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Finite-larmor-radius magnetohydrodynamic equations for microturbulence

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A set of nonlinear fluid equations which includes the effect of finite ion Larmor radius is derived to describe microturbulence [$k_\perp \rho_s \simeq O(1)$, $k_\parallel R \simeq O(1)$, $n_1/n_0 \simeq \rho_s/L_n$, and $B_1/B_0 \simeq \rho_s/R$] in an inhomogeneous plasma with a strong magnetic field of general geometry. Here ρ_s is the ion Larmor radius at the electron temperature, L_n is the density gradient scale length, R is the radius of curvature of the magnetic line of force, $\mathbf{k}$ is the wave vector, and n_1/n_0 and B_1/B_0 are relative levels of density and magnetic field perturbations.

There is increasing evidence that an inhomogeneous plasma in a strong magnetic field such that $\rho_i/L_\perp \equiv \epsilon_1 \ll 1$ (where ρ_i and $L_\perp$ are the ion Larmor radius and the scale length of the perpendicular inhomogeneity, respectively) is imbedded with intrinsic turbulence both in density[1] and magnetic field.[2] The wavenumber spectrum is anisotropic such that $k_\perp \gg k_\parallel$, where subscripts $\perp$ and $\parallel$ designate directions perpendicular and parallel to the magnetic field.[2] Observations commonly reveal that $k_\perp$ scales with ρ_i^{-1} or ρ_s^{-1}, where ρ_s [$= (T_e/m_i)^{1/2}/\omega_{ci}$] is the ion Larmor radius at the electron temperature T_e, while $k_\parallel$, although often not measurable, is believed to scale with R^{-1}, where R is a typical inhomogeneous scale length in the parallel direction.

It has been recognized[3] that when $k_\perp \sim \rho_s^{-1}$, mode couplings produce fully nonlinear effects even at fluctuation levels as small as $n_1/n_0 \simeq \epsilon_1$, where subscripts 1 and 0 indicate the perturbed and unperturbed quantities. The nonlinearity originates from the convective derivative of the $\mathbf{E}\times\mathbf{B}$ fluid velocity. A similar fully nonlinear effect also appears in the magnetic field perturbation $B_1/B_0 \simeq \epsilon_1\epsilon_2$ due to the bending of the magnetic field lines,[4] where $\epsilon_2 = L_\perp/L_\parallel$ ($\simeq a/qR$ in tokamak ordering, where a is the minor radius and q is the safety factor). Our purpose in this paper is to derive fully nonlinear fluid equations with two orderings, $\rho_s/L_\perp$ and $L_\perp/L_\parallel$, which are appropriate to study the turbulence often observed in a plasma with a strong magnetic field.

To take into account the effects of finite Larmor radius at $\omega \ll \omega_{ci}$ the most appropriate method is to use the gyrokinetic equations.[5] However, the resultant mode coupling equations are often too complex to be useful. In addition, the two-component approach, which is inevitable in this method, makes the derivation of the nonlinear mode coupling equations unnecessarily complicated. In this respect the magnetohydrodynamic equations are more convenient to use because only the pressure, current density, and velocity field of the "one" fluid enter as the kinematic variables. Strauss[6] has derived magnetohydrodynamic equations with the tokamak ordering, $a/R \ll 1$ by constructing an equation for the vorticity. These equations are useful in studying nonlinear problems including the effects of the curvature and shear of the toroidal field.

Attempts have been made to include the effect of the ion Larmor radius in the Strauss equations by adding the ion diamagnetic drift to the convective derivative of the ion equation of motion.[7–10] A method alternative to this is to include a perpendicular current which originates from the difference between the $\mathbf{E}\times\mathbf{B}$ drifts of ions and electrons.[11] Both methods can be shown to give the same result to order $(k_\perp \rho_i)^2$. However the latter method retains the favorable property of the magnetohydrodynamic equations in that the perpendicular velocity is given only by one fluid velocity, $\mathbf{E}\times\mathbf{B}$. Hence we adapt the latter method.

By recognizing that the curl of the magnetohydrodynamic equation of motion produces an equation which is equivalent to $\nabla_\perp \cdot \mathbf{J}_\perp + \nabla_\parallel \cdot \mathbf{J}_\parallel = 0$ we replace the equation of motion by $\nabla \cdot \mathbf{J} = 0$, where $\mathbf{J}_\perp$ is given by the guiding-center drift current (curvature, ∇B, and polarization current) plus the above-mentioned $\mathbf{E}\times\mathbf{B}$ current.

The electromagnetic field variables are the parallel component of the vector potential $A_\parallel$ and the scalar potential ϕ. Since $J_\perp \simeq \epsilon_1\epsilon_2 J_\parallel$, only $J_\parallel$ appears as the source of the electromagnetic field. The zeroth and the first moments of the electron drift kinetic equation produce the continuity equation for the electron $J_\parallel$ and Ohm's law. The parallel ion inertia current (which can be ignored in most cases) and the appropriate equations of state for electrons and ions close the set of equations.

We first introduce two small parameters:

$$\epsilon_1 = \rho_s/L_\perp \tag{1}$$

and

$$\epsilon_2 = L_\perp/L_\parallel . \tag{2}$$

Ordinarily

$$\epsilon_1 \ll \epsilon_2 \simeq \sqrt{\epsilon_1} . \tag{3}$$

We take coordinate z in the local direction of the unperturbed magnetic field and define $\hat{\mathbf{b}}$ as the unit vector in the local direction of the total magnetic field, then

$$J_\parallel = J_z + O(\epsilon_1\epsilon_2),$$
$$A_\parallel = A_z + O(\epsilon_1\epsilon_2), \tag{4}$$

while

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$$\nabla_\parallel \equiv \hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\nabla) = \hat{\mathbf{z}}\left(\frac{\partial}{\partial z} + \frac{B_{0y}}{B_0}\frac{\partial}{\partial y} + \frac{\mathbf{B}_\perp}{B_0}\cdot\nabla_\perp\right) + O(\epsilon_1\epsilon_2), \quad (5)$$

where B_{0y} represents the shear field and

$$\mathbf{B}_\perp(\mathbf{x},t) = \nabla_\perp \mathbf{A}_z \times \hat{\mathbf{z}} + O(\epsilon_1\epsilon_2) \quad (6)$$

is the perturbed magnetic field. Equation (5) shows that if $k_\perp B_\perp/B_0 \simeq k_z$, or if $B_\perp/B_0 \simeq \epsilon_1\epsilon_2$, the nonlinear term becomes of the same order as the linear term. Hence we retain the $\mathbf{B}_\perp\cdot\nabla$ term allowing $B_\perp/B_0 \simeq O(\epsilon_1\epsilon_2)$. The convective derivative due to the $\mathbf{E}\times\mathbf{B}$ drift, $\mathbf{v}_E$,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}_E\cdot\nabla = \frac{\partial}{\partial t} - \frac{\nabla\phi\times\hat{\mathbf{z}}}{B_0}\cdot\nabla \quad (7)$$

also gives a full nonlinear effect if $e\phi/T_e \simeq \epsilon_1$; thus we retain the $\mathbf{v}_E\cdot\nabla$ term allowing $e\phi/T_e \simeq \epsilon_1$.

Since the guiding-center current is a valid description to $O(\epsilon_1)$, $\nabla\cdot\mathbf{J} = 0$ gives at $O(\epsilon_1^2\epsilon_2)$

$$\begin{aligned}\hat{\mathbf{b}}\cdot\nabla(J_{zi} + J_{ze}) &= \frac{m_i n_0}{B_0^2}\frac{d}{dt}\nabla_\perp^2\phi + \frac{p_{i0}}{B_0^2\omega_{ci}}(\nabla\nabla_\perp^2\phi\times\hat{\mathbf{z}})\cdot\nabla\ln(p_{i0}+p_i) \\ &\quad + \sum_{j=i,e}\left(\nabla p_{\perp j}\cdot\frac{\nabla B_0\times\hat{\mathbf{z}}}{B_0^2} + \nabla p_{\parallel j}\cdot\frac{\mathbf{B}_0\times\mathbf{R}}{B_0^2 R^2}\right) \\ &\equiv -\nabla_\perp\cdot\mathbf{J}_\perp, \end{aligned} \quad (8)$$

where $\mathbf{R}/R^2 = -(\hat{\mathbf{z}}\cdot\nabla)\hat{\mathbf{b}}_0$ is the curvature of the unperturbed magnetic field, p_j and p_{j0} are the perturbed and unperturbed pressure of the jth species, and $\hat{\mathbf{b}}\cdot\nabla$ and d/dt are given by Eqs. (5) and (7). The second term on the right-hand side originates from the difference of $\mathbf{E}\times\mathbf{B}$ drift between electrons and ions, due to the fact that the ion sees the electrostatic field which is reduced by $\rho_i^2\nabla^2\phi$. We note that this term can also be constructed from the divergence of the polarization current in which the convective derivative due to the ion diamagnetic drift is retained. However, with the convective derivative in the first term given only by the $\mathbf{E}\times\mathbf{B}$ drift, this term should be retained explicitly as shown.

The Maxwell equations become, then,

$$\hat{\mathbf{b}}\cdot\nabla(\nabla_\perp^2 A_z) = \mu_0\nabla_\perp\cdot\mathbf{J}_\perp. \quad (9)$$

Equation (9) with (8) relates A_z and ϕ to the plasma pressure $p_\perp$ and $p_\parallel$.

By taking the first moment (with respect to $v_\parallel$) of the electron drift kinetic equation, and by ignoring the inertia term, we obtain the parallel component of Ohm's law.[4] To order $\epsilon_1^2\epsilon_2$, the equation becomes

$$\eta J_{ze} = -\frac{\partial A_z}{\partial t} - \hat{\mathbf{b}}\cdot\nabla\phi + \frac{1}{en}\hat{\mathbf{b}}\cdot\nabla(p_{\parallel e} + p_{0e}). \quad (10)$$

Here η is the resistivity, and J_{ze} is the z component of the electron current. In Eq. (10), we note that $\partial A_z/\partial t$ becomes dA_z/dt if $\nabla A_z\times\hat{\mathbf{z}}\cdot\nabla\phi$ is taken out of the second term and combined with $\partial A_z/\partial t$.

Depending on the choice of the equations of state, a relation between the pressure p and the number density n is needed. This may be obtained from the zeroth moment of the electron drift kinetic equation, which gives to the order $\epsilon_1^2\epsilon_2$

$$e\frac{d}{dt}(n_1 + n_0) + en_0\nabla\cdot\mathbf{v}_E = \hat{\mathbf{b}}\cdot\nabla J_{ze} + \nabla_\perp\cdot\mathbf{J}_{\perp e}. \quad (11)$$

Here $\nabla_\perp\cdot\mathbf{J}_{\perp e}$ is the electron portion of the right-hand side of Eq. (8).

We note here that from the convective derivative term of Eq. (11) the present ordering forces $k_\perp\rho_s e\phi/T_e \simeq O(\epsilon_1\epsilon_2)$. This is an electromagnetic ordering which allows either a longer wavelength $(k_\perp\rho_s < 1)$ or a smaller amplitude $(e\phi/T_e < \epsilon_1)$. On the other hand, the electrostatic ordering is $e\phi/T_e \simeq n_1/n_0 \simeq O(\epsilon_1)$ with $k_\perp\rho_s \simeq O(1)$.[3] In this case Eq. (10) gives the Boltzmann distribution, $n_1/n_0 = e\phi/T_e$ while Eq. (11) is balanced by the first term of Eq. (8) in the order ϵ_1^2. This gives the Hasegawa–Mima equation.[3]

We need equation(s) of state to eliminate the pressure. One common choice is isotropic incompressible ions,

$$\frac{dp_{\parallel i}}{dt} = \frac{dp_{\perp i}}{dt} \equiv \frac{dp_i}{dt} = 0, \quad (12)$$

and isotropic, isothermal electrons,[6]

$$\frac{dp_{\parallel e}}{dt} = \frac{dp_{\perp e}}{dt} = \frac{T_e}{e}\hat{\mathbf{b}}\cdot\nabla J_{ze}. \quad (13)$$

For simplicity, if we assume the ion motions to be two dimensional in the perpendicular plane; J_{zi} may be ignored compared with J_{ze}. Then, Eq. (13) and the Maxwell equation give

$$\hat{\mathbf{b}}\cdot\nabla(\nabla_\perp^2 A_z) = -\frac{e\mu_0}{T_e}\frac{dp_e}{dt}. \quad (14)$$

Equations (8)–(10), (12), and (13) form the complete set of equations we desire. We note that the nonlinear terms appears only through d/dt, $\hat{\mathbf{b}}\cdot\nabla$, and through the second term in Eq. (8).

Different physical problems can be treated by modifying the equation(s) of state. For example, when the isothermal condition is not applicable, one can use the Braginskii formula,[12] or when trapped particles become important one can use two equations of state for trapped and untrapped particles.

The local dispersion relation for a case $J_0 = \eta = 0$ may be obtained from Eqs. (8), (10), (12), and (13). For a simple curved field line,

$$\frac{\nabla p\cdot(\mathbf{R}\times\mathbf{B}_0)}{R^2B_0^2} \simeq \nabla p\cdot\hat{\mathbf{z}}\times\frac{\nabla B_0}{B_0^2} = -\frac{\partial p}{\partial y}\frac{1}{RB_0}, \quad (15)$$

where the x axis is taken in the direction of the radius of curvature and that of the pressure gradient. If we define the electron and the ion drift wave frequency,

$$\omega_e^* = \frac{k_y T_e}{eB_0}\frac{\partial}{\partial x}\ln p_{e0}$$

and

$$\omega_i^* = \frac{k_y T_i}{eB_0}\frac{\partial}{\partial x}\ln p_{i0}, \quad (16)$$

the local dispersion relation becomes

$$(\omega + \omega_e^*)\left[\omega^2 - \omega_i^*\omega - k_z^2 v_A^2 - \frac{2k_y^2}{m_i Rk_\perp^2} \times\left(T_i\frac{\partial}{\partial x}\ln p_{i0} + T_e\frac{\partial}{\partial x}\ln p_{e0}\right)\right]$$

$$= \frac{k_z^2 v_A^2}{\omega}(\omega - \omega_i^*)\left[k_\perp^2 \rho_s^2 \omega + \omega_i^*\left(R\frac{\partial}{\partial x}\ln p_{i0}\right)^{-1}\right], \quad (17)$$

where v_A is the Alfvén speed. Equation (17) is the dispersion relation for coupled ballooning and drift modes and agrees with that obtained by Strauss.[13] If one uses the equation of state for electrons in which $\nabla_\perp \cdot \mathbf{v}_{\perp e} \neq 0$ is included an additional term, $p_{e0} \nabla_\perp \cdot \mathbf{v}_{\perp e}$, appears in Eq. (13). This term modifies the dispersion relation obtained here. In particular a term which is proportional to the product of curvatures appears. This has also been noticed recently by Diamond.[14]

A set of nonlinear fluid equations is derived which is suitable to describe magnetohydrodynamic modes with finite-ion-Larmor-radius corrections. The equations are simple enough to be numerically soluble yet contain all the basic dynamics in a inhomogeneous plasma at low frequencies.

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Echo phenomenon associated with lower-hybrid wave launching

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Lower-hybrid waves at two different frequencies f_1 and f_2 are launched simultaneously from two localized antennas, and a third wave is observed to arise near the plasma edge at the frequency $f = f_2 - f_1$. This phenomenon can be explained by an echo effect near the plasma surface.

Lower-hybrid waves are important in thermonuclear fusion research primarily because of their possible applications to plasma current drive and heating in tokamaks. They can be launched by imposing a macroscopic electric field near the plasma surface with metal waveguide arrays at high power density ($\sim 10\ \mathrm{kW/cm^2}$). Reactor studies[1] show that in a tokamak fusion reactor with lower-hybrid-wave current drive and heating, about 1 % of the total first wall area will be used for wave launching. In other words, there will be more than one antenna, possibly driven at different frequencies; and it is interesting to examine how these waves, from various antennas, interact with each other. In this letter we present experimental data showing that lower-hybrid waves launched from two separate antennas at different frequencies f_1 and f_2 can excite a third wave at the beat frequency $f_2 - f_1$. This third wave has a parallel phase velocity near the electron thermal velocity so that it is strongly damped near the plasma surface via electron Landau damping. With respect to controlled thermonuclear reaction (CTR) applications including current-drive and plasma core heating, this nonlinear process may represent a source of operating inefficiency.

The experiment was performed in the Princeton ACT-1 device[2] working with a hydrogen plasma and the following parameters: magnetic field on axis $\simeq 2.8$ kG, neutral pressure $\lesssim 10^{-5}$ Torr (gauge), plasma density varying from $3 \times 10^8\ \mathrm{cm^{-3}}$ to $10^{10}\ \mathrm{cm^{-3}}$, and electron temperature ~ 1–2 eV. The plasma was produced by a heated tungsten filament as described previously.[2] Figure 1(a) is a schematic drawing of the experimental setup. The lower-hybrid waves were launched from electrostatic plates at the plasma edge. Two of 12 plates were driven at two frequencies f_1 and f_2. The excited lower-hybrid waves propagate along resonance cones as shown in Fig. 1(b). In addition to the excited waves at frequencies f_1 and f_2, the probe picks up a signal at the beat frequency $f = f_2 - f_1$ near the plasma edge. In order to avoid mixing effects inside the probe sheath, the driving signals were kept at very low power levels (approximately 1 mW). The probes detect the beat signal only when $f_2 > f_1$. When we switch the two oscillators, the signal disappears as shown in