omega$, where Ω is the rotation rate of the fluid. For plasma, we just need to replace Ω by Ω_i, which is the gyro frequency of ions. Since Ω or Ω_i is very large, the vorticity is along the rotation axis (in plasma, it is in the main field direction) so that the flow is quasi-two dimensional.

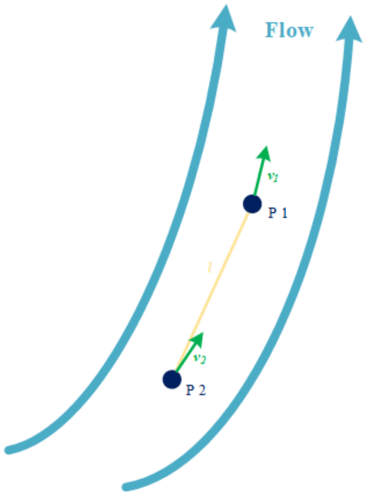

Fig 1 Two points in a flow

Now consider two point frozen into flow, as show in Fig 1. The displacement between these two points is $\boldsymbol{l}$ and their velocities are the velocities of flow at their locations. Obviously, $\boldsymbol{l}$ can vary with time, and its evolution can be described by

$$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{v}(\boldsymbol{x_2}) - \boldsymbol{v}(\boldsymbol{x_1}), \qquad (2-1)$$

or in a compact form

$$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{l} \cdot \nabla \boldsymbol{v} \qquad (2-2)$$

Eq (2-2) may remind us the local form of the Alfvén's theorem

$$\frac{d}{dt}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right) \cdot \nabla \boldsymbol{v}, \qquad (2-3)$$

which says for ideal plasma, magnetic flux is frozen into the field.
A straightforward derivation can prove that vorticity is also frozen into fluid/plasma. For flow, the equation of motion is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{\nabla p}{\rho} - 2\boldsymbol{\Omega}\times\boldsymbol{v}, \qquad (2-4)$$

where the last term on the RHS represents Coriolis force or Lorentz force. Because the definition of vorticity is $\nabla\times\boldsymbol{v}$, we can get the vorticity equation by simply taking the curl of Eq (2-4). N.B. $\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla(v^2/2) - \boldsymbol{v}\times\boldsymbol{\omega}$. When p is a function of ρ only, the vorticity equation is

$$\partial_t(\boldsymbol{\omega}+\mathbf{2\Omega}) = \nabla\times\big(\boldsymbol{v}\times(\boldsymbol{\omega}+2\boldsymbol{\Omega})\big). \qquad (2-5)$$

Combining Eq (2-5) with the continuity equation, we have

$$\frac{d}{dt}\frac{(\boldsymbol{\omega}+2\boldsymbol{\Omega})}{\rho} = \frac{(\boldsymbol{\omega}+\mathbf{2\Omega})}{\rho}\cdot\nabla\boldsymbol{v}. \qquad (2-6)$$

Again, the form of Eq (2-6) implies that $(\boldsymbol{\omega}+\mathbf{2\Omega})/\rho$ is frozen-in.
By the way, since $|\boldsymbol{\Omega}|$ is static and for when it is much larger than other frequencies in this problem, if we only keep the 0th order of Eq (2-6), it implies $\boldsymbol{\Omega}\cdot\nabla\boldsymbol{v}\approx 0$. This is a famous theorem in fluid dynamics called the **Taylor-Proudman Theorem**, i.e., to 0th order, flow is uniform along the direction of rotation axis. Even though there are corrections, the plasma is two-dimensional. Since we are considering a case with small Rossby number, there is no surprise to see this result.
Now consider a passive scalar ψ, which is conserved along trajectory ($d\psi/dt = 0$). Similarly, at two different points in the flow, $\delta\psi = \psi(\boldsymbol{x_2}) - \psi(\boldsymbol{x_1}) = \nabla\psi\cdot d(\boldsymbol{x_2} - \boldsymbol{x_1}) = \nabla\psi\cdot d\boldsymbol{l}$. Because ψ is invariant along trajectory, $\delta\psi = 0$, which further implies

$$\frac{d}{dt}(\nabla\psi\cdot d\boldsymbol{l}) = 0. \qquad (2-7)$$

Since $d\boldsymbol{l}$ and $(\boldsymbol{\omega}+\mathbf{2\Omega})/\rho$ satisfy the same equation (as shown in Eq (2-2) and Eq (2-6)), we can replace $d\boldsymbol{l}$ in Eq (2-7) by $(\boldsymbol{\omega}+\mathbf{2\Omega})/\rho$ and rewrite Eq (2-7) as

$$\frac{d}{dt}\left[\frac{(\boldsymbol{\omega}+\mathbf{2\Omega})\cdot\nabla\psi}{\rho}\right] = 0. \qquad (2-8)$$

Through Eq (2-8), we construct a conserved quantity called potential vorticity, and we always use letter q (sometimes ρ_q) to denote it. PV can be interpreted as "effective

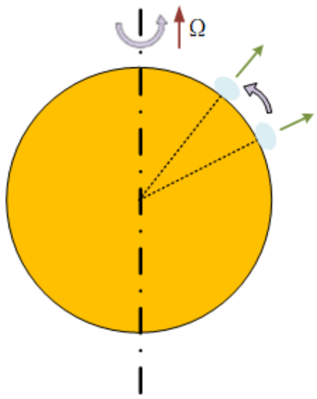

Fig 2 Displacement of a fluid element in latitude

charge" density.

2.3. Utility of PV

PV is a useful quantity. For example, as shown in Fig 2, if $\nabla\psi = \hat{\boldsymbol{r}}$, and we displace a fluid element in latitude, since $\boldsymbol{\Omega}\cdot\hat{\boldsymbol{r}}$ changes, $\boldsymbol{\omega}\cdot\hat{\boldsymbol{r}}$ must change, too. We can predict the flow change without detailed calculation.

2.4. Symmetry behind PV conservation

When there is a conserved quantity, there must be a symmetry behind it. This is what Noether's theorem tells us. So what is the corresponding symmetry of conserved PV? The answer is particle re-labeling symmetry. If we label each of particles at position s and time t in the flow and shift the position by δs, the thermodynamic state is invariant under this transformation, which explains why PV is conserved. For a one-component fluid whose thermodynamic state is determined by two scalar variables, specific volume v and specific entropy η, whereas the label represents a three-dimensional manifold, there exist one-dimensional relabeling transformations leading to the conservation of a scalar.

2.5. From Kelvin's theorem to Charney equation

Recalling Eq (2-5) and Alfvén's theorem

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{2-9}$$

we can conclude that the flux of total vorticity through any given moving plasma element does not change. Mathematically, we have the following equation

$$\frac{d}{dt}\left[\int d\boldsymbol{S}\cdot(\boldsymbol{\omega}+2\boldsymbol{\Omega})\right] = 0. \tag{2-10}$$

Specially, if $\boldsymbol{\Omega} = 0$, we have

$$\frac{d}{dt}\left[\int d\boldsymbol{S}\cdot\boldsymbol{\omega}\right] = \frac{d}{dt}\oint d\boldsymbol{l}\cdot\boldsymbol{v} = 0. \tag{2-11}$$

Eq (2-10) is called **Kelvin's theorem**: the total circulation, including a parcel component and a planetary component, is conserved.

By the way, when the system is non-isentropic, i.e., p is not a one-variable function of ρ, Eq (2-5) no longer holds, and Eq (2-10) fails as well. For this case, the evolution of vorticity is

$$\frac{\partial}{\partial t}(\boldsymbol{\omega}+2\boldsymbol{\Omega}) = \nabla\times\big(\boldsymbol{v}\times(\boldsymbol{\omega}+2\boldsymbol{\Omega})\big) + \frac{1}{\rho_s^2}\nabla\rho\times\nabla \mathrm{p} \tag{2-12}$$

Multiplying Eq (2-12) by ∇s, where $s(p,\rho)$ is entropy, we obtain

$$\begin{aligned}\nabla s\cdot\frac{\partial}{\partial t}\omega &= \nabla s\cdot[\nabla\times(\boldsymbol{v}\times\boldsymbol{\omega})] = -\nabla\cdot[\nabla s\times(\boldsymbol{v}\times\boldsymbol{\omega})]\\ &= -(\boldsymbol{\omega}\cdot\nabla s)\nabla\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\nabla(\boldsymbol{\omega}\cdot\boldsymbol{s}) + \boldsymbol{\omega}\cdot\nabla(\boldsymbol{v}\cdot\nabla s)\end{aligned} \tag{2-13}$$

In addition to Eq (2-13), we also have

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}s = 0 \tag{2-14}$$

Combining Eq (2-13), (2-14) and continuity equation, instead of Kelvin's theorem, what we get now is **Ertel's theorem**:

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}\cdot\nabla s}{\rho}\right)=0 \qquad (2-15)$$

According to Eq (2-8), the definition of PV, we can define $\psi = s$ and $\mathrm{PV} = (\boldsymbol{\omega}\cdot\nabla s)/\rho$. So PV is still conserved for non-isentropic ideal fluid, except a fact that the definition of PV is different from the definition in Kelvin's theorem.

Now go back to the equation of motion Eq (2-4). Because $Ro \ll 1$, which implies $d\boldsymbol{v}/dt \ll \boldsymbol{\Omega}\times\boldsymbol{v}$, we can neglect the tern on the LHS and balance the two terms on the RHS (so-called Geostrophic balance). Therefore, the solution to this equation is

$$\boldsymbol{v}\approx-\frac{\nabla_{\perp}p\times\hat{\boldsymbol{z}}}{2\Omega}. \qquad (2-16)$$

Now consider a tangent plane of a planet and call it "β-plane". If we displace the parcel on this plane, as shown in Fig 3, Eq (2-10) tells us

$$\frac{d\omega}{dt}=-\frac{2\Omega}{A}\sin\theta_0\frac{dA}{dt}=-2\Omega\sin\theta_0\frac{d\theta_0}{dt}\equiv-\beta v_y \qquad (2-17)$$

where $\beta = 2\Omega\sin\theta_0/R$, the gradient of Coriolis parameter. We can rewrite Eq (2-12) as

$$\frac{d}{dt}(\omega+\beta y)=\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)(\omega+\beta y)=0, \omega=\nabla_{\perp}^2\phi \qquad (2-18)$$

Eq (2-18) is what we call **Charney equation**. This equation is just an expression for the conservation of potential vorticity. As we can see, the latitudinal displacement may result in a change in relative vorticity. By linearizing Charney equation, we get an equation describing Rossby wave (azimuthally asymmetric vortex mode):

$$\partial_t\nabla^2\phi=-\beta\partial_x\phi. \qquad (2-19)$$

The corresponding dispersion relation is

$$\omega=-\beta k_x/k^2 \qquad (2-20)$$

When $k_x\to 0, \omega\to 0$. This reminds us of the zonal flow. In fact, the group velocity in the y-direction v_{gy} is $2\beta k_x k_y/k^4$, intimately relating latitudinal propagation to the Reynolds stress.

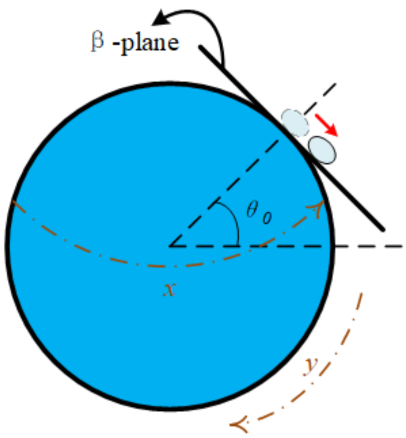

Fig 3 Displacement of a parcel in the β-plane

3. Hasegawa-Mima equation and electron drift wave

3.1. A novel way of deriving Hasegawa-Mima equation

To derive Hasegawa-Mima equation, all we need to do is replacing the Coriolis frequency $\mathbf{2\Omega}$ in Eq (2-8) by ion gyro frequency $\Omega_i \hat{\boldsymbol{z}}$. And the density has a small perturbation $\tilde{n}$. Taking $\nabla\psi = \hat{\boldsymbol{z}}$, PV conservation tells us

$$\frac{d}{dt}\left[\frac{\omega_z + \Omega_i}{n_0(r) + \tilde{n}}\right] = 0. \qquad (3-1)$$

Expand Eq (3-1) and simplify it as

$$\frac{d}{dt}\omega_z - \frac{\Omega_i}{n_0}\frac{\widetilde{dn_i}}{dt} = 0 \qquad (3-2)$$

where $\omega_z = (c/B_0)\nabla_\perp^2\phi, \boldsymbol{v} = -(c/B_0)\nabla\phi \times \hat{\boldsymbol{z}}, \widetilde{dn_i}/dt = d\tilde{n}/dt + \tilde{v}_r\partial_r n_0(r)$.
In addition, we assume both electrons and ions satisfy Boltzmann equation so that

$$\frac{\tilde{n}_i}{n_0} = \frac{\tilde{n}_e}{n_0} \approx \frac{|e|\tilde{\phi}}{T_0}. \qquad (3-3)$$

This requires $v_{thi} < \omega/k_\parallel < v_{the}$, so that electrons can reach equilibrium in one period of oscillation. Substituting Eq (3-3) into Eq (3-2), we get the famous **Hasegawa-Mima equation**

$$\frac{d}{dt}\left(\frac{|e|\hat{\phi}}{T} - \rho_s^2\nabla_\perp^2\frac{|e|\hat{\phi}}{T}\right) + v_*\partial_y\frac{|e|\hat{\phi}}{T} = 0, \qquad (3-4)$$

where $\rho_s^2 = c_s^2/\Omega_i^2, v_* = \rho_s c_s/L_n, L_n = -\partial_r n_0/n_0$.
Therefore, Hasegawa-Mima equation is equivalent to the statement that $PV = (\omega_z + \Omega_i)/(n_0(r) + \tilde{n})$ is conserved.

3.2. The usual derivation of Hasegawa-Mima equation

We start from the continuity equation for ions

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\boldsymbol{v}) = 0. \qquad (3-5)$$

By linearizing it, we get

$$\partial_t\tilde{n} + \tilde{v}_r\partial_r n_0 + n_0\nabla\cdot\tilde{\boldsymbol{v}} = 0. \qquad (3-6)$$

Again, we assume electrons satisfy Boltzmann distribution. The perturbed velocity is the sum of polarization velocity and $\boldsymbol{E}\times\boldsymbol{B}$ velocity, i.e.,

$$\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}_{\boldsymbol{E}\times\boldsymbol{B}} + \tilde{\boldsymbol{v}}_{pol} = -(c/B_0)\nabla\phi\times\hat{\boldsymbol{z}} + \rho_s^2 n_0\frac{d}{dt}E_\perp. \qquad (3-7)$$

And the divergence of $\tilde{\boldsymbol{v}}$ is

$$\nabla\cdot\tilde{\boldsymbol{v}} = \nabla\cdot\left(\tilde{\boldsymbol{v}}_{\boldsymbol{E}\times\boldsymbol{B}} + \tilde{\boldsymbol{v}}_{pol}\right) = -\rho_s^2 n_0\nabla\cdot\frac{d}{dt}\boldsymbol{E}_\perp. \qquad (3-8)$$

Combining Eq (3-6), (3-7) and (3-8), we get

$$\left(\partial_t - \frac{c}{B_0}\nabla\phi\times\hat{\boldsymbol{z}}\cdot\nabla\right)(\phi - \rho_s^2\nabla_\perp^2\phi) + v_*\frac{\partial\phi}{\partial y} = 0, \qquad (3-9)$$

which is same as Eq (3-4).
The linear wave we can obtain from Eq (3-9) is electron drift wave and its dispersion relation is

$$\omega = \frac{k_\theta v_*}{1 + k_\perp^2 \rho_s^2} = \frac{\omega_*}{1 + k_\perp^2 \rho_s^2}, \qquad (3-10)$$

which is different from Eq (2-15), the dispersion relation of Rossby wave. The "1" in the denominator of Eq (3-10) comes from Boltzmann electrons, which have no counterpart in fluids. And "$k_\perp^2 \rho_s^2$" is a representation of ions' inertia, which is important for the drift wave instability we'll discuss next.

3.3. Zonal flow

One might want to relate H-M equation to zonal flow. However, the basic property of zonal flow is contradictory to the setup of H-M equation. In H-M model we require $\omega/k_\parallel < v_{the}$ while in zonal flow, due to its symmetry in poloidal and toroidal directions, $k_\parallel = 0$. Therefore, electrons in zonal flow don't satisfy the Boltzmann relation. To get zonal flow, we turn to exploit the fact that plasma is quasi-neutral, which means for zonal flow

$$\nabla \cdot (\boldsymbol{J}_\perp + \boldsymbol{J}_\parallel) = 0 \qquad (3-11)$$

Since $k_\parallel = 0$, $\nabla_\parallel J_\parallel = 0$. Further, because $\boldsymbol{J}_{\boldsymbol{E}\times\boldsymbol{B}} = 0$, the only non-vanishing term of Eq (3-11) is $\nabla_\perp \cdot \boldsymbol{J}_{\boldsymbol{pol}} = 0$. The contribution to $\boldsymbol{J}_{\boldsymbol{pol}}$ is mainly from ions, because ions' inertia is much larger than electrons'. Finally, the equation describing zonal flow is

$$\frac{d}{dt} \rho_s^2 \nabla_\perp^2 \phi = 0, \qquad (3-12)$$

which is just the equation of 2D fluid.

One important observation is that zonal flow is not governed by the same equation as drift wave while Charney equation allows us to connect the Rossby wave to zonal flow fluently. So "Charney-Hasegawa-Mima equation" is actually a misnomer. In history, Charney first derived this equation in 1958, 20 years earlier than Hasegawa and Mima. At the same time as Hasegawa and Mima, Sagdeev et al. derived the same equation.

Why is zonal flow important? First, zonal flow is more easily excited in plasma than fluid since it has the minimal inertia. A huge difference between Eq (3-12) and Eq (3-4) is that there is no Boltzmann electron in Eq (3-12). So the potential vorticity of zonal flow is simply $q_r^2 \rho_s^2 \hat{\phi}_q$, as opposed to $(1 + k_\perp^2 \rho_s^2)\hat{\phi}_{\boldsymbol{k}}$. Low effective inertia means that large zonal flow velocities develop in response to drift wave drive, unless damping intervenes. Second, the shearing effect of zonal flow can repress turbulence, so it is very important for confinement. Third, zonal flow can't cause transport. Because for zonal flow $m = 0$, $\tilde{v}_r$, the radial flow fluctuation is zero. Thus zonal flow cannot tap expansion free energy stored in radial gradients. (See Pat Diamond, et al, 2005)

But can we relate zonal flow to PV conservation? The answer is yes! Recall Eq (3-2), the linearized version of PV conservation. Since zonal flow has no density perturbation, it is reasonable to consider the pure vortex mode, which means $\tilde{n} \to 0$ but $\omega_z \neq 0$. And the poloidal symmetry of zonal flow allows us to eliminate $\tilde{v}_r$. Then Eq (3-2) becomes

$$\frac{d}{dt}\omega_z = \frac{d}{dt}\nabla_r^2 \phi = 0, \qquad (3-13)$$

which is same as Eq (3-12). N.B. $\tilde{v}_r = 0$ means zonal flow cannot tap/relax free energy source like ∇n_0 and ∇T_0. In fact, zonal flow can only be excited by nonlinear interactions and, more specifically, driven by Reynolds force (i.e. vorticity flux). We will discuss this in detail later in this course. Another point worth mentioning here is that zonal flow is an "$\boldsymbol{E} \times \boldsymbol{B}$" flow. We should distinguish it from the physical mass flow.

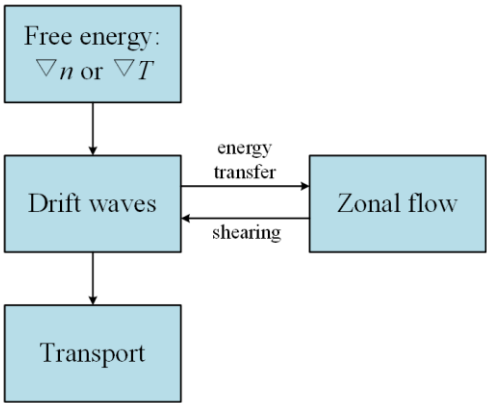

Fig 4 Branching of energy between zonal flow and transport

4. Hasegawa-Wakatani model and drift wave instability

4.1. Motivation and history

In section 3, we discussed the derivation of Hasegawa-Mima equation and the dispersion relation of electron drift wave. However, there is no instability in H-M equation, which means it can only describe stable drift waves. Hasegawa-Wakatani model is the prototype of drift instability. It illustrates the connection between relaxation/transport and zonal flow, and the branching of energy (as shown in Fig 4) between these two channels. If we understand H-W model well (collisional drift wave), it is easy to grasp collisionless drift wave (aka "Universal Mode"), DTEM, CTEM, etc. In history, Sagdeev and Moiseev first developed theory of collisional drift wave (i.e. Hasegawa-Wakatani) in 1960s, much earlier than Hasegawa and Wakatani. **It is a useful but not absolute rule of thumb that everything was done first in Russia.**

4.2. Derivation of Hasegawa-Wakatani model

To include instability into dynamical model, one may want to ask this question first: what is missing here? The clue is radial particle flux. In H-M model, we have zero radial particle flux because we assume electrons satisfy Boltzmann equation. The correlation between $\tilde{v}_r$ and $\tilde{n}$ is

$$\langle \tilde{v}_r \tilde{n} \rangle = c\langle \partial_y \tilde{\phi} \tilde{\phi} \rangle = c^* \langle k_y \tilde{\phi} \tilde{\phi} \rangle = 0. \qquad (4-1)$$

The reason that this correlation vanishes is because it is an odd moment of k_y. To get non-vanishing $\langle \tilde{v}_r \tilde{n} \rangle$, $\tilde{n}$ and $\tilde{\phi}$ should be weakly coupled. To get a non-vanishing $\tilde{n}\tilde{\phi}$ correlation, we need a shift in the phase between $\tilde{n}$ and $\tilde{\phi}$. This phase shift is fundamental to all electron drift waves. In this simple story, phase shift comes from parallel dissipation.

Now recall the fact that plasma is quasi-neutral, so we have

$$\nabla \cdot \boldsymbol{J} = 0 \qquad (4-2)$$

where $\boldsymbol{J} = \boldsymbol{J}_\parallel + \boldsymbol{J}_\perp = \boldsymbol{J}_\parallel + \boldsymbol{J}_{\boldsymbol{pol}} + \boldsymbol{J}_{\boldsymbol{PS}}$. In this case, we only consider the contribution of $\boldsymbol{J}_{\boldsymbol{pol}}$ to $\boldsymbol{J}_\perp$ and retain $\nabla_\parallel \boldsymbol{J}_\parallel$, which was neglected in Eq (3-11). We already know the expression for $\boldsymbol{J}_\perp$ in Eq (3-12), so we need to figure out what $\boldsymbol{J}_\parallel$ is. To be precise, $\boldsymbol{J}_\parallel$ is

$$J_\parallel = -n_0|e|(\tilde{v}_{\parallel e} - \tilde{v}_{\parallel i}). \qquad (4-3)$$

In this case, since we do not worry about acoustic wave coupling, we only retain $\tilde{v}_{\parallel e}$. For simplicity, instead of using Braginskii equations, we choose to get $\tilde{v}_{\parallel e}$ from a simple version of drift kinetic equation for electrons, which is

$$\frac{\partial f}{\partial t} + v_\parallel \hat{\boldsymbol{n}} \cdot \nabla f - \frac{c}{B}\nabla_\perp \phi \times \hat{\boldsymbol{z}} \cdot \nabla f - \frac{|e|E_\parallel}{m_e}\frac{\partial f}{\partial v_\parallel} = c(f). \qquad (4-4)$$

We choose to study electrons rather than ions because electrons' inertia is small, so they stay on the field line, making them easier to deal with. In Eq (4-4), $v_\parallel$ is the parallel velocity in phase space. Apart from collisions, Eq (4-4) is like Vlasov equation, including a parallel motion along the field line and a $\boldsymbol{E} \times \boldsymbol{B}$ drift. The velocity field $v_\parallel(\boldsymbol{r})$ can be calculated by taking the average over momentum space, i.e. $\hat{v}_\parallel = \int d^3 v_\parallel f$. After Multiplying Eq (4) by $v_\parallel$ and integrating it over momentum space, we get

$$\frac{m_e}{|e|}\left\{\frac{\partial \hat{v}_\parallel}{\partial t} - \frac{c}{B}\nabla_\perp \phi \times \hat{\boldsymbol{z}} \cdot \nabla \hat{v}_\parallel\right\} + \frac{m_e}{|e|}\boldsymbol{n} \cdot \nabla \int d^3 v v_\parallel^2 f + E_\parallel = -\frac{m_e}{|e|}\nu_{ei}\hat{v}_\parallel \qquad (4-5)$$

The term on the RHS of Eq (4-5) is the frictional losses due to collisions with ions. Again, utilizing the fact that electron inertia is small, we can drop the first term on the LHS and rewrite Eq (4-5) as

$$\frac{m_e}{|e|}\nabla_\parallel \langle v_\parallel^2 f\rangle + E_\parallel = -\frac{\nu_{ei} m_e}{|e|}\hat{v}_\parallel = -\frac{\nu_{ei} m_e}{n_0 |e|^2} J_{\parallel e} = \eta J_{\parallel e}. \qquad (4-6)$$

The first term on the LHS represents electron pressure. Electron pressure contribution to Ohm's law is complex, but for simplicity we approximate it to nT (where temperature is a constant). Therefore, the final expression for $J_{\parallel e}$ is

$$\tilde{J}_\parallel = -\frac{v_{the}^2}{\nu_{ei}}\nabla_\parallel\left(\tilde{\phi} - T\frac{\tilde{n}}{n_0}\right) = -D_\parallel \nabla_\parallel\left(\tilde{\phi} - \frac{T}{|e|}\frac{\tilde{n}}{n_0}\right). \qquad (4-7)$$

Plugging Eq (4-7) into Eq (4-2), we get the first Hasegawa-Wakatani equation

$$\rho_s^2 \frac{d}{dt}\nabla_\perp^2 \tilde{\phi} = D_\parallel \nabla_\parallel^2\left(\tilde{\phi} - \frac{T}{|e|}\frac{\tilde{n}}{n_0}\right). \qquad (4-8)$$

Obviously, to close this dynamical model, we need another equation for $\tilde{n}$. Starting from the continuity equation and linearizing it, we get

$$\frac{1}{n_0}\frac{\partial \tilde{n}}{\partial t} + \frac{\tilde{v}_r}{n_0}\frac{\partial n_0}{\partial r} + \tilde{v}_r \cdot \frac{\nabla \tilde{n}}{n_0} = -n_0 \nabla_\parallel \tilde{v}_{\parallel e}. \qquad (4-9)$$

Replacing the RHS of Eq (4-9) by electron current density, we get the second Hasegawa-Wakatani equation

$$\frac{1}{n_0}\frac{d}{dt}\tilde{n}+\frac{\tilde{v}_r}{n_0}\partial_r n_0(r)=D_\parallel\nabla_\parallel^2\left(\tilde{\phi}-\frac{T}{|e|}\frac{\tilde{n}}{n_0}\right). \qquad (4-10)$$

So finally, the complete Hasegawa-Wakatani model is

$$\begin{cases}\rho_s^2\frac{d}{dt}\nabla_\perp^2\tilde{\phi}=D_\parallel\nabla_\parallel^2\left(\tilde{\phi}-\frac{T}{|e|}\frac{\tilde{n}}{n_0}\right)\\ \frac{1}{n_0}\frac{d}{dt}\tilde{n}+\frac{\tilde{v}_r}{n_0}\partial_r n_0(r)=D_\parallel\nabla_\parallel^2\left(\tilde{\phi}-\frac{T}{|e|}\frac{\tilde{n}}{n_0}\right)\end{cases} \qquad (4-11)$$

Sometimes, by including viscosity and diffusion, Hasegawa-Wakatani can be modified to

$$\begin{cases}\rho_s^2\left(\frac{d}{dt}-\nu_0\nabla_\perp^2\right)\nabla_\perp^2\tilde{\phi}=D_\parallel\nabla_\parallel^2\left(\tilde{\phi}-\frac{T}{|e|}\frac{\tilde{n}}{n_0}\right)\\ \frac{1}{n_0}\left(\frac{d}{dt}-D_0\nabla_\perp^2\right)\tilde{n}+\frac{\tilde{v}_r}{n_0}\partial_r n_0(r)=D_\parallel\nabla_\parallel^2\left(\tilde{\phi}-\frac{T}{|e|}\frac{\tilde{n}}{n_0}\right)\end{cases} \qquad 4-11(b)$$

A few comments:

- Here we did not consider ion acoustic corrections.
- H-W is clearly a 2-field model with dissipative coupling.
- In reality, $D_\parallel \gg \nu_0 \gg D_0$.
- Frequently, a scale-invariant damping is invoked for zonal mode.

In Hasegawa-Wakatani model, there are two conserved quantities: energy E and potential enstrophy U. The conservation laws for these two quantities are

$$\frac{\partial E}{\partial t}\equiv\frac{1}{2}\frac{\partial}{\partial t}\int[n^2+(\nabla\phi)^2]dV$$

$$=-c_1'\int\left(\frac{\partial n}{\partial z}-\frac{\partial\phi}{\partial z}\right)^2dV-c_2\int(\nabla^2\phi)^2dV-\int n(\hat{\boldsymbol{z}}\times\boldsymbol{\kappa})\cdot\nabla\phi dV,$$

$$\frac{\partial U}{\partial t}\equiv\frac{1}{2}\frac{\partial}{\partial t}\int(\nabla^2\phi-n)^2=-c_2\int(n-\nabla^2\phi)\nabla^4\phi dV-\int n(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{\kappa}})\cdot\nabla\phi dV,$$

where $c_1'=T_e/e^2n_0\eta\omega_{ci}$, $c_2=\nu_0/\rho_s^2\omega_{ci}$. (See Hasegawa, Wakatani, 1983)

Now, very importantly, we should realize that H-W model is a set of coupled equations for $\tilde{n}$ and $\tilde{\phi}$. To measure how strong this coupling is, we need a dimensionless parameter called adiabaticity parameter α, which is the ratio of $k_\parallel^2 v_{the}^2/\nu_{ei}$ to ω, i.e., the ratio of the first term on the RHS of the second H-W equations to the first term on the LHS ($k_\parallel=1/Rq$). There are two important limits of H-W model: when $\alpha>1$, $\tilde{n}$ and $\tilde{\phi}$ are strongly coupled, corresponding to adiabatic limit (drift wave mode limit); when $\alpha<1$, $\tilde{n}$ and $\tilde{\phi}$ are weakly coupled, corresponding to hydrodynamical limit (convective cell limit). In each case there are two 2 linear modes, but usually the adiabatic limit is more interesting. Let's discuss them separately.

4.3. Adiabatic limit

In adiabatic limit, $\alpha>1$, which means fluid element diffuses $\lambda_\parallel$ faster than 1 oscillation. In this case, we can assume electrons are nearly Boltzmann but with a small correction, so

$$\frac{\tilde{n}}{n} \approx \frac{|e|}{T}\tilde{\phi} + \tilde{h} \qquad (4-12)$$

Plugging Eq (4-12) into the second equation of H-W, we can replace equations for $\tilde{n}$ and $\tilde{\phi}$ with equations for $\tilde{h}$ and $\tilde{\phi}$, which are

$$\begin{cases} \frac{\partial \tilde{h}_{\boldsymbol{k}}}{\partial t} + k_{\parallel}^2 D_{\parallel} \tilde{h}_{\boldsymbol{k}} = -\frac{|e|}{T}\frac{\partial \tilde{\phi}}{\partial t} - \frac{v_* |e|}{T}\frac{\partial \tilde{\phi}_{\boldsymbol{k}}}{\partial y} \\ \rho_s^2 \frac{\partial}{\partial t}\nabla_\perp^2 \tilde{\phi}_{\boldsymbol{k}} = k_{\parallel}^2 D_{\parallel} \tilde{h}_{\boldsymbol{k}} \end{cases} \qquad (4-13)$$

So

$$\tilde{h}_{\boldsymbol{k}} = \frac{i|e|}{T}\frac{(\omega - \omega^*)\tilde{\phi}}{\cancel{-i\omega} + k_{\parallel}^2 D_{\parallel}}. \qquad (4-14)$$

Here we drop $-i\omega$ in the denominator because $k_{\parallel}^2 D_{\parallel} \gg \omega$. Now, after adding a correction $\tilde{h}$, the radial particle flux is no longer zero and its expression is

$$\langle \tilde{v}_r \tilde{n} \rangle = \langle \tilde{v}_r \tilde{h} \rangle = \sum_{\boldsymbol{k}} -\rho_s c_s \left(\frac{|e|\tilde{\phi}_{\boldsymbol{k}}}{T}\right)^2 \frac{k_y(\omega - \omega^*)}{k_{\parallel}^2 D_{\parallel}} \qquad (4-15)$$

The parallel diffusion $k_{\parallel}^2 D_{\parallel}$ matters here, because without it radial flux vanishes. But to have instability which can relax density gradient, that's not enough. According to Eq (4-15), an outward radial particle flux requires $\omega < \omega^*$. To compare the magnitudes of ω and ω^*, we need to calculate ω first. As mentioned above, $\tilde{h}$ is a small correction and electrons are nearly Boltzmann, so by relating the two equations of H-W, we can obtain

$$\rho_s^2 \frac{d}{dt}\nabla_\perp^2 \tilde{\phi} \cong \frac{1}{n_0}\left(\frac{d\tilde{n}}{dt} + \tilde{v}_r \partial_r n_0\right) \qquad (4-16)$$

and

$$\partial_t \frac{|e|\tilde{\phi}}{T} - \rho_s^2 \frac{d}{dt}\nabla_\perp^2 \tilde{\phi} + v^* \partial_y \frac{|e|\tilde{\phi}}{T} = 0. \qquad (4-17)$$

Eq (4-17) is exactly Hasegawa-Wakatani equation. This is an important observation: in adiabatic limit, Hasegawa-Wakatani model reduces to Hasegawa-Mima equation! Obviously, ω_{real} is just the frequency of electron drift wave, which is $\omega^*/(1 + k_\perp^2 \rho_s^2)$. Fortunately, due to the ion inertia, ω is naturally smaller than ω^*, which allows us to have an outward radial flux and instability. Plugging the expression for ω into Eq -(4-15), we can see the radial particle flux is

$$\langle \tilde{v}_r \tilde{n}/n_0 \rangle = \sum_{\boldsymbol{k}} \rho_s^2 c_s^2 \left|\frac{|e|\tilde{\phi}_{\boldsymbol{k}}}{T}\right|^{\mathbf{2}} \frac{k_y^2 k_\perp^2 \rho_s^2 \omega^*}{k_{\parallel}^2 D_{\parallel}(1 + k_\perp^2 \rho_s^2)}. \qquad (4-18)$$

N.B.: to get relaxation/growth, we should have both parallel friction and $\omega < \omega^*$. The physics behind $\omega < \omega^*$ is that the energy gained from gradient relaxation exceeds the cost of "pumping" the potential.

The growth rate of drift wave instability can be calculated simply by perturbation method. Recall Eq (3-10), the dispersion relation of stable drift wave, we can rewrite is as

$$\frac{\omega^*}{\omega} - k_\perp^2 \rho_s^2 = 1. \qquad (4-19)$$

The “1” on the RHS comes from Boltzmann electron. Now we know electrons are nearly Boltzmann, but with a small correction, which is given by Eq (4-14). Add this correction back to Eq (4-19), and the more accurate dispersion relation is

$$\frac{\omega^*}{\omega} - k_\perp^2 \rho_s^2 = 1 + \frac{i(\omega - \omega^*)}{k_\parallel^2 D_\parallel}. \qquad (4-20)$$

To the 0th order, Eq (4-20) is equivalent to Eq (4-19). To the 1st order, perturbation method gives us the growth rate

$$\frac{\gamma}{\omega} = \frac{\omega^* k_\perp^2 \rho_s^2}{k_\parallel^2 D_\parallel (1 + k_\perp^2 \rho_s^2)^2}. \qquad (4-21)$$

Up to now, we have finished all the derivation for drift wave instability. This is also called the collisional/dissipative/resistive drift wave instability.

All the electron drift instabilities and trapped electron modes are, in some sense, the similar story. See table 2 for some hints.

Collisional drift wave	Phase shift is: $ik_\parallel^2 D_\parallel$
Collisionless drift wave	Dissipation comes from electron Landau resonance ($\omega \sim k_\parallel v_\parallel$)
Dissipative trapped electron mode	Phase shift is: $i\nu_{ee}$ (ν_{ee} is electron detrapping frequency)
Collisionless trapped electron mode	Dissipation comes from resonance between the energy-dependent processional drift and the frequency

table 2 Comments on some other modes

Another point is that not only can ∇n drive drift wave instability, but also can ∇T.

4.4. Hydrodynamical limit

In this case, oscillation is faster than collisional parallel diffusion. So we can no longer assume that electrons satisfy Boltzmann distribution. This case is similar to MHD. ω_{re} and ω_{im} are both comparable and proportional to $\sqrt{\alpha}$. A more accurate expression for the frequency at this limit is

$$\omega_{hydrodynamic} \cong \frac{1}{2}\left(-i\frac{\alpha(1 + k_\perp^2 \rho_s^2)}{k_\perp^2 \rho_s^2} + \sqrt{\frac{4i\alpha\omega^*}{k_\perp^2 \rho_s^2}}\right) \cong \sqrt{\frac{\omega^* \alpha}{2k_\perp^2 \rho_s^2}}(1 + i). \qquad (4-22)$$

Hydrodynamical limit is less important unless we have a high density and low temperature limit, at the edge.

4.5. Ohm’s law

The Ohm’s law is the key to all this. Even for simplest story, there are many terms in it. For example, in this case, we have

$$\left(-\frac{1}{c}\frac{\partial A_\parallel}{\partial t}\right) - \nabla_\parallel \phi + T\nabla_\parallel n = \eta J_\parallel \qquad (4-23)$$

① ② ③ ④

- When ① balances ②, we have ideal MHD, i.e. $E_\parallel = 0$.

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0$$

- When ② balances ④ (with ①), we have resistive MHD ($\alpha < 1$).

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J}$$

- When ② balances ③ (with ④), we have drift wave instabilities ($\alpha > 1$).

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J} - \frac{\nabla p_e}{ne}$$

- When we consider small scale things like ETG and EMHD, we have to consider electron inertia.

$$\frac{m_e}{e}\frac{d}{dt} v_\parallel + E_\parallel = \eta J_\parallel - \frac{\nabla_\parallel p_e}{ne}$$

Basically, the dominant balance in Ohm's law largely determines dynamics.