Physics 218C Week 7 Lecture Note (May 10th, 12th)

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Contents

1 Electron drift-wave and TEM

Last time, we talked about the three players in the game of electron drift-wave (DW):

- resonant electrons,
- waves, and
- zonal mode.

1.1 Review

We discussed collisionless drift waves ($v_{i,th} \leq \omega/k_{\parallel} \ll v_{th,e}$) with non-adiabatic density $\widetilde{h}$ such that $\widetilde{n}/n_0 \simeq \frac{e\widetilde{\phi}}{T} + \widetilde{h}$. Hereafter, we write $\frac{\widetilde{\phi}}{T} \equiv \widetilde{\Phi}$ and $\widetilde{n}/n_0 \equiv \widetilde{N}$ for simplicity. And we derived the energy theorem for the collisionless DW (energetics of the internal wave energy)

$$\frac{\partial}{\partial t}\mathscr{E}_{\omega} = \underbrace{-\int dx^3 \langle \widetilde{\Phi}\frac{\partial}{\partial t}\widetilde{h}\rangle}_{\substack{\text{electron cooling}\\ \text{into waves}}} \underbrace{-\int d^3x \langle v'_{E\times B}\rangle \langle \widetilde{v}_{x,E\times B}\widetilde{v}_{y,E\times B}\rangle}_{\substack{\text{wave stresses (Reynolds) power}\\ \text{exert on flow}}} + \text{dissipation}, \tag{1}$$

where $\mathscr{E}_{\omega}$ is the wave energy, $v'_{E\times B} \equiv \partial_x v_{E\times B}$, and under the cartisian coordinate where x is in radial and y is in poloidal direction. Note that is we include **ion Landau Damping**, where parallel compression is included ($\nabla_{\parallel}\widetilde{v}_{\parallel} \neq 0$), we have

$$\frac{\partial}{\partial t}\mathscr{E}_{\omega} = \underbrace{-\int dx^3 \langle \widetilde{\Phi}\frac{\partial}{\partial t}\widetilde{h}\rangle}_{\substack{\text{electron cooling}\\ \text{into waves}}} - \underbrace{\int d^3x \langle \widetilde{\Phi}\nabla_{\parallel}\widetilde{v}_{\parallel}\rangle}_{\substack{\text{compressional coupling}\\ \text{to waves}}} - \underbrace{\int d^3x \langle v_{E\times B}\rangle' \langle \widetilde{v}_{x,E\times B}\widetilde{v}_{y,E\times B}\rangle}_{\substack{\text{wave stresses (Reynolds) power}\\ \text{exert on flow}}} + \text{dissipation}. \tag{2}$$

The second term on RHS can be calculated via ion-drift kinetics response $\widetilde{f}$ such that

$$\widetilde{v}_{\parallel} \simeq \int d^3v\, v_{\parallel}\widetilde{f}. \tag{3}$$

This term have two contribution:

- generates Landau damping that stabilize the the wave, and
- geneartes acoustic wave radiation (important when consider wave energy in a bounded domain).

The acoustic wave radiation can be derive from $\widetilde{\Phi}\nabla_{\parallel}\widetilde{v}_{\parallel} = \nabla_{\parallel}(\widetilde{\Phi}\widetilde{v}_{\parallel}) - \widetilde{v}_{\parallel}\nabla_{\parallel}\widetilde{\Phi}$, and do the integral of reals space $\int d^3x$ on the former.

Now, we are interested in *resonant electrons*, which will produce wave energy evolution and create the collisionless trapped lectron mode. This is important to the DW **energy balance theorem**

$$\underbrace{\frac{\partial}{\partial t}\text{TWED}}_{\substack{\text{total wave}\\ \text{energy density}}} + \underbrace{\frac{\partial}{\partial t}\text{REED}}_{\substack{\text{resonant electron}\\ \text{energy density}}} + \underbrace{\frac{\partial}{\partial t}\text{RIED}}_{\substack{\text{resonant ion}\\ \text{energy density}}} + \underbrace{\frac{\partial}{\partial t}\text{ZED}}_{\substack{\text{zonal}\\ \text{energy density}}} = 0, \tag{4}$$

where total wave energy consist of electric energy density (EED) plus the non-resonant particle kinetic energy density (NRPKED). Hence, we need to study the *resonant kinetic electron energy evolution*, which amounts to a quasi-linear (QL) type equation for electrons.

1.2 Energy Theorem for Resonant Electrons

Recall the Vlasov equation

$$\frac{\partial}{\partial t}f + v_z \cdot \nabla_z f + v_\perp \cdot \nabla_\perp f + a_z \cdot \nabla_z f = 0, \tag{5}$$

where we have parallel velocity $v_z = v_\parallel$, $(E \times B)_y$ drift-wave velocity $-\nabla_\perp \phi \times \hat{z}/B_0$, and electron parallel acceleration $a_z = -|e|E_\parallel/m_e$. Hence, we have electron-drift (kinetic) equation

$$\frac{\partial}{\partial t}\widetilde{f} + v_\parallel \nabla_\parallel \widetilde{f} - \frac{1}{B_0}\nabla_\perp \phi \times \hat{z} \cdot \nabla_\perp f_0 - \frac{e}{m_e}E_\parallel \frac{\partial}{\partial v_\parallel} f_0 = 0, \tag{6}$$

where $f \equiv f_0 + \widetilde{f}$ is the mean distribution function and its perturbation. The last term in Eq.6 $\frac{e}{m_e}E_\parallel \frac{\partial}{\partial v_\parallel} f_0$ is important to the *parallel nonlinearity*, i.e. intrinsic rotation if look at the ion. Eq.6 can be decomposed in to two equations:

- the guiding center equation of plasma

$$\frac{\partial}{\partial t}\widetilde{f}_\perp - \frac{1}{B_0}\nabla_\perp \phi \times \hat{z} \cdot \nabla_\perp f_0 = 0, \tag{7}$$

  where $f_\perp$ evolved by $E \times B$ advection[1].

- And the 1D parallel kinetic equation (along B_0 field line)

$$\frac{\partial}{\partial t}\widetilde{f}_\parallel + v_\parallel \nabla_\parallel \widetilde{f} - \frac{e}{m_e}E_\parallel \frac{\partial}{\partial v_\parallel} f_{0,\parallel} = 0. \tag{8}$$

The linear response $\widetilde{f}$ is

$$\widetilde{f}_{\omega,k} = \frac{\widetilde{\phi} L_k}{-i(\omega - k_\parallel v_\parallel)} f_0, \tag{9}$$

where L_k is 2D operator that contains two pieces

$$L_k = \underbrace{-ik_\parallel \frac{|e|}{m_e}\frac{\partial}{\partial v_\parallel}}_{\substack{\text{parallel}\\ \text{acceleration}}} \underbrace{- \frac{ik_\theta}{B_0}\frac{\partial}{\partial x}}_{\substack{\text{radial}\\ \text{scattering}}}. \tag{10}$$

Since the non-resonant particle are weak, we consider resonant particle only. The Eq.9 gives

$$\widetilde{f}_{\omega,k} = \frac{|e|\widetilde{\phi}}{T}\left(1 - \underbrace{\frac{\omega - \omega_*}{\omega - k_\parallel v_\parallel}}_{\text{non-Boltzmann}}\right) f_0,, \tag{11}$$

[1]If we are in torous, we need to consider the magnetic drift term $\omega_D \cdot \nabla f_{0,\perp}$.

where$\omega_* \equiv -\frac{\partial n_0}{n_0 \partial x}\frac{k_y T_e}{|e|B}$. When $\omega - \omega_* < 0$ there is an instability, where fluctuation gains more energy from from relaxing in space and loses energy by heating. We also derived non-adiabatic response

$$\widetilde{h}_{k,\omega} \simeq \frac{-i\pi(\omega - \omega_*)}{|k_\| v_{th,e}|}\frac{|e|\widetilde{\phi}_k}{T}. \tag{12}$$

This non-adiabatic response is small because the electron transit time is short $\tau_\| \equiv \frac{1}{|k_\| v_{th,e}|} \ll 1$. The non-Boltzmann effect only has short time to react to the wave, and this is why the *trapped electron effect* is important (discuss in later classes).

How does energy transfer? Originally, energy stored in electron (thermally). Gradient of electron pressure $\nabla p_e \simeq T_e \nabla n$ drives the wave, by the electron-wave resonance (only resonant electron matters). Energy in waves goes to zonal structure (i.e. zonal flow...etc.), via Reynolds stress and $\nabla \cdot J_\perp$. And ultimately energy can be dissipated by damping (nonlinear damping and turbulent frictions) and we have the ion heating (see Fig.1).

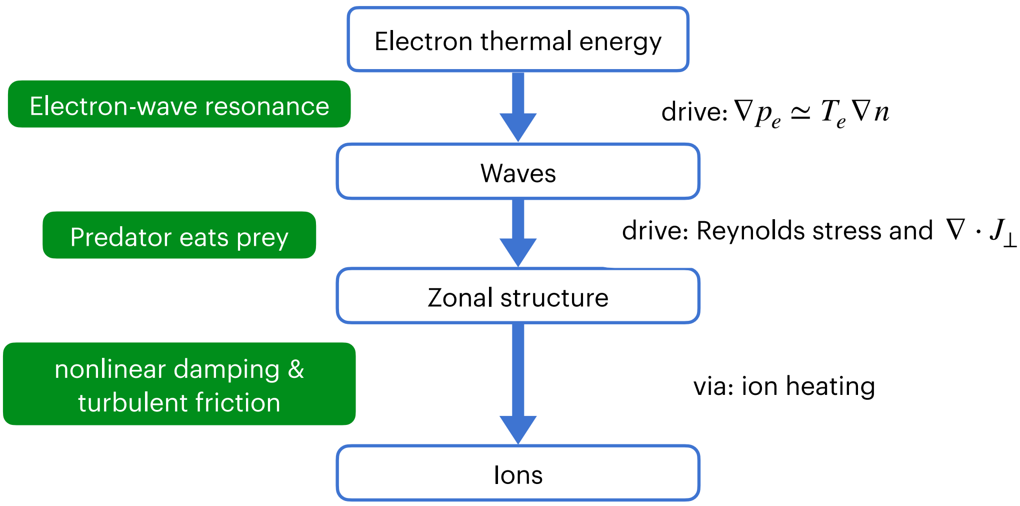

Figure 1: How energy transfer from resonant electrons to ions.

Now, we have evolution of mean distribution function

$$\frac{\partial}{\partial t} f_0 + \frac{\partial}{\partial x}\langle \widetilde{v}_x \widetilde{f}\rangle + \frac{\partial}{\partial v_\|}\langle \frac{-|e|}{m_e}\widetilde{E}_\| \widetilde{f}\rangle = 0. \tag{13}$$

From this, we can get the electron energy theorem

$$\underbrace{\frac{\partial}{\partial t}\varepsilon_e}_{\text{evol. of tot. electron kinetic energy density}} + \underbrace{\frac{\partial}{\partial x} Q_x}_{\text{divergence of turbulent electron heat flux}} - \underbrace{\langle \widetilde{E}_\| \widetilde{J}_{e,\|}\rangle}_{\text{turbulent heating/cooling of electron to waves}} = 0. \tag{14}$$

By applying volume integration this equation, we have

$$\underbrace{\frac{\partial}{\partial t}E_e}_{\substack{\text{tot. electron kinetic}\\ \text{energy}}} + \underbrace{Q_x|_{\substack{\text{fixed}\\ \text{boundary}}}}_{\substack{\text{heat flux}\\ \text{at the boundary}}} - \int d^3x\langle\widetilde{E}_{\parallel}\widetilde{J}_{e,\parallel}\rangle = 0, \tag{15}$$

where the heat flux, usually, in a fixed boundary is zero $Q_x = 0$.

1.3 Energy Theorem for waves

Similarly, we have wave energy theorem

$$\underbrace{\frac{\partial}{\partial t}\varepsilon_\omega}_{\substack{\text{evol. of tot. wave}\\ \text{energy density}}} + \underbrace{\frac{\partial}{\partial x}S_x}_{\substack{\text{div. of wave energy}\\ \text{density flux}}} + \underbrace{\langle\mathbf{E}\cdot\mathbf{J}\rangle}_{\substack{\text{wave-(tot.)particle}\\ \text{coupling}}} = 0, \tag{16}$$

where $S_x \equiv v_{gr}\varepsilon_\omega$ and $\langle\mathbf{E}\cdot\mathbf{J}\rangle$ can be decomposed into parallel and perpendicular pieces

$$\langle\mathbf{E}\cdot\mathbf{J}\rangle = \underbrace{\langle E_{\parallel}\cdot J_{e,\parallel}\rangle}_{\substack{\text{electron cooling}\\ \text{to waves}}} + \underbrace{\langle E_{\perp}\cdot J_{\perp}\rangle}_{\substack{\text{Reynolds power}\\ \text{to ZF}}}\ , \tag{17}$$

where $J_{\perp}$ is from the nonlinear polarization of ion velocity $v_i = v_{E\times B} + J_{\text{pol.}}$. And this **nonlinear polarization drift** $\langle E_{\perp}\cdot J_{\perp}\rangle$ is

$$\langle E_{\perp}\cdot J_{\perp}\rangle = \int d^3x\langle v_{E\times B}\rangle'\langle\widetilde{v}_{x,E\times B}\widetilde{v}_{y,E\times B}\rangle, \tag{18}$$

which is the second term on RHS of Eq. 1. One should notice that the Reynolds stress $\langle\widetilde{v}_{x,E\times B}\widetilde{v}_{y,E\times B}\rangle$ can be write down as vorticity flux such that $\langle\widetilde{v}_{x,E\times B}\widetilde{v}_{y,E\times B}\rangle = \langle\widetilde{v}_{x,E\times B}\nabla^2\widetilde{\phi}\rangle$. And one can find the linear response of electrical potential is in proportional to the resonance piece $\widetilde{\phi} \propto 1/-i(\omega - k_y v_{y,E\times B})$. This is very much the **Landau resonance effect**, and this is the origin of *'irreversibility'* of the DW/ZF coupling. And again, we apply the volume integration on the wave energy density equation (Eq.16) and obatin

$$\underbrace{S_x|_{\substack{\text{fixed}\\ \text{boundary}}}}_{\substack{\text{wave energy flux}\\ \text{out thru. boundary}}} + \underbrace{\int d^3x\langle E_{\perp}\cdot J_{\perp}\rangle}_{\substack{\text{output to}\\ \text{zonal flow}}} = -\underbrace{\int d^3x\langle E_{\parallel}\cdot J_{e,\parallel}\rangle}_{\substack{\text{input from}\\ \text{electron cooling}}}. \tag{19}$$

If $S_x = 0$, then we have

$$\int d^3x\langle E_{\perp}\cdot J_{\perp}\rangle + \int d^3x\langle E_{\parallel}\cdot J_{e,\parallel}\rangle = 0, \tag{20}$$

which is the competition between the electron cooling effect (energy input) and the Reynolds power exerts on zonal flow (energy output). In conclusion, there are three player in the energy transfer

- energy transfer from resonant electrons to the wave via electron cooling, and
- energy transfer from wave to zonal flow via Reynolds stress power via wave/zonal flow coupling.

See Fig.2 for details.
The same game is also in energetic particles (EP), Alfvén eigen mode (i.e. TAE), and zonal flow interaction.

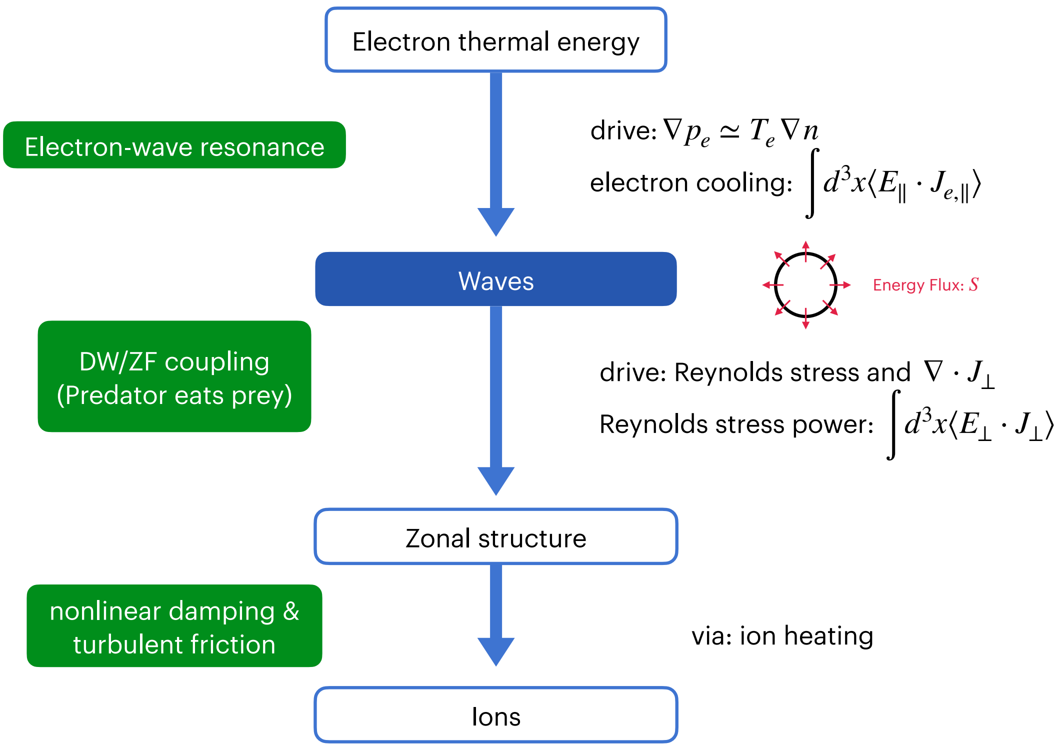

Figure 2: How energy transfer from resonant electrons to ions.

2 Evolution of Distribution Function

2.1 Quasi-Linear Equation

We start with QL equation for the electrons

$$\frac{\partial}{\partial t} f_0 = \sum_k L_k |\tilde{\phi}_k|^2 \left(\frac{i}{\omega - k_\parallel v_\parallel} \right) L_k f_0, \tag{21}$$

where L_k is the propagator in Eq.10. The above will lead us to

$$\frac{\partial}{\partial t} f_0 = \frac{\partial}{\partial r} D_{r,r} \frac{\partial}{\partial r} f_0 + \frac{\partial}{\partial r} D_{r,v_\parallel} \frac{\partial}{\partial v_\parallel} f_0 + \frac{\partial}{\partial v_\parallel} D_{v_\parallel,r} \frac{\partial}{\partial r} f_0 + \frac{\partial}{\partial v_\parallel} D_{v_\parallel,v_\parallel} \frac{\partial}{\partial v_\parallel} f_0. \tag{22}$$

This is the 2D quasi-linear equation due to two gradience in distribution function:

- in space: the radial diffusion,
- in velocity: the parallel velocity diffusion.

The propagator L_k will gives $L_k \propto (1 - (\omega - \omega_*))$ (see Eq.11). This indicates that to release the energy from the spatial gradient, the system pay the penalty for heating. The $D_{r,r}$ is basic $E \times B$ diffusivity

$$D_{r,r} = Re \left[\sum_k \frac{k_y^2}{B_0^2} |\tilde{\phi}_k|^2 \frac{i}{\omega - k_\parallel v_\parallel} \right], \tag{23}$$

where $\frac{i}{\omega - k_{\parallel} v_{\parallel}} = \pi \delta(\omega - k_{\parallel} v_{\parallel})$ for resonant particles. The $D_{v,v}$ is the velocity diffusion

$$D_{v,v} = Re\left[\sum_k \frac{|e|^2 k_z}{m_e^2} |\widetilde{\phi}_k|^2 \frac{i}{\omega - k_{\parallel} v_{\parallel}}\right]. \tag{24}$$

The cross-terms are the same

$$D_{r,v} = D_{v,r} = Re\left[\sum_k \frac{|e| k_y k_z}{B_0 m_e} |\widetilde{\phi}_k|^2 \frac{i}{\omega - k_{\parallel} v_{\parallel}}\right]. \tag{25}$$

The non-zero cross term $D_{r,v} = D_{v,r} \neq 0$ requires spectral asymmetry $\langle k_y k_z \rangle \neq 0$.

> *Homework: Do the ion drift kinetic equation and find out the cross-correlation $k_y k_z$. Recover the intrinsic equation from 2D QL equation.*

The validity of QL theory is Kubo number small $Ku < 1$

$$Ku \equiv \frac{\text{radial scattering length}}{\text{correlation length (eddy size)}} \equiv \frac{\widetilde{v}_x \tau_c}{\Delta_\perp} \tag{26}$$

$$\begin{aligned} Ku &\simeq k_y \rho_s C_s \frac{|e|\widetilde{\phi}}{T} \frac{L_n |k_x|}{\Delta k_y \rho_s C_s} \\ &\simeq \frac{k_y}{\Delta k_y} L_n |k_x| \frac{|e|\widetilde{\phi}}{T}. \end{aligned} \tag{27}$$

For the criterion of valid QL, we have $Ku < 1$ such that

$$\boxed{\frac{|e|\widetilde{\phi}}{T} < \frac{1}{k_x L_n} \frac{\Delta k_y}{k_y}.} \tag{28}$$

To examine validity of QL theory, one can also start from the auto-correlation rate and the trapping rate. Since we'll linearize our equation, the particle should behave diffusively instead of the non-linear scattering. This indicates that the auto-correlation rate larger than the trapping rate

$$1/\tau_{ac} > 1/\tau_{tr}. \tag{29}$$

The trapping rate 1D plasma is bounce frequency (see Fig.3). The auto-correlation rate can be calculated from

$$1/\tau_{ac} \simeq |\frac{d\omega}{dk_z} - \frac{\omega}{k_z}|\Delta k_z + |\frac{d\omega}{dk_y}|\Delta k_y, \tag{30}$$

if wave is non-dispersive, we'll have $|\frac{d\omega}{dk_z} - \frac{\omega}{k_z}| = 0$. Now, since we have the wave frequency only varies on poloidal direction $\omega = \omega(k_y)$, the $\frac{d\omega}{dk_z} = 0$. So we have

$$1/\tau_{ac} \simeq \frac{\omega}{k_z}\Delta k_z + |\frac{d\omega}{dk_y}|\Delta k_y \tag{31}$$

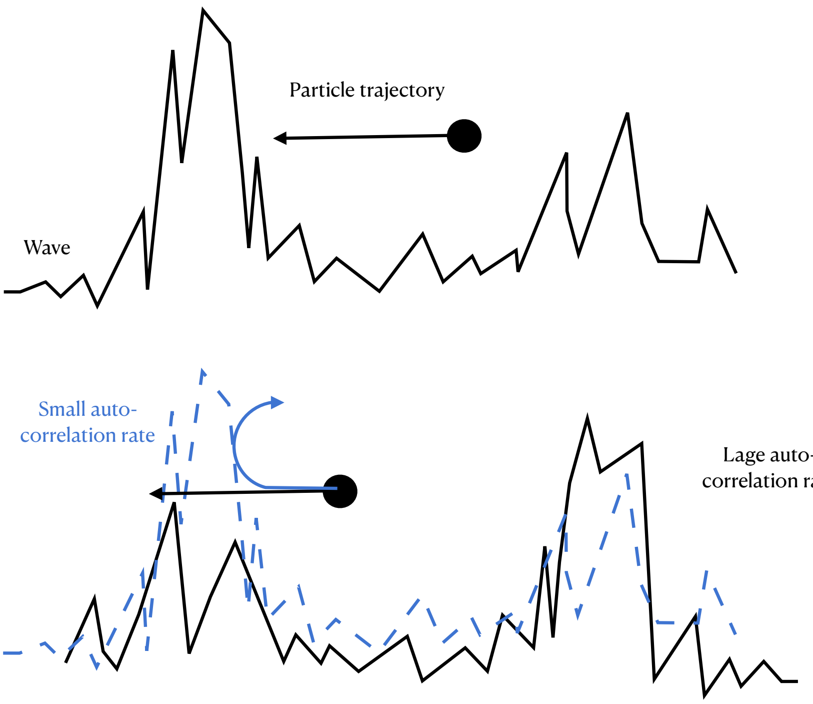

Figure 3: Auto-correlation rate. Top: wave, particle, and its trajectory at time $t = 0$. Bottom: At time $t = 1$. When the wave have fast auto-correlation rate, the particle won't be bounced. The 'trapping' of particle happens when bouncing rate is larger than the auto-correlation rate.

, where $\frac{\omega}{k_z}$ is parallel phase velocity and $|\frac{d\omega}{dk_y}| \sim v_*$, except $k_y\rho_s \simeq 1$. This is the $2D$ dimensional auto-correlation rate and more robust than that of 1D, which is dispersion sensitive. One can calculate the trapping rate

$$\begin{aligned} 1/\tau_{Tr} &\simeq k_\perp \tilde{v}_\perp \\ &\simeq \frac{k_y k_x}{B_0}\tilde{\phi} \\ &\simeq k_x C_s k_y \rho_s \frac{|e|\tilde{\phi}}{T} \end{aligned} \tag{32}$$

Hence, the condition for valid QL theory ($Ku < 1$) is

$$\boxed{\frac{\Delta k_y}{k_y}\frac{1}{k_x L_n} > \frac{|e|\tilde{\phi}}{T} \propto \frac{\tilde{n}}{n_0}}, \tag{33}$$

which reduces back to the criterion of $Ku < 1$ (see Eq.28). When we have $\Delta k_y \simeq k_y$, we obtain

$$\frac{n_0}{L_n} > k_x \tilde{n} \tag{34}$$

$$\underbrace{v_{x,0}\frac{\partial}{\partial x}n_0}_{\text{linear advection}} > \underbrace{\tilde{v}_x \nabla_x \tilde{n}}_{\text{nonlinerar advection}}, \text{ where } v_{x,0} \simeq v_{x,0}. \tag{35}$$

This leads us to the importance of **mixing length model**, where the term $1/k_x L_n$ is an *guideline* that can estimate whether the system *has large enough fluctuation to enter a* *strong nonlinear regime*. Hence, the $Ku \simeq 1$ level is equivalent to the $v_{x,0}\frac{\partial}{\partial x}n_0 \simeq \widetilde{v}_x \nabla_x \widetilde{n}$ in mixing length theory. More details of mixing length theory can be fond in Kadomtsev (1994).

Note that in the topic in tokamak, the idea of Kubo number Ku is obscure since it is the parallel streaming, instead of advections, that defines the 'response time'.

> *Homework: What is the mixing length result/estimation for ITG?*
>
> *Hint: What is mixed in ITG? Answer: ion temperature is mixed in ITG ($\frac{\widetilde{T}_i}{T_i} \simeq \frac{1}{k_x L_{T_i}}$). Start from mixing of ion temperature and then work out on potential fluctuation level (i.e. relate $\frac{\widetilde{T}_i}{T_i} \to \frac{|e|\widetilde{\phi}}{T}$).*

2.2 Saturation State

How does f_0 evolve when the growth is turned off? Consider the mean distribution function evolution again, we have

$$\frac{\partial}{\partial t} f_0 = \sum_k L_k |\widetilde{\phi}_k|^2 \left(\frac{i}{\omega - k_\parallel v_\parallel} \right) L_k f_0, \tag{36}$$

where

$$L_k = \underbrace{-ik_\parallel \frac{|e|}{m_e} \frac{\partial}{\partial v_\parallel}}_{\substack{\text{parallel} \\ \text{acceleration}}} - \underbrace{\frac{ik_\theta}{B_0} \frac{\partial}{\partial x}}_{\substack{\text{radial} \\ \text{scattering}}}$$

Taking integrate the $f_0^2/2$ over velocity and space, we obtain

$$\frac{\partial}{\partial t} \int \frac{f_0^2}{2} d^3v \cdot dr = -\sum_k |\widetilde{\phi}_k|^2 \int \left(\frac{i}{\omega - k_\parallel v_\parallel} \right) \frac{(L_k f_0)^2}{2} d^3v \cdot dr < 0, \tag{37}$$

indicating that $f_0^2/2$ will decay until

$$L_k f_0 = 0.$$

The $L_k f_0 = 0$ defines a 2D plateau. Recall that in 1 D we'll only have $\partial_{v_\parallel} f_0 = 0$, where the f_0 evolves to a flat spot. Now, we have 2D plateau due to having $\partial_{v_\parallel} f_0 = 0$ and $\partial_x f_0 = 0$. We should calculate leveled lines (curves) of constant f_0 in 2D plateau that satisfy

$$-ik_\parallel \frac{|e|}{m_e} \frac{\partial}{\partial v_\parallel} - \frac{ik_\theta}{B_0} \frac{\partial}{\partial x} \simeq 0,$$

which defines the structure of $f_0 = f_0(v_r, x)$. And we can get the level lines satisfy

$$x - \frac{k_y v_\parallel^2}{2\omega_k \omega_{c,e}} = \text{const.}, \tag{38}$$

where $\omega_{c,e} \equiv |e|B/m_e$ is cyclotron frequency of electron. This defines 2D QL plateau, i.e. can be view as a parabola in x and $v_\parallel$ space. The saturated f_0 will be a constant along the leveled curve lines. More over, we investigate the Eq.eq: 2D plateau and find that if there is an scattering in space δx, then will results in a scattering in energy $\delta v_{\parallel}^2$:

$$\delta x \simeq \frac{k_y \delta v_{\parallel}^2}{2\omega_k \omega_{c,e}}.$$

This is reflecting the fact that *transport in space will results in heating*, which can be expressed as

$$\delta x \simeq \frac{k_y \delta v_{\parallel}^2}{2\omega_k \omega_{c,e}} \tag{39}$$

$$\equiv \underbrace{\cancel{k_y} \frac{\alpha v_{e,th}^2 L_n}{\cancel{k_y} \rho_s C_s \Omega_{c,e}}}_{\text{electron adiabatic heating}} \simeq \alpha L_n \frac{v_{e,th}^2}{C_s^2} \frac{m_e}{m_i}, \tag{40}$$

The electron adiabatic heating is small since we have $\alpha \ll 1$ as the transit time $\tau_T \equiv 1/|l_{\parallel} v_{e,th}|$ is small. Hence,

$$\delta x \ll L_n.$$

This 2D plateau also imposes a **constraint on transport-heating relation**. The following helps us to sum up:

- transport accompanied by heating,
- QL plateau has weak saturation level ($\delta x \ll L_n$),
- the plateau can be reset by collisions, or external pumping or heating would avoid plateau,
- the formation of plateau is formed via zonal mode damping (by viscous friction, nonlinear heating...etc.).
- electron drift-wave is the prototype of this saturation.

3 Trapped Electron Mode (TEM)

Trapped Electron Mode (TEM) is from electron DW, with dissipation. It has separate response for trapped electron and circulating electrons. The trapping is due to the *radial variation* of the field, and occurs in the *bad curvature* region (i.e. in outer radius, see Fig.4). Trapped particles are bouncing poloidaly and toroidally in this banana region and have more perpendicular energy than toroidal. Trapped electron have narrower (and hence negligible) banana widths than those for ions. The population of trapped particle is small $n_{tr}/n_{tot} \simeq \sqrt{\varepsilon}$, where ε is the inverse aspect ratio. There are several reasons why trapped electrons are important:

- they have longer DW-electron coherence time.
- they make larger contribution to the growth—trapped electron exhibit more *fluid-like* response

Figure 4: The region that have trapped particles. Because of its shape, the region that has TEM is also called 'banana' region. Electrons and ions will bouncing back and forth in this banana region. Ion, with larger inertia, will drift in a bigger banana width a (in r-direction), i.e. $a \simeq \omega_{i,b} v_{r,mag} \propto \rho_{i,\theta}$, where $\omega_{i,b}$ is bouncing frequency of ions and $v_{r,mag}$ is radial component of magnetic drift.

- they, in some cases, can resemble the interchange (but usually irrelevant).

The story of trapped electron modes is all in timescales. The timescale ordering is

$$\underbrace{\omega_{e,b} \simeq k_{\parallel} v_{\parallel} \simeq \sqrt{\varepsilon}/\tau_{e,T}}_{\substack{\text{bouncing freq} \\ \text{for trapped particle}}} > \omega \simeq \omega_{e,*} > \underbrace{\omega_{e,D}}_{\substack{\text{magnetic drift} \\ \text{frequency}}}, \gamma_{e,e}, \tag{41}$$

where $\tau_{e,T}$ is $v_{e,th}/Rq$ and drift frequency is function of energy (i.e. $\omega_{e,D} = k_y \rho_s C_s / R = \omega_{e,D}(\sqrt{\varepsilon})$)—the *processional frequency*. The *processional frequency* describes particles drifting along toroidal field line. This is a slower process, so there is a *stronger coherence* between the wave and the particles, hence leading to the *strong electron TEM instability*. This electron TEM instability works precisely the same way the electron DW does, but is stronger. When $\omega_{e,D} > \gamma_{e,e}$, we have **collisionless TEM (CTEM)**; while if we have $\gamma_{e,e} > \omega_{e,D}$, we have **dissipative TEM (DTEM)**. If we average the bouncing back and forth, along the trapped particle orbit, Considering η, r, y_b, the Fourier transform of $k_{\parallel}$ (i.e. implication when there is a shear), radial variable (i.e. flux surface), the binormal variable (in poloidal dierction). Hence, one can have the total distribution function

$$f = \underbrace{\frac{|e|\tilde{\phi}}{T} f_0}_{\text{Boltzmann effect}} + \underbrace{\tilde{g}}_{\substack{\text{non-adiabatic of trapping} \\ \text{and circulating particles}}} \tag{42}$$

Now, we can have the electron drift-kinetic equation (DKE)

$$\boxed{-i(\omega - \omega_{e,D(\sqrt{\varepsilon})})\widetilde{g} + \underbrace{\frac{v_{\parallel}}{Rq}\frac{\partial}{\partial \eta}\widetilde{g}}_{\text{biggest term}} + \gamma_{e,e}\widetilde{g} = i\frac{|e|}{T}(\omega - \omega_{*})\widetilde{\phi} f_0,} \tag{43}$$

where $k_{\parallel}\frac{\partial}{\partial \eta}\widetilde{g}$ is $k_{\parallel}v_{\parallel}$.

To solve this electron DKE, the perturbation is a way. The zeroth order $k_{\parallel}v_{\parallel}g_0 = \frac{v_{\parallel}}{Rq}\frac{\partial g_0}{\partial \eta} \rightarrow 0$, which indicates that g_0 is independent of position (η) along the field line (i.e. $\parallel$ direction). We can have

$$g = \cancelto{0}{g_0} + g^{(1)} + g^{(2)} + \ldots \tag{44}$$

The first order equation of DKE is

$$-i(\omega - \omega_{e,D}(\sqrt{\varepsilon}))\widetilde{g_0} + \frac{v_{\parallel}}{Rq}\frac{\partial}{\partial \eta}\widetilde{g}^{(1)} + \gamma_{e,e}\widetilde{g_0} = i\frac{|e|}{T}(\omega - \omega_{*})\widetilde{\phi} f_0, \tag{45}$$

Now, one can average over the bouncing trajectory, which is along η (parallel) direction. So, we have the trajectory average operator

$$\underbrace{\int_{-\eta_b}^{\eta_b} dz \int dy}_{\text{forward}} + \underbrace{\int_{\eta_b}^{-\eta_b} dz \int dy}_{\text{backward}} \equiv \langle\ \rangle_b = \oint \frac{dl}{|v_{\parallel}|}, \tag{46}$$

where the bracket indicate the *bouncing* average that also average over a broad poloidal angle—hence, we can approximate $\langle \phi \rangle_b = \phi_0$ for simplicity. We apply this to the first order DKE (see Eq.45), we have

$$-i(\omega - \langle \omega_{e,D}(\sqrt{\varepsilon})\rangle_b)\langle g_0\rangle_b + \cancelto{0}{\langle \frac{v_{\parallel}}{Rq}\frac{\partial}{\partial \eta}\widetilde{g}^{(1)}\rangle_b} + \gamma_{e,e}\langle g_0\rangle_b = i\frac{|e|}{T}(\omega - \omega_{*})\langle\widetilde{\phi}\rangle_b f_0, \tag{47}$$

where the second term on LHS becomes zero because the forward and backward integration cancels for $v_{\parallel}$ will change its sign.

$$\langle g_0\rangle_b = i\frac{|e|}{T}\left[\frac{(\omega - \omega_{*})\langle\widetilde{\phi}\rangle_b f_0}{-i\omega(\omega - \langle \omega_{e,D}\rangle_b + i\gamma_{e,e})}\right] \tag{48}$$

where ω_{*} is $\omega_{*} = \omega_{*}(1 + \frac{L_n}{L_{Te}}(\varepsilon - \frac{1}{2}))$. For trapped particles $L_n/L_{Te} \gg 1$—the electron temperature gradient juices up the instability process from the temperature gradient, because the *temperature gradient favors the higher energy particle*. Eq. 48 indicates that **the response time of bounce average distribution function $\langle g_0\rangle_b$ to the bounce average potential $\langle\widetilde{\phi}\rangle_b$ is controlled by the bounce-averaged curvature drift (i.e. $\langle\omega_{e,D}\rangle_b$) or by the collisions (i.e. $\gamma_{e,e}$)**. The drift is for the CTEM case, while the collision is for the DTEM case. One should notice that both have some properties

- Both the instability from electron DW and CTEM are due to the non-Boltzmann effect. For electron drift wave, the non-Boltzmann factor is $(\frac{\omega-\omega_*}{k_\parallel v_{e,th}}$. For the CTEM, the non-Boltzmann factor is $(\frac{\omega-\omega_*}{\omega_{e,D}} e^{-R/L_n}\sqrt{\frac{R}{L_n}}\ )$.
- Drift (depends on $\omega_{e,D}$, i.e. collisionless trapped electron): the wave particle coherence time ($\tau_c \simeq 1/\omega_{e,D}$) is very long (have longer time to growth the wave). This indicates that the instability due to the collisionless TEM is stronger than the electron DW ($\tau_{tr} \simeq 1/(k_\parallel v_{e,th})$), such that $1/\omega_{e,D} > 1/(k_\parallel v_{e,th})$.
- the collisionless TEM (or even TIM) is an excellence playground for the **wave-particle resonance** effect.

Hence, the **electron DW in magnetic confinement devices usually refers to CTEM**.

3.1 Collisionless Trapped Electron Mode (CTEM)

One could derive the trapped particle density from Eq.48

$$\widetilde{n}_{Tr} \simeq \int d^3v \langle g_0\rangle_b, \tag{49}$$

while the density of circulating particle is

$$\widetilde{n}_{cir} \simeq \frac{|e|\widetilde{\phi}}{T}\frac{\omega-\omega_*}{|k_\parallel v_{e,th}|} f_0 \to 0, \tag{50}$$

for small transit frequency $\tau_{tr} \simeq 1/|k_\parallel v_{e,th}| \ll 1$.
If we combine Eq.42 and Eq.48 and ignore the circulating (un-trapped) particles, we have

$$\underbrace{\left[\frac{\omega_*}{\omega} - k_\perp^2\rho_s^2\right]\widetilde{\phi}}_{\substack{\text{diamagnetic vel.}\\ \text{and FLR}}} = \underbrace{\widetilde{\phi}}_{\substack{\text{Boltzmann}\\ \text{term}}} \underbrace{- \int_{tr} d^3v \left[\frac{(\omega-\omega_*)\langle\widetilde{\phi}\rangle_b f_0}{\omega - \langle\omega_{e,D}\rangle_b + i\gamma_{e,e}}\right]\langle\widetilde{\phi}\rangle_b}_{\substack{\text{non-Boltzmann}\\ \text{trapped electrons}}}. \tag{51}$$

By draping the collision frequency (for CTEM requires $\langle\omega_{e,D}\rangle_b > \gamma_{e,e}$), we obtain

$$\frac{\omega_*}{\omega} - k_\perp^2\rho_s^2 = \left[1 + \sqrt{\varepsilon}\frac{i\pi(\omega-\omega_*)}{|\langle\omega_{e,D}\rangle_b|} \underbrace{e^{-\omega/\langle\omega_{e,D}\rangle_b}}_{\substack{\text{resonance on the}\\ \text{fat tails of PDF}}} \sqrt{\frac{\omega}{\langle\omega_{e,D}\rangle_b}}\right], \tag{52}$$

where $\sqrt{\varepsilon}$ is the fraction of trapped particle, $\frac{i\pi(\omega-\omega_*)}{|\langle\omega_{e,D}\rangle_b|} \simeq i\pi(\omega-\omega_*)\tau_c$, the fat tail is due to small $\langle\omega_{e,D}\rangle_b$, and $\sqrt{\frac{\omega}{\langle\omega_{e,D}\rangle_b}}$ is an enhancement factor from volume integral. The fat tail demonstrate the *trade-off* that because of small $\langle\omega_{e,D}\rangle_b$, we have long time to growth the wave, but the particle will move to higher frequency in distribution function hence have a fat tail, on which particle has shorter processional timescale. Notice that for $\int d^3v$, the temperature gradient weighted more then 1D, because $\omega_* = \omega_*(1 + \frac{L_n}{L_{Te}}(\varepsilon - \frac{3}{2}))$.

Finally, we obtain the growth rate of CTEM

$$\frac{\gamma_{tr}}{\omega} \simeq \left[\frac{i\pi(\omega - \omega_*)}{|\langle \omega_{e,D} \rangle_b|} e^{-R/L_n} \sqrt{\frac{R}{L_n}} \gg \frac{\omega - \omega_*}{k_\| v_{e,th}}. \right] \tag{53}$$

This indicates that CTEM is electron DW, but more *virulent* than eDW due to long coherence time due to $\langle \omega_{e,D} \rangle_b$.

Homework: calculate (a) χ_e and D_n for CTEM via QL theory, (b) CTEM correction to ITG growth, (c) ITG-driven particle flux using CTEM response.

Note that the CTEM nonlinear evolution is prime candidate for the study of strong wave-particle interaction. The dispersion in the wave-particle resonance is from the doppler shift of the frequency ($\Delta(\omega - kv)$)

$$\Delta|(\omega - \omega_{e,D})| = \frac{d\omega}{dk_y}\Delta k_y - \Delta\langle \omega_{e,D} \rangle_b \tag{54}$$

$$= |\frac{d\omega}{dk_y} - \frac{\omega}{k_y}|\Delta k_y, \tag{55}$$

where $\frac{d\omega}{dk_y} - \frac{\omega}{k_y}$ has 1D structure because we eliminate the degree of freedom by bouncing average. Hence, we have auto-correlation time is

$$1/\tau_{ac} = |\frac{d\omega}{dk_y} - \frac{\omega}{k_y} \underbrace{|\Delta k_y}_{\text{spread of poloidal spectrum}} . \tag{56}$$

This indicates that because of bouncing average, this dispersion relation has 1D structure and is **sensitive to the dispersion**. In particular, if a system with long wavelength in y-direction (i.e. $\rho_s k_y \ll 1$ or with TIM), the coherence time will become very long and will lead to the strong resonance effect (i.e. dispersion becomes weak) and strong wave-particle interaction.

3.2 Dissipative Trapped Electron Mode (DTEM)

In DTEM, we have collision rate larger than the processional drift rate

$$\gamma_{e,e} > \langle \omega_{e,D} \rangle_b, \tag{57}$$

High collision The growth rate of DTEM is

$$\frac{\gamma_{tr}}{\omega} \simeq \left[\frac{i\pi(\omega - \omega_*)}{\gamma_{e,e}} \right] \simeq 1/n_0. \tag{58}$$

Recall that before the experiment of ITG is conducted, the poster child of Alcator C pellet experiment in 1984. When they shot in pellets, they recovered the linear scaling in confinement. Probably what they were seeing is the recover of the $1/n$ scaling. They peak the n profile and hence kill the ITG. This is probably the high density (after the peaking) drove the system in to the DTEM regime, and hence gives the $1/n$ type of

growth. The relation $\chi_e \propto 1/n$ in heat pulse propagation (HPP) post-pellet experiment might just the DTEM in action.
The competition between CTEM and DTEM is extremely sensitive to the temperature, because

$$\langle \omega_{e,D} \rangle_b \propto T^{1/2} \tag{59}$$

$$\gamma_{e,e} \propto T^{-3/2}, \tag{60}$$

indicating that a small temperature change can knock the system from one regime to the other, since $\langle \omega_{e,D} \rangle_b / \gamma_{e,e} \simeq T^2$.

4 Geometry and Dynamics in Tokamak

Until now, we have been talk about physics of plasma in *homogeneous* media in a slab. *Homogeneous* refers to modes like **k**. However, in fusion plasma (i.e. tokamak) has highly non-trivial geometry which enters physics! The different aspect of geometry have a compatibility issues. In this chapter, we'll talk about the geometry in a sense of magnetic field and $E \times B$ effect, on an equal footing.
The elements in geometry are

- Resonances and its surface. There can be spatial resonances (magnetic geometry) and also the wave-particle resonances (which involves in $E \times B$ shear).
- Shears. Shear will affect how you represent modes. Shear can be (a) magnetic, leading us to quasi-modes (or so called twisted slicing modes) which is an ancestor of ballooning mode. or can due to (b) $E \times B$ shear.
- Toroidicity (or so called ballooning). It is coupled block modes—an analogy of this is the modes of chain of a spring. The toroidicity is related to trapped particle.

A paper from Roberts & Taylor (1965) focuses on the central physics of the magnetic shear and how it works. Connor et al. (1978) discuss the ballooning formalism. Goldreich & Lynden-Bell (1965) investigate velocity shearing and shearing coordinates. Rhines & Young (1982) discuss the interaction of fluctuation (diffusive scattering) and shears.

4.1 Resonances

Demand the poloidal and toroidal periodicity, we have

$$\widetilde{\phi} = \widetilde{\phi}_{m,n}(r)e^{i(m\theta n\phi)}, \tag{61}$$

where m and n are poloidal and toroidal mode number and m/n is the pitch of the mode. The field lines should have their pitch too, because they wind. The equation of magnetic field line is

$$\frac{dr}{B_r} = r\frac{d\theta}{B_\theta} = \frac{Rd\phi}{B_z}, \tag{62}$$

where B_r, B_θ, and B_z are radial, poloidal, and toroidal magnetic field, respectively. We also have

$$\theta = \phi/q(r) + \theta_0, \tag{63}$$

where q is the **safety factor**, a variable defines the winding rate (i.e. the pitch) of the magnetic field lines

$$q(r) = \frac{rB_z}{RB_\theta(r)}. \tag{64}$$

The $q = q(r)$ leading to the magnetic shear because pitches varies radially. At the region where *the pitch of magnetic field lines matches the pitch of the mode* is by definition **rational surfaces** (RS). At rational surface, we have

$$q(r) = \frac{m}{n}. \tag{65}$$

At the rational surface, $k_\parallel \to 0$. This can be derived

$$k_\parallel = \frac{\mathbf{k}\cdot\mathbf{B}}{|B_0|} = \frac{m}{q(r)} - n, \tag{66}$$

And when at rational surface, we obtain

$$k_\parallel|_{@RS} = (\frac{m}{q(r)} - n)|_{m=qn} = 0, \tag{67}$$

$$\mathbf{k}\cdot\mathbf{B} = 0. \tag{68}$$

The properties, due to the fact that $k_\parallel \to 0$, for the RS are

- there is **no Landau damping** at RSs. This is because that the resonant factor

$$\frac{1}{\omega - k_\parallel v_\parallel} \to \frac{1}{\omega}, \text{ since } k_\parallel \to 0. \tag{69}$$

- RSs can grow instabilities. Ideal MHD energy principal yields

$$\delta W = \int d^3x \left[\frac{(\nabla \times \xi \times B_0)^2}{4\pi} + \text{ compression term } + \nabla p \text{ term } + \text{ current term } + \text{ curvature term } . \right] \tag{70}$$

  where δW is the energy and ξ is a displacement. The $\frac{(\nabla \times \xi \times B_0)^2}{4\pi} > 0$ means it is stable, and is proportional to $B_0^2 k_\parallel^2 \xi^2$. This indicates that **there is no penalty for bending field lines perpendicularly**—$\delta W > 0$ when bending a field line to with a displacement ξ.

- most of modes are related to/situated at RSs.

- Tearing, reconnection can occur at RSs. And this is also where magnetic islands form. At low q (i.e. $q = 3/2, 5/3$), tearing and NTM (MHD instability that leads to disruptions) can occur.

- At RSs, resonances are coupled to toroidicity.

- At RSs, resonances overlap and creates stochasticity and chaos.

Hence, a lot of fun can happen at RSs.
When study the physics in a tokamak, it is important to know what happen *along the total magnetic field lines*. Hence, an operator

$$\mathbf{B}\cdot\nabla \tag{71}$$

is important. This is because that **everything in magnetic confinement is sort of a variant on shear ALfvén wave**. Hence, the operator $\mathbf{B}\cdot\nabla$ that is from shear Alfvén wave (i.e. $\omega = k_\parallel^2 v_A^2$) is of importance. It can also appear as kinetic version, i.e. $v_\parallel \hat{n}\cdot\nabla$, where $\hat{n} = \mathbf{B}/|B|$. Moreover, this operator can also be expressed as a Vlasov-type operator

$$\mathbf{b}\cdot\nabla = \frac{\partial}{\partial z} + \frac{1}{Rq(r)}\frac{\partial}{\partial\theta} + \underbrace{\tilde{\mathbf{b}}\cdot\nabla}_{\substack{\text{perturbation}\\ \text{term}}} , \tag{72}$$

where b indicate the normalized magnetic field $\mathbf{b} = \mathbf{B}/|B|$ and the second term $\frac{1}{Rq(r)}\frac{\partial}{\partial\theta}$ indicates that the magnetic shear defines the a variable of winding rate. This operator analogies to

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \langle v_{E\times B,y}\rangle \frac{\partial}{\partial y} + \underbrace{\widetilde{\mathbf{v}}\cdot\nabla}_{\text{vel. scattering term}} \quad . \tag{73}$$

Both Eq.72 and Eqq.73 are important operator in physics of fusion plasma. These two operators also show that *the velocity shear and geometry of magnetic fields are closely linked.*
Noted that the velocity shear can enter the particle resonance

$$\frac{1}{\omega - k_{\parallel}v_{\parallel}} \rightarrow \frac{1}{\omega - k_y\langle v_{E\times B,y}\rangle - k_{\parallel}v_{\parallel}}. \tag{74}$$

4.2 Shears

The importance of shears is that they introduce a variable for winding of magnetic filed and flow rate. And that says the excitation have to twist. These *twists* affect the coherence of fluctuation in space and time—**fluctuations want to align with the shear.** But with demands of periodicity, the fluctuation cannot really align with the shear. The representation of fluctuations encounters difficulty because the direction of the flow or the direction of magnetic fields changes as a function of position due to the shear. Hence, the **shearing coordinates**, a natural way to describe fluctuation in shears, are critical to describe fluctuations. However, in this coordinate, we will loose normal mode description. Instead, we are taking about shearing 'quasi-modes' in the shearing coordinate. But should we care? The fixation of eigen-modes is artifact of linear theory; however, in real world, there is no linear theory. Hence, the shearing coordinate are useful.
We start in a cartesian coordinate where x, y, and z represent radial, poloidal, and toroidal direction, respectively. We have Vlasov equation

$$(\underbrace{\frac{\partial}{\partial t} + \langle v_{E\times B,y}\rangle \frac{\partial}{\partial y}}_{\text{shearing term}} + \widetilde{\mathbf{v}}\cdot\nabla)\rho = 0, \tag{75}$$

and we want to **eliminate the shearing term**.

4.2.1 Shearing Coordinate (for flow)

The method is to change the coordinate to a coordinate that are co-moving with the shear flow in velocity $\langle v_{E\times B,y}\rangle$. Hence, we have shearing coordinate (or tilting coordinate). So we define that in shearing coordinate (x', y', z') from old coordinate (x, y, z)

$$x' = x \tag{76}$$

$$y' = y - \langle v_{E\times B,y}\rangle' \cdot t \tag{77}$$

$$z' = z, \text{ and} \tag{78}$$

$$t' = t. \tag{79}$$

And the derivatives in shearing coordinate is

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}-\frac{\partial\langle v_{E\times B,y}\rangle'}{\partial x}t\frac{\partial}{\partial y'} \tag{80}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial y'} \tag{81}$$

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}. \tag{82}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t'}-\langle v_{E\times B,y}\rangle'\frac{\partial}{\partial y'}. \tag{83}$$

Hence, in shear coordinate, we have

$$\frac{\partial}{\partial t}+\langle v_{E\times B,y}\rangle' x\frac{\partial}{\partial y}=\frac{\partial}{\partial t'}\ \text{(easy operator)}, \tag{84}$$

indicating that **shearing coordinate eliminates fast variation (shearing)** term. In this shearing coordinate, the fast shearing mode is absorbed in to the *phase* of shear flow, i.e.

$$e^{i\mathbf{k}\cdot\mathbf{x}}\to e^{i\mathbf{k}'\cdot\mathbf{x}'-ik_y't\partial_x\langle v_{E\times B,y}\rangle'} \tag{85}$$

This can also be illustrated in Eikonal equation

$$\frac{\partial}{\partial t}k_x=-\frac{\partial}{\partial x}(\omega+\mathbf{k}\cdot\mathbf{v}). \tag{86}$$

Here, $k_x=k_x^{(0)}-k_yt\partial_x v_{E\times B,y}$, which matches Eq.105. Moreover, the physical picture of the term

$$k_x=k_x^{(0)}-k_yt\partial_x v_{E\times B,y} \tag{87}$$

is the **shear tilting and thinning effect** (see Week 4 Lecture note). This indicating that the shearing coordinate follow the eddy as it is tilted in a shear flow, and naturally account for the thinning and during the tilt—**the shearing coordinate is natural to describe how the eddy is tilted in the flow**.

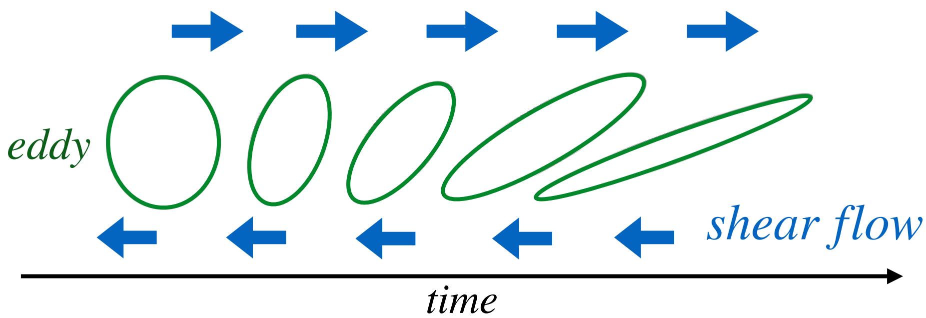

Figure 5: Shear-eddy tilting feedback loop. The $E\times B$ shear generates the $\langle k_xk_y\rangle$ correlation and hence support the non-zero Reynolds stress. The Reynold stress, in turn, modifies the shear via momentum transport. Hence, the shear flow reinforce the self-tilting.

Now, how to describe the diffusion/advection in a shear flow? One can write down

$$\left[\frac{\partial}{\partial t}+v_{E\times B,y}\frac{\partial}{\partial y}+D(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\right]c=0, \tag{88}$$

where $c = c_0 e^{i\mathbf{k}\cdot\mathbf{x}}$ is an arbitrary parameter. One can find that

$$c \propto exp\left[\frac{-k_y'^2 t^3 (\partial_x v_{E\times B,y})^2 D}{3}\right], \tag{89}$$

indicating that the shear enhance the decorrelation decay. In other words, *the shear stretch the eddy and amplify the effect of diffusion*. More details can be font in Goldreich & Lynden-Bell (1965).

4.2.2 Twisted Slicing Coordinate (for B-field)

Now, we are considering the shear of magnetic fields. This topic has been studied in the paper of Roberts & Taylor (1965). The shearing in magnetic fields creates 'twisted slicing modes'. Recall that the operator

$$\nabla_{\parallel} = \frac{\partial}{\partial z} + \frac{1}{Rq(r)}\frac{\partial}{\partial y} \tag{90}$$

where $1/Rq(r)$ can be rewritten as

$$\frac{1}{Rq(r)} \simeq \left(\frac{r}{Rq(r_0)} + \frac{x}{L_s}\right)\frac{\partial}{\partial y}. \tag{91}$$

In above equation, $1/L_s$ is magnetic shear length that can be defined as

$$\frac{1}{L_s} \simeq -\frac{\hat{s}}{Rq}, \tag{92}$$

where $\hat{s} \equiv r\partial_r q/q$ is the **shear parameter**. Hence, the parallel operator can be written as

$$\nabla_{\parallel} = \frac{\partial}{\partial z} + \frac{x}{L_s}\frac{\partial}{\partial y} \tag{93}$$

And know, we can have shearing coordinate to describe the shearing magnetic field. We will do the same derivation as in the shear-flow coordinate. The only difference is the tilting is in x and z direction, instead of x and t in the shear-flow coordinate. Hence, the relation between the shear-flow and mag.-shear coordinate is

$$t \rightarrow z \tag{94}$$

$$x/L_s \rightarrow \frac{\partial \langle v_{E\times B,y}\rangle'}{\partial x} t \tag{95}$$

The 'twisted slicing' coordinate is

$$x' = x \tag{96}$$

$$y' = y - \frac{x}{L_s} z \tag{97}$$

$$z' = z, \text{ and} \tag{98}$$

$$t' = t. \tag{99}$$

And the derivatives in 'twisted slicing' coordinate is

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} - \frac{x'}{L_s}\frac{\partial}{\partial y'} \tag{100}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \tag{101}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - \frac{x'}{L_s}\frac{\partial}{\partial y'} \tag{102}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}. \tag{103}$$

Hence, in 'twisted slicing' coordinate, we have

$$\frac{\partial}{\partial t} + \frac{x}{L_s}\frac{\partial}{\partial y} = \frac{\partial}{\partial z'} \text{ (easy operator)}. \tag{104}$$

The 'twisted slicing' coordinate annihilate the leading behavior of $\nabla_{\parallel}$. Similarly, we can obtain the rapidly evolving phase factor

$$e^{i\mathbf{k}\cdot\mathbf{x}} \rightarrow e^{i\mathbf{k}'\cdot\mathbf{x}' - ik_y'\frac{x}{L_s}z}. \tag{105}$$

This is where the idea of ballooning representation comes from. And similarly, the 'effective' Eikonal equation

$$\frac{\partial}{\partial z}k_x = -\frac{\partial}{\partial x}(k_y'\frac{B_y}{B_0}), \tag{106}$$

indicating that the 'effective' Eikonal k_x evolving in z. And the magnetic diffusion (D_M) can be found

$$\left[\frac{\partial}{\partial z} + \frac{x}{L_s}\frac{\partial}{\partial y} + D_M(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\right]c = 0, \tag{107}$$

where c becomes

$$c \propto exp\left[\frac{-k_y'^2 z^3 D_M}{3L_s^2}\right], \tag{108}$$

indicating that the shear enhances the decoorelation (amplifies the magnetic diffusion). Then, one can have the decoreelation rate

$$1/\tau_c = \left(\frac{k_y^2 v_{e,th}^2 D}{3L_s}\right)^{1/3}. \tag{109}$$

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