TURBULENCE IN TOROIDAL SYSTEMS

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INTRODUCTION

Toroidal systems represent one of the traditional approaches in research carried out for the purpose of producing controlled thermonuclear reactions. One typical system is the toroidal discharge (Zeta, Tokomak, etc.) in which the plasma is confined, in the final analysis, by the magnetic field associated with the longitudinal current; another system is the stellarator, in which confinement is achieved in the absence of a current.

Unfortunately, the experimental results have shown that plasma confinement in toroidal systems is significantly poorer than would be expected on the basis of classical considerations. In experiments that have been carried out up to the present time on many devices over a wide range of plasma parameters, it has been found that even in the absence of macroscopic plasma instabilities there are so-called "anomalous-diffusion" mechanisms that lead to the relatively rapid loss of plasma particles and energy. This fact has served to stimulate intensive theoretical research which has shown that certain effects that had not been considered earlier, i.e., collisions and the finite value of the Larmor radius, can cause a wide class of dissipative and drift instabilities. At the present time, a fairly complete linear theory has been developed to describe the instabilities characteristic of an inhomogeneous plasma. Therefore, it is now appropriate to evaluate the ultimate threat posed by instabilities as far as plasma confinement in toroidal systems is concerned. An investigation of plasma stability in the linear approximation could resolve this problem and the answer would be most satisfactory – if it could be shown that it is possible to build a toroidal system which is free from instabilities. Unfortunately, the hope of producing such a system in practice has all but vanished. However, the growth rates for many instabilities are very small and hence one might expect the averaged macroscopic effects associated with the enhanced diffusion and thermal conductivity (due to these instabilities) to be relatively small. In other words, giving up the idea of achieving absolute stability one can still proceed with the purpose of finding those conditions for which instabilities exist, but are not serious. In order to carry out this program, first of all it is necessary to know the averaged macroscopic particle flux and heat flux produced by the instabilities. In other words, it is necessary to investigate macroscopic effects associated with plasma turbulence. The present review is devoted to this topic.

Although most of the results that are obtained have a broader range of applicability, many of the actual analyses have been carried out using the example of a circular torus with axial symmetry in which a longitudinal current flows, i.e., a system such as Tokomak. This approach is due not only to the great simplicity of such a system, but also to the fact that it has certain advantages as compared with more complicated configurations. The chief of these advantages is the possibility of obtaining appreciable shear by virtue of the large curvature, i.e., the rather large ratio of the minor radius of the torus a to the major torus R_0. Another advantage is the fact that the mean magnetic field exhibits a minimum at the magnetic axis in systems of this kind.

In §1 we consider briefly the question of plasma equilibrium in a torus and introduce coordinates that will be found convenient for the further analysis; these coordinates become the usual cylindrical coordinates if the torus is "straightened," i.e., when $R_0 \to \infty$. The following four sections are devoted to an investigation of the various instabilities that are characteristic of a plasma in a toroidal system. These sections contain some new results: for example, an instability due to trapped particles and an instability associated with finite orbits; in the main, however, these sections are of a review nature in that many of the instabilities that are described have been investigated earlier. On the other hand, the analysis of these instabilities is somewhat different from the one that is generally used, since it is closely related to the subsequent analysis of the nonlinear oscillations produced by the instabilities.

Nonlinear plasma phenomena in toroidal discharges are investigated in §§6-10. In §7 we investigate the plasma convection that develops as a result of the current-convective instability. Although the associated anomalous thermal conductivity is not important at high temperatures (above 50 eV), we investigate the nonlinear convection in detail since this concrete example can be easily generalized to demonstrate the general nature of nonlinear phenomena in toroidal systems. Specifically, as we show in §7, the convection that develops in a toroidal plasma exhibits the feature that, in addition to the random nature of the process, which is characteristic of turbulence, there are certain elements of order which play a significant role. The order is imposed by the magnetic field: in toroidal systems with shear dissipative instabilities can develop only for perturbations that are highly elongated along the magnetic field; hence, the existence of shear in the lines of force means that the perturbations are localized in the radial direction in such a way that the pitch of the perturbation follows the pitch of the lines of force as closely as possible.

The localization of the perturbations means essentially that only those perturbations interact which have a common point of localization (the point at which the pitch of the perturbation coincides with the pitch of the lines of force); this interaction leads to the formation of nonlinear convection cells in the plasma. The interaction between different cells can be characterized as follows: for a given cell, all cells of larger scale size effectively represent a complicated macroscopic flow that does not have a strong effect on the flow pattern within the cell; on the other hand, all cells of smaller scale size cause an enhanced thermal conductivity, which can be taken into account by the introduction of an appropriate thermal conductivity coefficient. This coefficient is different for cells of different size, since contributions to the thermal conductivity come only from cells which are smaller than the one being considered. The largest value of the effective coefficient of thermal conductivity then obtains in cells of the maximum scale size; obviously, this effective thermal conductivity is approximately equal to the macroscopic thermal conductivity that characterizes the plasma as a whole.

A similar pattern holds for other instabilities, and this suggests the following method for determining the effective thermal conductivity χ and diffusion D. Assume that at the outset we take account of the appropriate effects in the equations of motion and choose χ or D in such a way that the perturbations with minimum localization are neutrally stable, while all more localized solutions are damped. In this way, in considering the largest scale perturbations, we have taken account of all perturbations of smaller scale. As a result, the effective coefficients that are found, D and χ, will differ from the true values only by the contribution due to the perturbations of largest scale size. But, since D and χ are found under the assumption that the growth rates for these perturbations are close to zero, the corresponding contribution from the large-scale perturbations will be small. In other words, the values that are obtained for the effective coefficients must be close to the true values as determined by all of the perturbations. This method, which is used as the basis for the nonlinear analysis, then also yields a linear formulation with the effective coefficients included beforehand.

In view of this approach, we must change the method of investigating the instability in the linear approximation. The point here is that if the plasma is unstable, it is not meaningful to seek asymptotic solutions of the linearized equations for $t \to \infty$, since the oscillations become nonlinear very rapidly. Hence, in the presence of instabilities, the linear solutions are not treated as ends in themselves, but rather as stepping stones toward the nonlinear investigation. As such a stepping stone we frequently find it convenient to replace the localized solutions by somewhat less exact solutions in the semiclassical approximation; in this approximation the frequency of the oscillations depends on r, so that the corresponding solutions are not characteristic functions of the linear equations. In any case, in analyzing the instability in the linear approximation we find it desirable to introduce the possibility of including the effective transport processes associated with the small-scale oscillations.

Sections 2-5 contain the results of the linear stability theory. In §2 we present the stabilization conditions for hydromagnetic instabilities in toroidal systems. In subsection 1 of §2 we obtain a stability criterion for small-scale perturbations (m >> 1) which generalizes the familiar Suydam criterion to a toroidal geometry. In subsection 2, we discuss the stability of a plasma with respect to the lowest modes (screw instability).

In the first subsections of §3 we derive equations that describe the dissipative instabilities of a high-temperature plasma. We consider the current-convective, drift-dissipative, and gravitational instabilities The last cannot exist in a minimum-B system (such as Tokomak) but is included for reasons of completeness. Main interest attaches to the growth rates and the regions of localization, the basic quantities that are needed for the subsequent nonlinear theory. A distinguishing feature of the dissipative instabilities is their strong dependence on the shear θ . A small reduction in shear leads to a sharp increase in the transport coefficients associated with these instabilities.

In subsection 6 of § 3, and in subsequent subsections, we deal with the collisionless drift instability. The advantage of high shear is especially marked for this instability. In particular, the results show that only two instabilities can develop in a plasma in which $\theta \gg (m_e/m_i)^{1/2}$: these are the temperature-drift instability and another instability that has much in common with it, the electron-temperature instability; the latter leads to a much smaller macroscopic coefficient. It is shown that in equilibrium with a low plasma pressure only the electrostatic drift instabilities can develop.

In § 4 we consider instabilities associated with trapped particles. In subsections 1 and 2 we consider collision-free and collision-dominated instabilities that arise in toroidal systems by virtue of the existence of local traps for particles; these traps tend to separate the particles into two classes – trapped particles and free particles. It will be evident that this instability is most dangerous in toroidal systems with low-density plasmas. In subsection 3 it is shown that particle drift in a bumpy magnetic field leads to effects that are equivalent to transverse ion inertia, i.e., these effects can determine the spatial structure of the instabilities. This mechanism is especially important in small-shear systems (stellarator).

Finally, in §5 we present the basic results pertaining to the high-frequency drift cyclotron and ion-acoustic instabilities. A plasma is also subject to the development of a drift cyclotron instability, but this instability is relatively easily stabilized by collisions. The ion-acoustic instability with $kd_e \sim 1$ (d_e is the Debye radius) can be excited only in a nonisothermal plasma with $T_i \ll T_e$ in the presence of a longitudinal current.

The results of the nonlinear theory are summarized in § 11, in which we present equations that take account of turbulence effects such as anomalous diffusion and anomalous thermal conductivity. In conclusion, we discuss the possibility of realizing controlled thermonuclear reactions in the presence of losses due to plasma turbulence.

§1. EQUILIBRIUM

1. Equilibrium of an Ideal Plasma. Coordinate System

A large number of original research papers and reviews have considered the equilibrium state of plasma in a toroidal system. In the present review we shall consider equilibrium only to the extent to which it is necessary in carrying out the subsequent investigation of stability.

For reasons of simplicity, we consider a system such as Tokomak, i.e., a circular torus with an axis of symmetry and a strong longitudinal magnetic field H_ζ. The magnetic field due to the longitudinal current is denoted by H_ϑ^0 and it is assumed that $H_\vartheta^0 \ll H_\zeta$. Under the assumptions of ideal magnetohydrodynamics, to obtain equilibrium it is sufficient that the gradient of the plasma pressure be balanced by the force associated with the magnetic field:

$$\nabla p = \frac{1}{c}[\mathbf{jH}] = \frac{1}{4\pi}[\operatorname{curl}\mathbf{H},\ \mathbf{H}], \tag{1.1}$$

where p is the plasma pressure and $\mathbf{j}$ is the current density.

In order to simplify the analysis, we assume that the toroidal features are not important, i.e., the minor radius of the toroidal pinch a is much smaller than the major radius R_0. Following Shafranov [1], we introduce a special curvilinear coordinate system which becomes the usual cylindrical coordinate system when $R_0 \to \infty$. We denote the new coordinates by r, ϑ and ζ, and denote the usual coordinate system by r', ϑ', and z'. It is assumed that the toroidal plasma is produced by bending a cylindrical plasma of circular cross

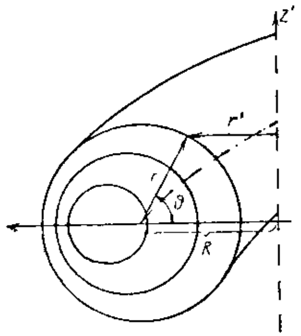

Fig. 1. Coordinate system.

section. Then, as an approximation, it can be assumed that magnetic surfaces in the cross section ϑ' = const are a system of nested circles. The radius of the circles r can conveniently be taken as one of the curvilinear coordinates, so that the equation r = const defines the magnetic surfaces. The second coordinate is taken to be the quantity ϑ, which is related to the azimuthal angle of the minor circle α (Fig. 1) by the expression $\vartheta = \alpha - \delta(\alpha)$; the small correction $\delta(\alpha)$ will be treated below. Assume that the center of the circle r = const, which represents the cross section of a given magnetic surface, is located at a distance $R = R_0 + \Delta(r)$ from the axis of symmetry (where R_0 = const, while the small quantity Δ takes account of the displacement of the magnetic surfaces due to the distortion). Then, taking the third coordinate to be the azimuthal angle ϑ', we can establish the following relation between the cylindrical coordinates r', ϑ', and z' and the new coordinates r, ϑ, and ζ :

$$\left.\begin{array}{c} r' = R_0 + \Delta(r) - r\cos\alpha = R_0 + \Delta(r) - r\cos(\vartheta + \delta); \\ \vartheta = \zeta;\ z' = r\sin\alpha = r\sin(\vartheta + \delta). \end{array}\right\} \quad (1.2)$$

If the toroidal features of the problem are weak, i.e., if $\varepsilon = a/R_0 \ll 1$, the quantities δ and Δ/a are smallness parameters of order ε. The magnitude of the displacement $\Delta(r)$ is determined, as we shall see below, by the equilibrium condition and the arbitrary quantity δ can be chosen in such a way that the lines of force of the unperturbed magnetic field are straight lines in the coordinates ϑ and ζ . Below we will take account only of first-order corrections in ε. For small values of ε the longitudinal (in the sense of a straight pinch) magnetic field (to an accuracy of order ε) is given by

$$H_\zeta = H_0\left(1 + \frac{r}{R}\cos\alpha\right) \approx H_0\left(1 + \frac{r}{R_0}\cos\vartheta\right), \quad (1.3)$$

where H_0 is the value of the field at the magnetic axis.

In this approximation, the azimuthal magnetic field can be written in the form

$$H_\vartheta = H_\vartheta^0(1 + \Lambda(r)\cos\vartheta), \quad (1.4)$$

where Λ is a small asymmetry parameter that depends on the distribution of plasma pressure and current density.

Assuming, in accordance with (1.2), that the quantity δ is a function of r and ϑ, we can now write an expression for the square of the element of length dl^2:

$$dl^2 = dr'^2 + r'^2 d\vartheta'^2 + dz'^2 = \sum g_{ik}dx^i dx^k, \quad (1.5)$$

where $dx^i = \{dr, d\vartheta, d\zeta\}$, while g_{ik} is the metric tensor; to order ε the only known nonvanishing components of this tensor are

$$\left.\begin{array}{l} g_{11} = 1 - 2\Delta'\cos\vartheta;\ g_{12} = r\left(\Delta'\sin\vartheta + r\dfrac{\partial\delta}{\partial r}\right); \\ g_{22} = r^2\left(1 + 2\dfrac{\partial\delta}{\partial\vartheta}\right);\ g_{33} = R_0^2\left(1 - \dfrac{2r}{R_0}\cos\vartheta\right), \end{array}\right\} \quad (1.6)$$

where $\Delta' = d\Delta/dr$.

The equation for the lines of force is given by

$$\frac{d\vartheta}{d\zeta} = \frac{H^2}{H^3},$$

where H^2 and H^3 are the second and third contravariant components of the magnetic field, these being given by

$$H^2 = H_\vartheta/\sqrt{g_{22}} = \frac{1}{r}H_\vartheta^0\left(1 + \Lambda\cos\vartheta - \frac{\partial\delta}{\partial\vartheta}\right); \quad (1.7)$$

$$H^3 = H_\zeta/\sqrt{g_{33}} = \frac{1}{R_0}H_0\left(1 + \frac{2r}{R_0}\cos\vartheta\right). \quad (1.8)$$

In stability investigations it is convenient to choose δ in such a way that the lines of force are straight lines in the coordinates ϑ and ζ, i.e., this quantity is chosen in such a way that the ratio H^2/H^3 is independent of ϑ and ζ. It then follows from Eqs. (1.7) and (1.8) that we must choose

$$\delta = (\Lambda - 2r/R_0)\sin\vartheta. \quad (1.9)$$

Further, using the equation

$$\operatorname{div}\mathbf{H} = \frac{1}{\sqrt{g}}\cdot\frac{\partial}{\partial\vartheta}\sqrt{g}H^2 = 0 \quad (1.10)$$

and taking account of Eqs. (1.6) and (1.7), we find

$$\Lambda = \frac{d\Delta}{dr} + \frac{r}{R_0};\qquad (1.11)$$

$$\sqrt{g} = R_0 r\left(1 - 2\frac{r}{R_0}\cos\vartheta\right),\qquad (1.12)$$

where g = Det g'_{ik}. Now, taking account of the relations in Eqs. (1.9) and (1.11) we obtain the final expression for the components of the metric tensor:

$$g_{11} = 1 - 2\Delta'\cos\vartheta;\qquad (1.13)$$

$$g_{12} = \left(\Delta'' r^2 + \Delta' r - \frac{r^2}{R_0}\right)\sin\vartheta;\qquad (1.14)$$

$$g_{22} = r^2\left[1 + 2\left(\Delta' - \frac{r}{R_0}\right)\cos\vartheta\right];\qquad (1.15)$$

$$g_{33} = R_0^2\left(1 - 2\frac{r}{R_0}\cos\vartheta\right).\qquad (1.16)$$

The lines of force are now straight in the coordinate system that has been developed:

$$\vartheta = \frac{H_\vartheta^0 R_0}{rH_0}\zeta + \text{const} = \frac{\zeta}{q(r)} + \text{const},\qquad (1.17)$$

while the magnetic field has the following contravariant H^i and covariant $H_i = \Sigma g_{ik}H^k$ components:

$$H^i = \left\{0;\ \frac{H_\vartheta^0}{r}\left(1 + 2\frac{r}{R_0}\cos\vartheta\right);\ \frac{H_0}{R_0}\left(1 + 2\frac{r}{R_0}\cos\vartheta\right)\right\};\qquad (1.18)$$

$$H_i = \left\{H_\vartheta^0\left(\Delta'' r + \Delta' - \frac{r}{R_0}\right)\sin\vartheta;\ rH_\vartheta^0(1 + 2\Delta'\cos\vartheta);\ R_0 H_0\right\}.\qquad (1.19)$$

In view of the fact that the third component H_3 and the external magnetic field contain a small correction δH_3 which arises from the azimuthal current, the electric current can be written in the form

$$j^i = \frac{c}{4\pi}(\text{curl}\,\mathbf{H})^i = \left\{0;\ j^2;\ \frac{c}{4\pi\sqrt{g}}\left(\frac{\partial H_2}{\partial r} - \frac{\partial H_1}{\partial\vartheta}\right)\right\}.\qquad (1.20)$$

It follows from the condition $\operatorname{div}\mathbf{j} = 0$ that $j^2 = I(r)/\sqrt{g}$, where I depends only on the magnetic surface.

Substituting the expression for **j** in the equilibrium equation

$$\nabla p = \frac{1}{c}[\mathbf{jH}]\qquad (1.21)$$

and carrying out an expansion in the small parameter a/R_0, in the zeroth approximation we obtain an equation that coincides with the equilibrium equation for a straight cylinder:

$$\frac{IH_0}{cR_0} = 4\pi\frac{dp^0}{dr} + \frac{H_\vartheta^0}{r}\frac{d}{dr}(rH_\vartheta^0),\qquad (1.22)$$

while the following approximation yields an equation for the quantity $\Lambda = \Delta' + r/R_0$:

$$\frac{d}{dr}\left(rH_\vartheta^2\Lambda - \frac{r^2}{R_0}H_\vartheta^2\right) = 8\pi\frac{r^2}{R_0}\cdot\frac{dp}{dr} - \frac{r}{R_0}H_\vartheta^2.\qquad (1.23)$$

For simplicity, we have omitted the subscript 0 on H and p. Then we can obtain an expression for Λ [1, 2]:

$$\Lambda = \frac{r}{R_0}\left\{1 + \frac{8\pi p}{H_\vartheta^2} - \frac{1}{r^2 H_\vartheta^2}\int_0^r (16\pi p + H_\vartheta^2)\,r\,dr\right\}.\qquad (1.24)$$

Knowing the quantity Λ we can then easily determine the displacement Δ. If it is assumed that Δ vanishes when r = b, where b is the radius of the chamber, then

$$\Delta = -\int_r^b\left(\Lambda - \frac{r}{R_0}\right)dr.$$

Thus, the relations obtained above determine uniquely the geometry of the pinch and the corresponding coordinate system for any distribution of p and H_ϑ along the radius r.

2. Drift Flows in an Equilibrium Plasma

Above we have only considered the so-called features of the equilibrium associated with the forces, i.e., we have determined the conditions for which the pressure gradient in the plasma is balanced by the magnetic field. In order to obtain a more complete picture of the equilibrium state of the plasma we must also find the thermal fluxes and the plasma flow rate (including diffusion). If the plasma is dense, in which case the mean free path λ does not exceed the length over which there is a significant change in the plasma parameters along the magnetic field $\pi q R_0$, we can apply the equations of two-fluid hydrodynamics [3] for this purpose:

$$\frac{\partial n}{\partial t} = -\operatorname{div} n\mathbf{V}_i = -\operatorname{div} n\mathbf{V}_e; \tag{1.25}$$

$$m_i n\frac{d_i\mathbf{V}_i}{dt} = -\nabla p_i + en\left(\mathbf{E}+\frac{1}{c}[\mathbf{V}_i\mathbf{H}]\right) - en\frac{1}{\sigma}\mathbf{j} - \operatorname{Div}\pi_i - \mathbf{R}_T; \tag{1.26}$$

$$m_e n\frac{d_e\mathbf{V}_e}{dt} = -\nabla p_e - en\left(\mathbf{E}+\frac{1}{c}[\mathbf{V}_e\mathbf{H}]\right) + en\frac{1}{\sigma}\mathbf{j} - \operatorname{Div}\pi_e + \mathbf{R}_T; \tag{1.27}$$

$$\frac{3}{2}n\frac{d_iT_i}{dt} + p_i\operatorname{div}\mathbf{V}_i = -\operatorname{div}q^i + Q_\Delta; \tag{1.28}$$

$$\frac{3}{2}n\frac{d_eT_e}{dt} + p_e\operatorname{div}\mathbf{V}_e = -\operatorname{div}q^e - Q_\Delta + \frac{j^2}{\sigma} + \frac{1}{en}\mathbf{j}\mathbf{R}_T. \tag{1.29}$$

In the continuity equations (1.25), the ion density and the electron density are assumed to be equal (quasineutrality); Eqs. (1.26) and (1.27) represent the equations of motion; Eqs. (1.28) and (1.29) represent the heat-balance equations; $\mathbf{V}_j$ is the mean velocity of particles of species j; m_i is the ion mass; m_e is the electron mass; $\frac{d_j}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_j\nabla$; $\sigma = \frac{e^2n\tau_e}{m_e}$ is the plasma conductivity; π_j is the viscosity tensor; $\mathbf{R}_T$ is the thermal force; $\mathbf{q}^j$ is the thermal flux; Q_Δ represents the heat exchange between the electrons and ions.

We first consider the heat-balance equations (1.28) and (1.29). As is well known, the heat flux carried by charged particles in a plasma in a strong magnetic field is made up of a transverse magnetized heat flux $\mathbf{q}_\perp = -\varkappa_\perp\nabla_\perp T$, a drift heat flux $\mathbf{q}_d = \frac{5}{2}\frac{cnT}{eH^2}[\mathbf{H}\nabla T]$ and the longitudinal heat flux $\mathbf{q}_\parallel = -\varkappa_\parallel\nabla_\parallel T$.

In the case of interest here, a plasma in which Z = 1 (Z is the atomic number) $\varkappa_{\perp i} = 9\frac{nT_i}{m_i\Omega_i^2\tau_i}$, $\varkappa_{\parallel i} = 3.9\frac{nT_i\tau_i}{m_i}$, $\Omega_i = \frac{eH}{m_ic}$, τ_i is the mean ion–ion collision time. Since the longitudinal thermal conductivity of the electrons $\varkappa_\parallel \sim n\lambda_e v_e$ is overwhelmingly large, the electron temperature can be assumed to be constant along the lines of force and thus, also on the magnetic surfaces. The ion temperature can also be regarded as constant on the magnetic surfaces to a first approximation; however, because of the drift thermal flux $\mathbf{q}_d$ the surface T = const is somewhat different from the magnetic surface, so that to a first approximation, $T_i = T_0(r) + T'(r)\sin\vartheta$, where $T' \ll T_0$. The quantity T' can be determined from the condition $\operatorname{div}(\mathbf{q}_d + \mathbf{q}_\parallel) = 0$,

and it is found that the longitudinal thermal conductivity tends to remove the departure of T from the magnetic surface caused by the drift. In the equation for $\mathbf{q}_d$ we neglect the small quantity T', and since nT_0 is a function of r only, we can write this condition in the form

$$5\frac{cnT_0}{eH^3}[\mathbf{H}\nabla H]\nabla T_0 + \frac{H_\vartheta^{0\,2}}{r^2H_0^2}\sin\vartheta\cdot\varkappa_\parallel T' = 0. \tag{1.30}$$

Here, we have made use of the relations $\nabla T_0\operatorname{curl}\mathbf{H} = 0$ and $\nabla_\parallel = \frac{H_\vartheta^0}{H_0r}\times\frac{\partial}{\partial\vartheta}$. In the gradient ∇H we need only take account of the inhomogeneity of the longitudinal field caused by the toroidal geometry of the system, so that, in the coordinates r, ϑ, and ζ we find that $\nabla H \approx \{\cos\vartheta, -\sin\vartheta, 0\}\frac{H_0}{R_0}$. Consequently, when $H_\vartheta^0 \ll H_z \approx H_0$

$$T' = -\frac{5cnT_0H_0r^2}{eH_\vartheta^{0\,2}\varkappa_\parallel R_0}\cdot\frac{dT_0}{dr}. \tag{1.31}$$

In order-of-magnitude terms we find $T' \sim T_0/\Omega_i\tau_i \ll T_0$. On the other hand, the electron temperature T_e can obviously be obtained from the expression $T_e' \sim T_0/\Omega_e\tau_e \sim (m_e/m_i)^{1/2}T_i' < T_i'$.

The existence of T' leaves the appearance of a radial component of q_{dr} in the drift flux $\mathbf{q}_d$:

$$q_{dr} = -\frac{5}{2}\cdot\frac{cnT_0}{eHr}T'\cos\vartheta. \tag{1.32}$$

Now we average (1.32) over ϑ with the weighting factor $[1-(r/R_0)\cos\vartheta]$ in order to take account of the toroidal geometry; recalling that $H \approx H_0\cdot[1+(r/R_0)\cos\vartheta]$, replacing T' by Eq. (1.31), and adding the result to $q_{\perp r}$, we obtain the following expression for the radial heat flux [4]:

$$q_r = -\frac{2nT_i}{m_i\Omega_i^2\tau_i}(1+1.6q^2)\frac{dT_i}{dr}, \tag{1.33}$$

where $q = rH_0/R_0H_\vartheta^0$ is the "stability margin" for the screw instability. The second term in the curved brackets in Eq. (1.33) takes account of the additional flux caused by the toroidal drift of the particles [5].

We now consider the equation of motion for the electrons (1.27) in which we shall neglect inertia, viscosity, and the longitudinal heat force $\mathbf{R}_{T\parallel}$

$$\nabla p_e = -en\left(\mathbf{E}+\frac{1}{c}[\mathbf{V}_e\mathbf{H}]\right) + \frac{en}{\sigma_\parallel}\mathbf{j}_\parallel + \frac{en}{\sigma_\perp}\mathbf{j}_\perp -$$

$$-\frac{3}{2}\cdot\frac{n}{\Omega_e\tau_e}\,[\mathbf{h}\nabla T_e], \tag{1.34}$$

where $\mathbf{h} = \mathbf{H}/H$, and in the frictional term we have taken account of the difference between the longitudinal and transverse conductivities $\mathbf{j}_{\parallel} = \mathbf{h}(\mathbf{h}\mathbf{j})$, $\mathbf{j}_{\perp} = \mathbf{j} - \mathbf{j}_{\parallel}$. Using Eq. (1.34), and taking account of the equilibrium equation (1.1), we find

$$\mathbf{v}_e = \frac{c}{H^2}\left[\mathbf{E} + \frac{\nabla p_e}{en},\ \mathbf{H}\right] - \frac{c^2}{\sigma_\perp}\cdot\frac{\nabla p}{H^2} + \frac{3}{2}\cdot\frac{c^2 n}{\sigma_\perp H^2}\nabla_\perp T_e + \gamma_e\mathbf{H}, \tag{1.35}$$

where the second term describes the diffusion and the third term describes the thermal diffusion in the plasma. The term $\gamma_e\mathbf{H}$ takes account of the possibility of electron flow along the lines of force. Multiplying Eq. (1.34) by $\mathbf{H}$ under the assumption that $\mathbf{H}\nabla p_e = 0$, we have

$$\mathbf{j}\mathbf{H} = \sigma_\parallel \mathbf{E}\mathbf{H}. \tag{1.36}$$

Since $T_e = T_e(r)$, the quantities $\sigma_\parallel$ and $\sigma_\perp$ are functions of r only. Taking $\mathbf{j}_\parallel = \alpha\mathbf{H}$, and assuming that under stationary conditions $E_z \approx E_0[1 + (r/R_0)\cdot\cos\vartheta]$, from Eq. (1.36) we find

$$\alpha = \frac{\sigma_\parallel}{H}\left\{E_0\left(1 + \frac{r}{R_0}\cos\vartheta\right) + \frac{H_\vartheta}{H_0}E_\vartheta\right\}. \tag{1.37}$$

On the other hand, using the equation $\operatorname{div}\mathbf{j} = 0$ and taking account of the equilibrium equation (1.1) and the relation $\operatorname{div}\mathbf{H} = 0$ we have

$$\mathbf{H}\nabla\alpha \approx H_\vartheta\frac{1}{r}\frac{\partial\alpha}{\partial\vartheta} = -\operatorname{div}\mathbf{j}_\perp = \frac{2c\,[\mathbf{H}\nabla p]\,\nabla H}{H^3}, \tag{1.38}$$

whence it follows that $\alpha = \alpha_0 + \dfrac{2cr}{H_0 H_\vartheta^0 R_0}\cdot\dfrac{dp}{dr}\cos\vartheta$. Substituting this expression in Eq. (1.37), we find

$$\alpha_0 = \frac{\sigma_\parallel E_0}{H_0};\quad E_\vartheta = \frac{2crH_0}{\sigma_\parallel H_\vartheta^{02} R_0}\cdot\frac{dp}{dr}\cos\vartheta. \tag{1.39}$$

Knowing the quantity E_ϑ we can find the radial component of the electric drift in Eq. (1.35). We can then average this quantity with respect to ϑ with the weighting factor $[1-(r/R_0)\cos\vartheta]$ (in order to take account of the increase of the surface in the outer part of the torus), thus obtaining the diffusion rate [6]

$$v_r^e = -\frac{c^2}{H^2}\left(\frac{1}{\sigma_\perp} + \frac{2q^2}{\sigma_\parallel}\right)\frac{dp}{dr} + \frac{3}{2}\cdot\frac{c^2 n}{\sigma_\perp H^2}\cdot\frac{dT}{dr}. \tag{1.40}$$

Here, the second term in the curved brackets takes account of the effect of toroidal drift on diffusion. Using the relation in (1.35) and $\operatorname{div}\mathbf{v} = 0$ it is a simple matter to find the components of the electron velocity v_ϑ^e and v_ζ^e.

The ion velocity $\mathbf{v}_i$ differs from $\mathbf{v}_e$ by the known quantity $\mathbf{j}/en_0$. If the value of $\mathbf{v}_i$ in the equilibrium equation (1.1) is large enough, it is then necessary to take account of the ion inertia, and this leads to some displacement of the surface p = const with respect to the magnetic surfaces [7].

We note further that in the presence of a temperature gradient in the plasma there will be an azimuthal electric field which is significantly larger than that given in (1.39). The point here is that if the perturbation in temperature is taken into account (1.31), it then follows from the equilibrium condition $\mathbf{H}\nabla p = 0$ that

$$2n'T_0 + T'n_0\sin\vartheta = 0,$$

where n' is the density perturbation. Since the electrons exhibit a Boltzmann distribution in the first approximation, taking (1.31) into account we find

$$E_\vartheta = -\frac{T_0}{en_0 r}\cdot\frac{\partial n'}{\partial\vartheta} = \frac{5cT_0H_0r}{2e^2H_\vartheta^{02}\varkappa_\parallel R_0}\cdot\frac{dT_0}{dr}\cos\vartheta. \tag{1.39a}$$

This quantity is $(m_i/m_e)^{1/2}$ times larger than the azimuthal field given by (1.39).

3. Particle Drift Trajectories

We now wish to consider the motion of individual particles in a torus. As is well known, the motion of the guiding center of a charged particle is described by the equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{h}v_\parallel + \frac{c\,[\mathbf{h}\nabla\varphi_0]}{H} + \frac{mc}{2eH^2}\left(v_\perp^2 + 2v_\parallel^2\right)[\mathbf{h}\nabla H]. \tag{1.41}$$

This relation together with the energy conservation relation

$$\frac{mv^2}{2} + e\varphi_0 = \text{const} \tag{1.42}$$

and the conservation of magnetic moment

$$\mu = \frac{v_\perp^2}{H} = \text{const} \tag{1.43}$$

determines the motion completely.

Here we shall consider the simple case $\varphi_0 = 0$. Furthermore, under the assumption that $\varepsilon = r/R_0 \ll q^2$, in ∇H we need only introduce the gradient of the longitudinal field $H_\zeta = H_0[1 + (r/R_0)\cos\vartheta]$. In this approximation the covariant components of ∇H are given by

$$(\nabla H)_i \approx \left\{\frac{H_0}{R_0}\cos\vartheta;\ -\frac{rH_0}{R_0}\sin\vartheta;\ 0\right\}\left(1+\frac{2r}{R_0}\cos\vartheta\right). \quad (1.44)$$

Introducing the quantity ξ, the departure from the line of force on the magnetic surface, by means of the relation $\xi = \zeta - q\vartheta$, and taking account of the approximation in (1.44), to accuracy of first order in r/R_0, we find from (1.41)

$$\frac{d\vartheta}{dt} = \frac{1}{R_0 q}\left(1+\frac{r}{R_0}\cos\vartheta\right)v_\parallel =$$

$$= \pm\frac{v}{R_0 q}\left(1+\frac{r}{R_0}\cos\vartheta\right)\sqrt{1-\frac{\mu H_0}{v^2}\left(1+\frac{r}{R_0}\cos\vartheta\right)}; \quad (1.45)$$

$$\frac{dr}{dt} = -\frac{mc}{2eH_0R_0}\left(v^2+v_\parallel^2\right)\sin\vartheta\left(1+\frac{r}{R_0}\cos\vartheta\right); \quad (1.46)$$

$$\frac{d\xi}{dt} = -\frac{mc}{2eH_0R_0r}\left(v^2+v_\parallel^2\right)\left(q\cos\vartheta+q'r\vartheta\sin\vartheta\right)\left(1+\frac{r}{R_0}\cos\vartheta\right), \quad (1.47)$$

where $q' = dq/dr$.

The departure from the line of force is small, so that the quantity r in Eq. (1.45) can be regarded as a constant. Thus, the equation for the longitudinal motion (1.45) can be solved independently of the equations for the departure from the line of force.

It is evident from the longitudinal equation (1.45) that the most important effect that arises in the transition to toroidal geometry is the appearance of trapped particles, which oscillate between the magnetic mirrors. These particles have a small longitudinal velocity $v_\parallel = \sqrt{\varepsilon}\,v$, i.e., for these particles, $\mu H_0/v^2$ is approximately unity. For this reason, the dependence of the radical on ϑ in Eq. (1.45) becomes extremely important. The weak dependence on ϑ contained in the factor $(1 + \varepsilon\cos\vartheta)$, can be neglected. We now introduce a spherical coordinate system v, ψ_π, and α in velocity space at the point $\vartheta = \pi$. Then $v_\perp^2/v^2 = \mu H_0(1-\varepsilon)/v^2 = \sin^2\psi_\pi$, and Eq. (1.45) assumes the form

$$\frac{d\vartheta}{dt} \approx \pm\frac{v}{R_0 q}\sqrt{\cos^2\psi_\pi - \varepsilon\sin^2\psi_\pi(1+\cos\vartheta)}, \quad (1.48)$$

where $\varepsilon \doteq r/R_0$.

It is then evident that for small values of $\cos^2\psi_r$ the radical can vanish for certain values of ϑ, i.e., the particles are reflected from a magnetic mirror. We now introduce an additional angle $\gamma_\pi = (\pi/2) - \psi$. Since the angle γ is small for the trapped particles, for these particles we can write as an approximation $\cos^2\psi_\pi \approx \gamma_\pi^2$, $\sin^2\psi_\pi \approx 1$. Introducing the new variable $\varkappa^2 = \gamma_\pi^2/2\varepsilon$, we can now write Eq. (1.48) in the following form for particles with small longitudinal vicinity:

$$\frac{d\vartheta}{dt} = \pm\frac{v\sqrt{\varepsilon}}{R_0 q}\sqrt{2\varkappa^2 - 1 - \cos\vartheta}. \quad (1.49)$$

Evidently, a turning point $\vartheta = \vartheta_0(\varkappa)$ occurs at $1 = \cos\vartheta_0 = 2\varkappa^2$. This turning point appears when $\varkappa < 1$. Thus, the value $\varkappa = 1$ distinguishes the trapped particles from the free particles. Using Eq. (1.49), we can now find the oscillation period of the trapped particles τ,

$$\tau = 4\frac{R_0 q}{v\sqrt{\varepsilon}}\int_{\vartheta_0}^{\pi}\frac{d\vartheta}{\sqrt{2\varkappa^2-1-\cos\vartheta}} = \frac{4R_0 q}{v\sqrt{\varepsilon}}\sqrt{2}K(\varkappa), \quad (1.50)$$

where K is a complete elliptic integral of the first kind.

For the free particles ($\varkappa > 1$),

$$\tau = 4\frac{R_0 q}{v\sqrt{\varepsilon}}\int_{0}^{\pi}\frac{d\vartheta}{\sqrt{2\varkappa^2-1-\cos\vartheta}} = \frac{4\sqrt{2}R_0 q}{v\sqrt{\varepsilon}\,\varkappa}K\left(\frac{1}{\varkappa}\right). \quad (1.51)$$

In Fig. 2 we show the dependence of the quantity $\omega_0 = 2\pi/\tau$ on $\varkappa$. For large values of $\varkappa$ the particles move along the magnetic field in essentially free fashion and $\omega_0 \approx \varkappa(v\sqrt{2\varepsilon}/2R_0q)$. When $\varkappa \to 1$, the frequency of gyration of the particles in the ϑ direction is reduced to zero and the particles are trapped, in which case ω_0 becomes the angular frequency of the oscillations between the mirrors.

Evidently, the presence of an inhomogeneity along the magnetic field causes an essential change in the nature of the particle motion. In a uniform magnetic field there can be particles that move arbitrarily slowly, these particles being capable of resonant interactions with slow waves which lead to damping when $\omega/k_\parallel v_i \to 0$; in the present case (cf. Fig. 2), there are almost no particles that move with slow average velocities, since ω_0 tends to zero logarithmically when $\varkappa \to 1$ (in other words, the number of slow particles is exponentially small). Thus, there is no reason to assume that the oscillations

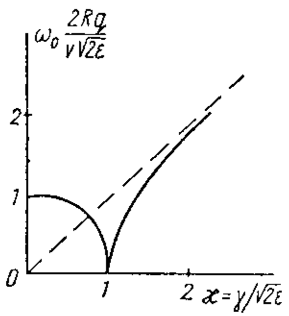

Fig. 2. Mean angular velocity of a particle in the azimuthal direction.

will be damped when $\omega/k_\parallel v_i \to 0$. Actually, as we shall see below, in toroidal geometry there is an instability that arises specifically from the presence of trapped particles. In order to treat this instability we must consider the transverse motion in addition to the longitudinal motion. This feature is the basic reason for our interest in trapped particles.

In the case of particles with small $v_\parallel$, in Eqs. (1.46) and (1.47) we can neglect $v_\parallel^2$ as compared with v^2; then, from Eq. (1.45),

$$\Delta r = \pm \int \frac{mcvq}{2eH_0\sqrt{\varepsilon}} \cdot \frac{\sin\vartheta\, d\vartheta}{\sqrt{2\varkappa^2-1-\cos\vartheta}} = \pm \frac{mcvq}{eH_0\sqrt{\varepsilon}}\sqrt{2\varkappa^2-1-\cos\vartheta}. \qquad (1.52)$$

The particle trajectory in the r, ϑ plane is shown in Fig. 3. In its motion along the magnetic field (v > 0) an ion drifts outward from the magnetic surface and in its reverse motion it drifts inward. The quantity Δr is of opposite sign for the electrons.

It will be evident from Eq. (1.52) that the displacement of the ions in the radial direction is of order $\Delta r \sim \rho_i q/\sqrt{\varepsilon}$. We assume that this quantity is smaller than a; if this condition does not hold, a significant fraction of the ions can escape to the walls even in the absence of collisions or instabilities.

For the free particles characterized by $\varkappa \gg 1$ we can assume that $v_\parallel$ = const, in which case Eqs. (1.45) and (1.46) yield the relation

$$\Delta r = -\frac{q}{2\Omega v_\parallel}(v^2+v_\parallel^2)\cos\vartheta, \qquad (1.53)$$

whence it is evident that the displacement of the free particles along r is smaller than the displacement of the trapped particles, the ratio being approximately $\sqrt{R_0/r}$ [Eq. (1.53) obviously holds only when $v_\parallel \gg v\sqrt{r/R_0}$].

In what follows we shall require the quantity $\Delta\xi$, which is the displacement of the trapped particles along the plasma (i.e., along ζ) in one oscillation period. It is evident from Eqs. (1.45) and (1.46) that this displacement can be written in the form

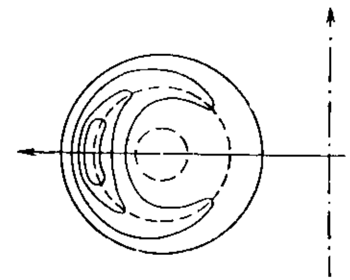

Fig. 3. Particle trajectory in the r, ϑ plane.

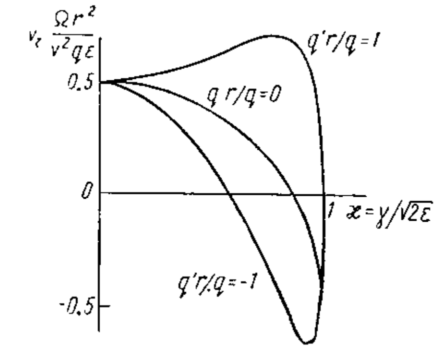

Fig. 4. Mean velocity associated with the magnetic drift as a function of the position of the turning point ϑ_0.

$$\Delta\xi \approx -2\frac{q}{m\Omega r}\frac{d}{dr}J_\parallel. \qquad (1.54)$$

Here,

$$J_\parallel = \oint mv_\parallel dl = 4\int_0^{\vartheta_0} mv_\parallel qR_0(1-\varepsilon\cos\vartheta)\,d\vartheta \approx 8\sqrt{2\varepsilon}\,qR_0 v\{E(\varkappa)-(1-\varkappa^2)K(\varkappa)\}, \qquad (1.55)$$

where $E(\varkappa)$ is a complete elliptic integral of the second kind. The quantity $J_\parallel$ is the longitudinal invariant, which is equal to twice the integral of the longitudinal momentum $mv_\parallel$ taken along the line of force between the turning points. We note that a relation such as (1.54) can be obtained in general form for any quasiperiodic motion [8] and applies to more complicated magnetic configurations.

Carrying out the differentiation in Eq. (1.54) and using (1.51) we can find v, the mean drift velocity of the trapped particles along ζ:

$$v_\zeta = \frac{\Delta\xi}{\tau} = \frac{v^2 q\varepsilon}{\Omega r^2}G(\varkappa), \qquad (1.56)$$

where

$$G(\varkappa) = G_1(\varkappa) + \frac{2q'r}{q}G_2(\varkappa) = \left(\frac{E}{K}-\frac{1}{2}\right) +$$

$$+\frac{2q'r}{q}\left(\frac{E}{K}-1+\varkappa^2\right). \qquad (1.57)$$

The dependence of v_ζ on the angle ϑ_0 for different q' is shown in Fig. 4. When ϑ_0 is close to π, the charged particle is close to the outer part of the toroidal surface r = const, i.e., the region in which the magnetic field falls off in the outward direction. When q' = 0, the particle executes a drift in this region, which is unfavorable from the point of view of stability ($v_\zeta > 0$). As the quantity ϑ_0 is reduced, the velocity v_ζ is reduced, and when $\vartheta_0 \approx 0.85$ the latter changes sign: the corresponding particles spend a large part of the time in the region $\vartheta < \pi/2$, in which the magnetic field increases outward from the magnetic surface. When $\vartheta_0 \to 0$, the velocity v_ζ tends to some finite value corresponding to the drift velocity for $\vartheta = 0$, where the particles spend the largest fraction of time.

If $q' \neq 0$, the expression for v_ζ contains an additional term which arises as follows: when $v_{\|} > 0$, the particle is located in the region $\Delta r > 0$, where the pitch of the line of force for q' > 0 is larger than the pitch at the point r, and when $v_{\|} < 0$ the particle moves in the region with smaller pitch. As a result, when q' > 0, the contribution to the drift associated with this effect is unfavorable, and when q' < 0 it is favorable.

When $q' \to -\infty$, the fraction of particles that execute unfavorable drifts approaches zero.

4. Equilibrium of a Rarefied Plasma

If the mean free path is larger than $2\pi Rq$, in its motion along a line of force a particle will not collide in one circuit around the minor azimuth and the problem of equilibrium is no longer amenable to hydrodynamic analysis. In this case the equilibrium state is described by the kinetic equations for the electrons and ions:

$$\mathbf{v}\nabla f_{0j}+\frac{e_j}{m_j}\left\{\mathbf{E}+\frac{1}{c}[\mathbf{vH}]\right\}\frac{\partial f_{0j}}{\partial \mathbf{v}}=st_j, \qquad (1.58)$$

where st is the collision term.

When the number of collisions is small, Eq. (1.58) can be solved by an expansion in st, neglecting this term in the zeroth approximation. In this case, Eq. (1.58) can be written in the form $df_{0j}/dt = 0$, where d/dt is the total derivative along the trajectory. The general solution of this equation is in the form of an arbitrary function of the integrals of motion. In other words, f_{0j} is a constant along the particle trajectories. The trajectory of an individual particle can be written in the form

$$\mathbf{r}=\mathbf{r}_0+\frac{[\mathbf{hv}]}{\Omega_j}, \qquad (1.59)$$

where $\Omega_j = e_jH/m_jc$; $\mathbf{r}_0$ is the coordinate of the center of the guiding center, and $\mathbf{v}$ is the particle velocity. As we have shown in the preceding subsection, the guiding center departs only negligibly from the magnetic surface: the displacement Δr is given by Eq. (1.53). Thus, $r = r_{00} + \Delta r + [\mathbf{hv}]_r/\Omega_j$, where r_{00} = const is the magnetic surface close to which the given particle moves. Using Eqs. (1.41) and (1.42), we find

$$f\left(\mu H_0, v^2+\frac{2e_j}{m_j}\varphi_0, r\right)=A=\text{const}, \qquad (1.60)$$

where μ, φ_0, and r are taken on a given trajectory.

In the zeroth approximation $r = r_{00}$, $\mu H_0 = v_\perp^2$, and we can take $\varphi_0(r_{00}) = 0$ without loss of generality; then, in Eq. (1.60), we find $A = f_0 \cdot (v_\perp^2, v^2, r_{00})$ (it is assumed that the equilibrium state is symmetric with respect to ϑ). In the next approximation we must substitute the following in (1.60):

$$r=r_{00}+\Delta r+\frac{[\mathbf{hv}]_r}{\Omega_j};\ \mu H_0=v_\perp^2\left(1-\frac{r}{R_0}\cos\vartheta\right);\ \varphi_0=\frac{\partial\varphi_0}{\partial r}\Delta r+\varphi',$$

where φ' takes account of the possibility of an azimuthal field. We now expand this expression and take the first term in the expansion:

$$f=f_0(v_\perp^2, v^2, r)-\frac{[\mathbf{hv}]_r}{\Omega_j}\cdot\frac{\partial f_0}{\partial r}-\Delta r\frac{\partial f_0}{\partial r}+v_\perp^2\frac{r}{R_0}\cos\vartheta\frac{\partial f_0}{\partial v_\perp^2}-$$
$$-\frac{2e_j}{m_j}\cdot\frac{\partial\varphi_0}{\partial r}\Delta r\frac{\partial f_0}{\partial v^2}-\frac{2e_j}{m_j}\varphi'\frac{\partial f_0}{\partial v^2}, \qquad (1.61)$$

where, for simplicity, we have omitted the subscript on r_{00}, the coordinates corresponding to the magnetic surface being considered.

Since we are only interested in systems in which the plasma is confined for many collision periods, the function f_0 can be taken as a Maxwellian, i.e., $f_0 = F(r)\exp(-m_jv^2/2T_j)$. Then Eq. (1.61) assumes the simpler form

$$f=f_0(v^2, r)-\frac{[\mathbf{hv}]_r}{\Omega_j}\cdot\frac{\partial f_0}{\partial r}-\Delta r\frac{\partial f_0}{\partial r}+\frac{e_j}{T_j}\frac{d\varphi_0}{dr}\Delta r f_0+\frac{e_j}{T_j}\varphi' f_0. \qquad (1.62)$$

Here, the last term corresponds to a simple Boltzmann distribution with respect to the angle ϑ, while the second term describes the so-called Larmor current. Great interest attaches to the third and fourth terms, which are associated with the motion of the guiding centers along the drift trajectory. The displacement Δr is very small for electrons, and hence these particles can be assumed to move along the magnetic surfaces, i.e., $\Delta r_e = 0$. On the other hand, for the ions we find that the displacement $\Delta r \sim q\rho_i$, and this can lead to a significant distortion of the Maxwellian function (especially in the region of trapped particles where $\Delta r \sim q\sqrt{R/r}$). If the Larmor radius ρ_i is large enough, some of the ions can be lost directly to the walls of the chamber. Under these conditions the plasma becomes charged negatively, and the equilibrium is maintained by virtue of confinement of ions to the drift trajectories of the electric field [in Eq. (1.62) this effect is described by the fourth term]. Under these conditions the plasma is confined by the electrons, and hence it is very important that there be magnetic surfaces in the system. In the general case of a toroidal equilibrium (not axisymmetric) the possibility is not excluded that there will be some distortion of the magnetic surfaces for which the ions will not feel the small fluctuations of the magnetic field, but in which electrons (in the absence of collisions) can move freely to the walls, because of random wandering of the lines of force. In this case, the plasma is not charged negatively and the containment time will be determined by the ion drift. In the presence of collisions this effect is important only if there is a strong disturbance of the magnetic surfaces.

In the next approximation in st it is possible to take account of collisional transport effects. The order of magnitude of these effects is the same as that found in subsection 2 [4, 5].

§2. HYDROMAGNETIC INSTABILITY

1. Flute Instability

If a dense plasma executes rapid motions, the magnetic field can be regarded as frozen into the plasma matter so that large-scale instabilities that are reasonably rapid can be investigated in the approximation of ideal magnetohydrodynamics. In this approximation a toroidal plasma is subject to two instabilities, the flute instability and the screw instability (cf. the review in [9]). The usual criterion for the stability of a current-carrying plasma with respect to a localized perturbation of the flute type is taken to be the Suydam criterion for a straight pinch [10]:

$$-8\pi p' < \frac{rH_z^2}{4}\left(\frac{q'}{q}\right)^2, \tag{2.1}$$

where $q = rH_z/R_0H_\vartheta$ is the so-called stability margin with respect to the screw instability and the primes denote differentiation with respect to r.

However, the condition in (2.1) will generally not apply in toroidal systems such as Tokomak, which have a strong longitudinal magnetic field, i.e., $H_z \gg H_\vartheta$. The point here is that in the final analysis the flute instability develops because of the curvature of the lines of force. In a straight discharge the radius of curvature $R_s = rH_z^2/H_\vartheta^2$ and when $H_z \gg H_\vartheta$, this quantity is appreciably greater than r. It is clear that when $R_s > R_0$ the curvature of the lines of force due to the toroidal geometry will start to play a role. This curvature changes sign as a function of the minor azimuthal angle ϑ. In the outer region the lines of force are convex, and in the inside region they are concave with respect to the plasma. Hence, for small $\beta = 8\pi p/H^2$ the curvature exhibits an effect which represents an average over ϑ; for large β there is a transition to the so-called ballooning mode [11, 12], in which the perturbation in the outer region is greater than in the inner one.

Within the framework of ideal hydrodynamics an instability starts when an initial axisymmetric equilibrium state is characterized by the possibility of formation of an equilibrium configuration characterized by a perturbed magnetic field. Hence, to find a stability criterion it is sufficient to investigate the solution of the equilibrium equation

$$\nabla p = \frac{1}{c}[\mathbf{jH}] = \frac{1}{4\pi}[\text{curl}\,\mathbf{H},\mathbf{H}] \tag{2.2}$$

and to determine under what conditions there are equilibria that are close to the axisymmetric equilibrium.

In the case being considered here, in which $H_z \gg H_\vartheta$, it is found convenient to replace Eq. (2.2) by certain relations that follow from it:

$$\mathbf{H}\nabla p = 0;\ \mathbf{j}_\perp = \frac{c\,[\mathbf{H}\nabla p]}{H^2}, \tag{2.3}$$

where $\mathbf{j}_\perp$ is the component of the current density which is perpendicular to the direction $\mathbf{H}$. The total vector $\mathbf{j}$ can be written in the form

$$\mathbf{j} = \mathbf{j}_\perp + \alpha\mathbf{H},\ \alpha = \frac{c}{4\pi H^2}\,\mathbf{H}\,\text{curl}\,\mathbf{H}. \tag{2.4}$$

Using the condition $\text{div}\,\mathbf{j} = 0$ taking account of the fact that $\text{div}\,\mathbf{H} = 0$, and making use of Eqs. (2.2)-(2.4), we find

$$\mathbf{H}\nabla\alpha + \text{div}\,\mathbf{j}_\perp = \mathbf{H}\nabla\alpha + 2c\frac{[\mathbf{H}\nabla H]}{H^3}\nabla p = 0. \tag{2.5}$$

We now consider the initial equilibrium state characterized by p^0, H_z^0, and H_ϑ^0 and the linearized equations (2.2)-(2.5) under the assumption that the quantities p', H'_z, and H'_ϑ are small. We take account of the fact that

$$H^0_\vartheta \ll H^0_z,\ p_0 \leqslant H^{0^2}_\vartheta/8\pi.$$

Since H_z is large, the perturbation H'_z can be neglected (if this is not the case, we would have a very large perturbation in the magnetic pressure $H'_z H^0_z/4\pi$). For this reason we can neglect the perturbation in H and take this quantity to be H^0_z.

If the effect of the toroidal geometry is small, the longitudinal magnetic field goes as 1/R as a function of distance from the axis of symmetry of the torus; thus, to first order in r/R_0, in accordance with Eq. (1.3), we find

$$H^0_z \approx H_0\left(1 + \frac{r}{R_0}\cos\vartheta\right),$$

where H_0 is the longitudinal field at the magnetic axis. Thus, in the first approximation in the toroidal parameter in Eq. (2.5) we can take the quantity ∇H to be $H_0 \nabla \frac{r}{R_0}\cos\vartheta$, while H^3 is replaced by H^3_0. The second term in Eq. (2.5) then takes account of the curvature of the lines of force in the toroidal geometry. In Eq. (2.4), the equation for α, we can neglect the toroidal geometry and assume, as an approximation, that r, ϑ, and ζ coincide with the cylindrical coordinates for a straight plasma.

In view of the foregoing considerations we can now write the linearized equation (2.5) in the form

$$\mathbf{H}^0\nabla\alpha' + H'_r\frac{d\alpha_0}{dr} + \frac{2[\mathbf{H}_0\mathbf{e}_x]}{H^2_0 R_0}\nabla p' + \frac{2[\mathbf{H}'\mathbf{e}_x]}{H^2_0 R_0}\nabla p_0 + b = 0, \quad (2.6)$$

where $\mathbf{e}_x = \nabla r\cos\vartheta$ is a unit vector directed along the x axis, while the last term, b, takes account of second-order terms in r/R_0.

Since ∇p_0 is directed along the radius r, the next-to-last term in (2.6) is proportional to H'_z and is negligibly small.

Since $H^0_z \gg H^0_\vartheta$, we find

$$\alpha_0 \approx \frac{1}{H_0} j_0 = \frac{c}{4\pi H_0}\cdot\frac{1}{r}\cdot\frac{d}{dr} rH^0_\vartheta, \quad (2.7)$$

while the perturbation α' satisfies the relation

$$\alpha' \approx \frac{c}{4\pi}\cdot\frac{\mathbf{H}_0}{H^2_0}\operatorname{rot}\mathbf{H}' = \frac{c}{4\pi H_0}\left\{\frac{1}{r}\cdot\frac{\partial}{\partial r} r\mathbf{H}'_\vartheta - \frac{1}{r}\cdot\frac{\partial}{\partial\vartheta} H'_r\right\}. \quad (2.8)$$

Since H'_z is small, the equation of continuity $\operatorname{div}\mathbf{H}' = 0$ assumes the form

$$\frac{1}{r}\cdot\frac{\partial}{\partial r} rH'_r + \frac{1}{r}\cdot\frac{\partial}{\partial\vartheta} H'_\vartheta = 0. \quad (2.9)$$

It is then evident that the perturbation in the magnetic field can be expressed in terms of a stream function ψ:

$$rH'_r = \frac{\partial\psi}{\partial\vartheta};\ H'_\vartheta = -\frac{\partial\psi}{\partial r}. \quad (2.10)$$

Substituting these expressions in Eq. (2.8) and using the result for α' in Eq. (2.6), we find

$$\mathbf{H}^0\nabla\Delta_\perp\psi + \frac{4\pi}{cr}\cdot\frac{dj_0}{dr}\cdot\frac{\partial\psi}{\partial\vartheta} + \frac{8\pi}{cR_0}\left\{\sin\vartheta\,\frac{\partial p'}{\partial r} + \cos\vartheta\,\frac{1}{r}\cdot\frac{\partial p'}{\partial\vartheta}\right\} + \frac{4\pi b}{c} = 0, \quad (2.11)$$

where

$$\Delta_\perp\psi = \frac{1}{r}\cdot\frac{\partial}{\partial r} r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2\psi}{\partial\vartheta^2}.$$

In order to find the quantity b, we assume that the flute instability develops from perturbations that are highly elongated along the magnetic field, in which case $\mathbf{H}^0\nabla\alpha' \to 0$, $\mathbf{H}^0\nabla\ p' \to 0$, i.e., α' and p' are slowly varying functions along the lines of force. But Ho $\mathbf{H}^0\nabla = H_0\frac{\partial}{\partial l}$, where $\frac{\partial}{\partial l}$ is the derivative along the lines of force; consequently, neglecting $\nabla\alpha_0$ we find from Eq. (2.5),

$$\alpha' = -\int\frac{\operatorname{div}\mathbf{j}_\perp}{H}dl, \quad (2.12)$$

where H is the unperturbed magnetic field. Since p' is a slowly varying function of l, we can carry out the integration in Eq. (2.12) over a bounded range of l assuming p' to be constant. If this interval corresponds to one or more orbits with respect to ϑ, then we can carry out an averaging over l and separate the required second-order terms in r/R_0. In computing the integral in Eq. (2.12) we make use of the curvilinear coordinate system introduced in § 2, in which the magnetic surfaces coincide with the coordinate surfaces r = const and in which the ratio $H^3/H^2 = q(r)$ is independent of ϑ. In this coordinate system, Eq. (2.12) can be written in the form

$$\alpha' = -\int \frac{cdl}{H\sqrt{g}} \left\{ \frac{\partial}{\partial r} \cdot \frac{1}{H^2} \left(H_2 \frac{\partial p'}{\partial \zeta} - H_3 \frac{\partial p'}{\partial \vartheta} \right) + \frac{\partial}{\partial \vartheta} \cdot \right.$$

$$\left. \frac{1}{H^2} \left(H_3 \frac{\partial p'}{\partial r} - H_1 \frac{\partial p'}{\partial \zeta} \right) + \frac{\partial}{\partial \zeta} \cdot \frac{1}{H^2} \left(H_1 \frac{\partial p'}{\partial \vartheta} - H_2 \frac{\partial p'}{\partial r} \right) \right\}, \quad (2.13)$$

where the H_i are the covariant components of the unperturbed magnetic field. Since p' is essentially constant along the lines of force, as an approximation we can write $\frac{\partial p'}{\partial \vartheta} \approx -q \frac{\partial p'}{\partial \zeta}$. We assume further that along the lines of force $\frac{dl}{H} = \frac{d\vartheta}{H^2} = \frac{d\zeta}{H^3}$, where the H^i are the contravariant components of $\mathbf{H}^0$, so that, in accordance with Eq. (1.10), the quantity $\sqrt{gH^2}$ is independent of ϑ. Then, Eq. (2.13) can be written in the form

$$\alpha' = -\frac{c}{\sqrt{g}H^2} \int \left\{ \frac{\partial p'}{\partial \vartheta} \cdot \left(\frac{1}{q} \frac{\partial}{\partial r} \frac{H_2}{H^2} + \frac{\partial}{\partial r} \frac{H_3}{H^2} \right) + \right.$$

$$\left. + \frac{\partial p'}{\partial r} \cdot \frac{\partial}{\partial \vartheta} \cdot \frac{H_3}{H^2} - \frac{\partial p'}{\partial \zeta} \cdot \frac{\partial}{\partial \vartheta} \cdot \frac{H_1}{H^2} \right\} d\vartheta. \quad (2.14)$$

Since p' varies slowly along the lines of force, the derivatives with respect to p' in the integrands can be taken to be constant and an average can be taken over ϑ. In this case, the second and third terms drop out and the first reduces to the expression

$$\alpha' = -\frac{c}{\sqrt{g}H^2 2\pi} \int \frac{\partial p'}{\partial \zeta} \cdot \left(\frac{\partial U}{\partial r} + \frac{2\pi r^2}{R_0 q^3} \frac{dq}{dr} \right) d\vartheta, \quad (2.15)$$

where

$$U = \lim_{\Theta \to \infty} \frac{2\pi}{\Theta q} \int_0^{\Theta} \frac{H_2 + qH_3}{H^2} d\vartheta = \int \frac{dl}{H}. \quad (2.16)$$

We take account of the fact here that $H_2 d\vartheta + qH_3 d\vartheta = H_2 d\vartheta + H_3 d\zeta = \mathbf{H}dl$ and $H_2 \cong rH^0_\vartheta$. When $H^0_\vartheta \ll H^0_z$, we can write $U \approx 2\pi R_0/H_0$, $\sqrt{g} = rR_0$, $d\vartheta/H^2 = dl/H_0$, so that Eq. (2.15) can be written in the form

$$\alpha' = -\int \frac{c\,[\mathbf{H}^0 \nabla p']}{H_0^3} \left(\frac{\nabla U}{U} + \frac{r^2}{R_0^2} \frac{\nabla q}{q^3} \right) dl, \quad (2.17)$$

where H_0 can be regarded as constant and equal to the value of the field at the magnetic axis, since ∇U is a second-order quantity in r/R_0. Taking account of the fact that $q^2 \gg 1$, we neglect the second term in the curved brackets in the integrand in Eq. (2.17). Differentiating Eq. (2.17) along the lines of force, we find $\mathbf{H}^0 \nabla \alpha' + b = 0$; where

$$b = \frac{cH^0\,[\nabla p' \nabla U]}{H_0^2 U}. \quad (2.18)$$

This is the desired value of the second-order terms in ϑ as averaged over the angle r/R_0 in Eq. (2.6). The calculation of this quantity by means of a direct expansion in r/R_0 would require much more extended calculations.

Now, by adding to Eq. (2.11) the linearized first equation in (2.3), i.e.,

$$\mathbf{H}^0 \nabla p' + \frac{1}{r} \cdot \frac{dp_0}{dr} \cdot \frac{\partial \psi}{\partial \vartheta} = 0, \quad (2.19)$$

we obtain two equations for p' and ψ.

We note that for a straight pinch ($R_0 \to \infty$) with $H^0_\vartheta \ll H^0_z$, by taking account of small quantities of order H^2_ϑ/H^2_z we would obtain an equation of the form (2.11) in which the last two terms would be replaced by the expression $\frac{4\pi}{cR_s} \cdot \frac{1}{r} \cdot \frac{\partial p'}{\partial \vartheta} = \frac{4\pi H^{0\,2}_\vartheta}{cH^{0\,2}_z} \cdot \frac{1}{r^2} \cdot \frac{\partial p'}{\partial \vartheta}$. It is evident that the toroidal geometry starts to play a rolé when $R_0 < R_s$. By virtue of the periodicity in ϑ and z, an arbitrary solution of the system of equations (2.11) and (2.19) can be written in the form $\psi = \psi_{mn}(\vartheta, r)\exp(im\vartheta - in\zeta)$ and similarly for p', where the functions $\psi_{mn}(\vartheta, r)$ and $p_{mn}(\vartheta, r)$ can be regarded as having a minimum number of nodes in ϑ. In other words, if ψ_{mn} and p_{mn} are expanded in Fourier series in ϑ, i.e., $\psi_{mn} = \sum_l \psi_l \exp(il\vartheta)$, then ψ_l must be a diminishing function of the index l.

Substituting this expansion in Eqs. (2.11) and (2.19), and converting from the stream function ψ to the quantity $\xi_l = \psi_l/k_l$, after eliminating $k_l = \frac{m+l}{r}\frac{H^0_\vartheta}{H^0_z} - \frac{n}{R_0}$, we obtain a system of equations for the harmonics ξ_l:

$$k_l \Delta_\perp k_l \xi_l + \frac{4\pi(m+l)}{crH_0} \cdot \frac{dj_0}{dr} k_l \xi_l - A \left\{ -\frac{2R_0}{R_s}(m+l)^2 \xi_l + \right.$$

$$+ (m+l-1)^2 \xi_{l-1} + (m+l+1)^2 \xi_{l+1} + r\frac{\partial}{\partial r}[(m+l-1)\xi_{l-1} -$$

$$\left. - (m+l+1)\xi_{l+1}] - \frac{4U'R_0}{U}(m+l)^2 \xi_l \right\} = 0. \quad (2.20)$$

Here, we have introduced the notation $A = -\frac{4\pi}{R_0 r^2 H^2} \cdot \frac{dp^0}{dr}$ and have included the small term $\sim 1/R_s$ for completeness. We are interested in localized flute perturbations which develop close to the singular point $r = r_0$ at which the longitudinal wave number $k_0 = \frac{m}{r} \cdot \frac{H_\vartheta^0}{H_z^0} - \frac{n}{R_0}$ vanishes. In the vicinity of this point we can expand k_0 in $x = r - r_0$: $k_0 = (m/r^2)\theta x$, where $\theta = r^2 q'/R_0 q^2$, $q' = dq/dr$. Furthermore, in Eq. (2.20) we can neglect the second term compared with the following terms.

The width of the localized short-wave perturbations $\Delta x \sim r/m$. As we shall see below, it is sufficient to consider the case $rq'/q \ll 1$. In this case, the singular point for the perturbation $l = 1$ is determined by the relation $k_1 = \frac{m+1}{r} \cdot \frac{H_\vartheta^0}{H_z^0} - \frac{n}{R_0} \approx \left(\frac{mq'}{q} x + 1\right) \frac{1}{r} \frac{H_\vartheta^0}{H_z^0} = 0$ and lies far beyond the limits of the localization of the perturbation, so that k_1 can be regarded as a constant equal to $H_\vartheta^0/H_z^0 r$. When $\beta_\vartheta = \frac{8\pi p}{H_\vartheta^2} \ll \frac{R_0}{r}$, a condition which is almost always satisfied, the harmonics ξ_l fall off rapidly with the harmonic number l. For this reason, in the system of equations in (2.20) we need only consider the first two harmonics. Applying the operator $\Delta_\perp$ to Eq. (2.11) for the fundamental, and then eliminating ψ_1 and ψ_{-1} by means of Eq. (2.11) for the first harmonics, we obtain the following equation for ψ_0:

$$\Delta_\perp x \Delta_\perp x \psi_0 - (\gamma + \delta)\Delta_\perp \psi_0 = 0, \quad (2.21)$$

where

$$\delta = \frac{1}{2}\left(\frac{r}{R_0} \cdot \frac{8\pi p'}{H_\vartheta^{02}}\right)^2 \left(\frac{q}{q'}\right)^2; \quad \gamma = -\frac{8\pi p'}{rH_z^{02}}\left(\frac{q}{q'}\right)^2 - \frac{4\pi p'}{H_\vartheta^{02}} \cdot \frac{U'}{U}\left(\frac{q}{q'}\right)^2,$$

where the primes denote differentiation with respect to r. In order to solve this problem, we shall find it convenient to carry out a Fourier transformation with respect to x: $\psi_0(x) = \int \psi_0(k)\exp(ikx)dk$. In the Fourier representation, Eq. (2.21) becomes

$$\frac{d^2U}{d\varkappa^2} - \frac{1 - (\gamma + \delta)(1 + \varkappa^2)}{(1 + \varkappa^2)^2} U = 0, \quad (2.22)$$

where $\varkappa = rk/m$. This equation is essentially the same as the Suydam equation [9], differing from the latter only in the fact that γ is replaced by $\gamma + \delta$. This equation will have characteristic solutions only when $\gamma + \delta > 1/4$. Thus, the stability condition for the flute instability is of the form $\gamma + \delta < 1/4$ or, in inverted form,

$$-\frac{8\pi p'}{rH_z^{02}} - \frac{4\pi p' U'}{H_\vartheta^{02} U} + \frac{1}{2}\left(\frac{8\pi p'}{H_\vartheta^{02}}\right)^2 \frac{r^2}{R_0^2} < \frac{1}{4}\left(\frac{q'}{q}\right)^2. \quad (2.23)$$

As has been pointed out in [13, 14], toroidal systems with a strong longitudinal magnetic field have a minimum of U in the sense that U increases toward the periphery of the plasma. The quantity U'/U is computed in [14], and is found to be

$$-\frac{U'}{U} = \frac{4r}{R_0^2}\left(\Lambda + \frac{5}{4}\right) + \frac{r^2}{R_0^2}\Lambda', \quad (2.24)$$

where Λ is the asymmetry coefficient of the azimuthal magnetic field [cf. Eq. (1.4)]. In general, the quantity U' is negative.

It follows from Eqs. (2.23) and (2.24) that the flute instability is stabilized in Tokomak systems when $\beta_\vartheta = \frac{8\pi p}{H_\vartheta^{02}} < 1$ and when the variation of p with r is not too strong.

2. Screw Instability

In considering perturbations with small azimuthal numbers m in Eq. (2.11) we must take account of the second term, which corresponds to the screw instability. Assuming that $\beta_\vartheta = \frac{8\pi p}{H_\vartheta^2} \ll 1$, in which case we can neglect the third term in Eq. (2.11), and using the relations in (2.18) and (2.19), we have

$$\frac{1}{r} \cdot \frac{d}{dr} r \frac{d\psi}{dr} - \frac{m^2}{r^2}\psi = V\psi, \quad (2.25)$$

where

$$V = \frac{4\pi}{c}\frac{dj_0}{dr} \cdot \frac{1}{H_\vartheta^0\left(1 - \frac{n}{m}q\right)} + \frac{4\pi U' p'}{H_\vartheta^{02} U\left(1 - \frac{n}{m}q\right)^2}. \quad (2.26)$$

Equation (2.25) is analogous to the Schrödinger equation with a potential V.

When $p' = dp/dr < 0$ and $U' < 0$, which corresponds to a minimum in H, the second term in Eq. (2.26) is positive and approaches infinity as $(r - r_0)^{-2}$ as r approaches the singular point r_0, where $1 - (n/m)q(r_0) = 0$. This strong singularity completely divides two regions : $r - r_0 > 0$ and $r - r_0 < 0$ (cf., for example, the review in [9]).

Thus, the solution that satisfies the boundary conditions $\psi(r = 0) < \infty$, $\psi(r = b) = 0$ (where b is the chamber radius) can appear only by virtue of the first term in Eq. (2.26), if the latter is negative. Since $\frac{4\pi}{c} j_0 = \frac{1}{r}\frac{\partial}{\partial r} rH_\vartheta^0$, the first term is of order $1/r^2$, and for small m (say m = 1, 2) the instability is determined by the range of r. However, if the stability margin q is appreciably greater than unity, when $n/m \sim 1$, the potential $V \sim 1/r^2q^2$ is small and the instability must vanish for small values of m. In this case, the instability can develop for a rapid change of current density j_0 with radius r, that is, if the quantity dj_0/dr becomes sufficiently large in some region. The instability is also favored by values of m and n for which $1-(n/m)q$ is small in a region in which j_0 changes rapidly, i.e., in a region that contains a singular point $r = r_0$ corresponding to the point at which $1-(n/m)q$ vanishes. Near the singular point we can write $1-(n/m)q = -(q'/q)x$, $x = r-r_0$, i.e.,

$$V = \frac{A}{x} + \frac{B}{x^2},$$

where

$$A = \frac{4\pi}{c} \cdot \frac{dj_0}{dr} \cdot \frac{q}{q'} \cdot \frac{1}{H_\vartheta^0}; \quad B = \frac{4\pi U' p' q^2}{H_\vartheta^{0\,2} U q'^2}. \tag{2.27}$$

If $\beta_\vartheta \ll 1$, then B is small in the second term in V and can only cause ψ to vanish when $x = 0$, but has no effect on the potential in the main region. For large values of m, which correspond to the region in which the difference $1-(n/m)q$ vanishes, the instability is localized. In this case, r can be replaced by r_0 and Eq. (2.25) assumes the form of a Schrödinger equation with a Coulomb potential

$$\frac{d^2\psi}{dx^2} = \left(\frac{m^2}{r_0^2} + \frac{A}{x}\right)\psi. \tag{2.28}$$

From the condition that the first level must vanish, which for $A < 0$ corresponds to the solution $\psi = x \exp\left(-\frac{m}{r_0}x\right)$, we find the stability condition

$$\frac{4\pi r_0}{cH_\vartheta^0}\left|\frac{dj_0}{dr}\right| < 2\left|\frac{q'}{q}\right| m. \tag{2.29}$$

For large values of m this condition can easily be satisfied if the current density varies sufficiently smoothly.

Thus, if the current density exhibits a smooth profile, and if the stability margin $j_0(r)$ is large enough, the screw instability can also be stabilized. Consequently, we may assume that hydromagnetic instabilities are not dangerous.

§3. DRIFT AND DISSIPATIVE INSTABILITIES

1. Choice of Parameters and Localization Width for the Perturbations

The instabilities we shall analyze below are characterized by being highly elongated along the lines of force and occupy a localization region that is much smaller than the radius of the plasma. For this reason, we can carry out a simplification by replacing the toroidal plasma by a straight pinch of length $L = 2\pi R$ and radius a surrounded by an ideally conducting chamber of radius b. The toroidal feature can be simulated by joining the ends. We introduce a cylindrical coordinate system r, ϑ, and z in which the z axis is along the cylinder axis. We assume that the longitudinal magnetic field H_z is strong and that the quantity $q = rH_z/RH_\vartheta$, whose value at the edge of the plasma is sometimes called the stability margin for the screw instability, is between 1 and 10. Since the ratio a/R cannot exceed $^1/_2$-$^1/_3$ because of geometric and constructional considerations, the ratio $H_\vartheta/H_z \sim 10^{-1} \ll 1$. The quantity $\iota = \frac{2\pi}{q} = \frac{LH_\vartheta}{rH_z}$ represents the rotational transform, i.e., the angle through which the line of force rotates in azimuth in a length L. The rate of change of the angle ι with respect to radius characterizes the shear of the lines of force. We shall find it convenient to introduce the quantity $\theta = \frac{r^2}{R} \cdot \frac{d\iota}{dr}$, which we will simply call the shear. We shall assume at the outset that

$$\theta > \left(\frac{m_e}{m_i}\right)^{1/2}. \tag{3.1}$$

This condition, as will be shown below, means that it is not necessary to consider the collisionless drift instability in which $k\rho_i \gtrsim 1$, where k is the wave number, $\rho_i = \sqrt{\frac{T}{m_i\Omega_i^2}}$, $\Omega_i = \frac{eH}{m_ic}$ (according to some estimates [15], these instabilities should lead to a diffusion characterized by a coefficient $D \sim \sqrt{\frac{m_e}{m_i\beta}}\rho_i v_i$ when $\beta = \frac{8\pi p}{H^2} > \frac{m_e}{m_i}$, where $v_i = \sqrt{\frac{2T}{m_i}}$). In what follows, in making estimates we will frequently assume that $\theta \sim 10^{-1}$.

From the equilibrium condition with respect to the major radius in a toroidal system it follows that $\frac{a}{R}\beta < \frac{H_\vartheta^2}{H^2} < \theta^2 \ll 1$, and we assume below that

$$\beta < \theta^2 \frac{R}{a}. \tag{3.2}$$

This condition means that we can neglect nonelectrostatic features of the drift waves.

We shall limit ourselves to the case of a strong magnetic field:

$$\frac{\rho_i}{a} < \left(\frac{m_e}{m_i}\right)^{1/2}. \tag{3.3}$$

In investigating plasma stability it is natural to guide our thinking by the conditions which hold in existing toroidal devices or in future thermonuclear devices. For the first case we have $T \sim 10\text{-}100$ eV, $n \sim 10^{13}$ cm^{-3}, $a \sim 10$ cm, $H \sim 10^4$ g: for the second case we have $T \sim 10$ keV, $n \sim 10^{15}$ cm^{-3}, $a \sim 10^2$ cm, $H \sim 10^5$ э. In each case the mean free path $\lambda_e \approx 3 \cdot 10^{-12} \cdot (T^2/n)$ for Coulomb collisions is large, so that we can limit ourselves to the region

$$\frac{\lambda_e}{a} > 1.$$

In what follows it will be found convenient to use another dimensionless parameter to characterize the collisional environment $S = \lambda_e \rho_i / a^2$. This parameter indicates the ratio of the ion gyrofrequency $v_i \rho_i / a^2$ to the collision frequency $\nu_i \approx v_i/\lambda_i \approx v_i/\lambda_e$. We shall assume that

$$S = \frac{\lambda_e \rho_i}{a^2} > \sqrt{\frac{m_e}{m_i}}. \tag{3.4}$$

On the other hand, we shall take

$$S < \sqrt{\frac{m_i}{m_e}}, \tag{3.4a}$$

because, if this is not the case the plasma is subject to instabilities associated with trapped particles (cf. §4).

In present-day devices the parameter $S \sim 10^{-1}\text{-}10$ and for thermonuclear devices, $S \sim 1$.

We shall also introduce a velocity associated with the current $u = j/en$. To avoid two-stream instabilities, we must make u smaller than v_e:

$$\frac{u}{v_e} < 1. \tag{3.5}$$

Furthermore, in order to avoid perturbations associated with cyclotron waves we must satisfy the condition $u < (v_e/3)(T_i/T_e)^{3/2}$. In order to avoid Alfven waves [16], it is desirable to have $u < c_A$, where $c_A = H/\sqrt{4\pi n m_i}$ is the Alfven velocity. As an approximation the condition $u < c_A$ can be written in the form

$$H_p \equiv \pi a^2 n \frac{e^2}{m_i c^2} > \frac{H_\vartheta^2}{H_z^2}. \tag{3.6}$$

Everywhere below we shall assume that the relations in (3.1)-(3.6) are satisfi[ed.] Specific cases in which these inequalities do not hold will be discussed quali[ta]tatively.

Drift instabilities develop from perturbations that are highly elongated along the magnetic field. For the case of a cylindrical plasma column we can write the perturbation in the form $\exp\left(-i\omega t + im\vartheta - i\frac{2\pi n}{L}z\right)$. The derivative along the lines of force for these perturbations is given by $\mathbf{h}\nabla = \frac{H_\vartheta}{rH}(m - nq)$. At points $r = r_0$, where $q(r_0) = m/n$, this derivative vanishes, i.e., the perturbation is constant along the lines of force. It is precisely at these points that we then find the development of perturbations characterized by specific values of m and n. Writing $\mathbf{H}\nabla = iHk_\parallel$, we can obtain a projection of the wave number of the magnetic field $k_\parallel$ close to the point $r = r_0$: $k_\parallel = k_y \theta x/r$, where $k_y = m/r_0$, $x = r - r_0$. If θ is not too small, as we assume below, it is sufficient to consider the case of small x.

The most dangerous instabilities are the electrostatic instabilities characterized by $\mathbf{E} = -\nabla\varphi$, in which the lines of force of the magnetic field remain fixed. If the longitudinal phase velocity of a wave $\omega/k_\parallel$ is appreciably smaller than the electron thermal velocity $v_e = \sqrt{\frac{2T}{m_e}}$, the electrons can set up a Bolzmann distribution along the lines of force, i.e., the perturbation in electron density n_e' will be given by

$$n_e' = \frac{ne\varphi}{T_e}.$$

On the other hand, if $\omega/k_\parallel \gg v_i = \sqrt{\frac{2T}{m_i}}$, we can neglect the longitudinal motion of the ions; the transverse motion of the ions (when $\omega \ll \Omega_i = eH/m_i c$) is given by the drift motion $\mathbf{v}_\perp = c[\mathbf{h}\nabla\varphi]/H$, where $\mathbf{h} = \mathbf{H}/H$. If we assume, as an approximation, that $H \approx$ const, then from the

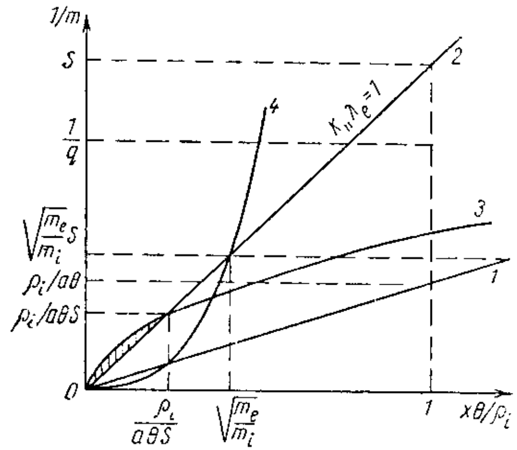

Fig. 5. Region of localization of various instabilities.

ion equation of continuity we find

$$n_i' = -\frac{k_y c}{\omega H n}\cdot\frac{dn}{dr}\varphi.$$

Using the neutrality condition $n_i' = n_e'$, from the foregoing we find

$$\omega = \omega^* = -\frac{k_y cT}{cHn}\cdot\frac{dn}{dr}.$$

Thus, in this approximation the density perturbation propagates in the azimuthal direction in the form of a wave with frequency ω^*. In order-of-magnitude terms, $\omega^* \sim (m/a^2)\rho_i v_i$, where $\rho_i = v_i/\Omega_i$ is the mean ion Larmor radius. These waves are called drift waves [17]. The longitudinal phase velocity of a drift wave $\omega/k_\parallel \sim v_i\rho_i/\theta x$; consequently, $\omega/k_\parallel > v_i$ when

$$x < \frac{\rho_i}{\theta}. \tag{3.7}$$

If $x > \rho_i/\theta$, the ions can also establish an equilibrium along the lines of force and a perturbation arises only in the presence of curvature of the lines of force. Under these conditions the instability (if it actually occurs) is a magnetohydrodynamic nonelectrostatic instability, i.e., it is either a screw instability or a flute instability or some combination of these (when the finite conductivity is taken into account). Thus, the condition in (3.7) defines the region of characteristic drift waves.

If account is taken of the dissipative effects that have been neglected above, the drift waves grow in time. The particular instability that become important depends on precisely which dissipative effect plays the predomina role.

We now wish to consider the regions of localization associated with th various instabilities. In Fig. 5 we have plotted the quantity $x\theta/\rho_i$ along the abscissa axis, where x is the width of the region of localization; along the or nate axis we have plotted $m^{-1} = r/k_y$, the inverse azimuthal number for the appropriate perturbation. All of the drift instabilities lie in the region $x\theta/\rho_i < 1$, regardless of the number m. However, in the case of collisionless instabilities the basic contribution to the transport process is associated with perturbation characterized by $k_y x \sim 1$, so that the important collisionless pe turbations are grouped around the line $1/m = a/x$ (Fig. 5, curve 1). If a localization $x\theta/\rho_i \sim 1$ is possible, then $1/m = \rho_i/a\theta$. We assume that this qua tity is smaller than the maximum allowed value of the quantity 1/m for the localized solution, which is 1/q. Along the ordinate axis (Fig. 5) we have al so plotted the quantity $\frac{1}{m} = S\sqrt{\frac{m_e}{m_i}} = \frac{\lambda_e\rho_i}{a^2}\sqrt{\frac{m_e}{m_i}}$. For this value of m and higher values, as will be shown in §4, there are collisionless instabilities associated with trapped particles. If the loss due to this instability is to be smaller than the loss due to the drift instabilities, then the quantity $\sqrt{\frac{m_e}{m_i}}S$ must not exceed $\rho_i/a\theta$, i.e., $\lambda_e/a \lesssim \frac{1}{\theta}\sqrt{\frac{m_i}{m_e}}$, a condition which imposes a limitation on the plasma density from below (for a given temperature). On the other hand, as λ_e is reduced the collision-dominated dissipative instabilities become important. These instabilities are characterized by the condition $k_\parallel\lambda_e < 1$, i.e., they lie in the region above curve 2 (Fig. 5) which corresponds to $k_\parallel\lambda_e = 1$. As we shall see in §7, the cells associated with the collision-dominated instabilities can overlap the entire pinc and can have a macroscopic effect on the diffusion of thermal conductivity when $xm^2 \sim a$, i.e., when the number of cells $\sim m^2$ multiplied by the mean cell width x becomes comparable with a. In other words, a contribution to th transport process comes only from perturbations characterized by $\frac{1}{m} < \sqrt{\frac{a}{x}}$ i.e., perturbations below curve 3 (Fig. 5). Thus, the important dissipative instabilities exhibit a localization which is smaller than $x \sim \rho_i^2/a\theta^2 S$, correspon ing to the cross-hatched region in Fig. 5. Actually, the localization of the dissipative instabilities can be even smaller; when $S \sim 1$, dissipative instabilities are not important in systems with reasonable shear, i.e., systems in which ($\theta \sim 10^{-1}$). When $S \gtrsim \frac{1}{\theta}\sqrt{\frac{\rho_i}{a}}$ these are known to fall in the localization region $x \leqslant \rho_i$.

Curve 4 (Fig. 5) shows the functional dependence $\frac{1}{n} = S\sqrt{\frac{m_i}{m_e}}\cdot\frac{x^2\theta^2}{\rho_i^2}$, corresponding to the relation $\chi_{\|} k_{\|}^2 = \omega^*$, where $\chi_{\|} = \lambda_e v_e$ is the longitudinal thermal conductivity of the electrons. Below this curve the electron temperature can be regarded as constant along the lines of force.

2. Equations for the Dissipative Hydromagnetic Instabilities

The stability conditions for an ideal plasma represent necessary, but not sufficient, conditions for the stability of a real plasma. If these are satisfied, the plasma may not be subject to fast hydromagnetic instabilities; however, if account is taken of the dissipative terms that have been neglected earlier (friction, viscosity), the plasma can still be subject to slow dissipative instabilities. In order to examine these instabilities, it is convenient to use the equations of two-fluid hydrodynamics (1.25)-(1.29) [18].

Together with the Maxwell equations

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{j};\ \operatorname{div}\mathbf{H} = 0;\ \mathbf{H} = \operatorname{curl}\mathbf{A};\ \mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad (3.8)$$

these equations form a complete system. In order to simplify the calculations we shall replace Eqs. (1.25)-(1.29) by somewhat simpler equations and then show under what conditions these simpler equations are valid. The changes introduced by the neglected terms can then be evaluated.

We first combine the equations of motion (1.26) and (1.27), neglecting the electron inertia compared with the ion inertia, also neglecting the viscosity tensor π. This procedure provides an equation that describes the motion of the plasma as a whole

$$m_i n \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c}[\mathbf{jH}]. \quad (3.9)$$

In the electron equation (1.27) we neglect the inertia and express $\mathbf{v}_e$ in terms of $\mathbf{v} = \mathbf{v}_i$ and $\mathbf{j}$ so that the resulting equation represents Ohm's law for the plasma:

$$\frac{1}{\sigma}\mathbf{j} = \mathbf{E} + \frac{1}{c}[\mathbf{vH}] - \frac{1}{enc}[\mathbf{jH}] + \frac{\nabla p_e}{en}. \quad (3.10)$$

If the Hall current $\frac{\sigma}{ecn}[\mathbf{jH}]$ and the electron pressure gradient are also neglected, (3.10) becomes the usual form of Ohm's law for a conducting fluid:

$$\frac{\mathbf{j}}{\sigma} = \mathbf{E} + \frac{1}{c}[\mathbf{vH}]. \quad (3.11)$$

In place of the two equations of continuity for the ions and electrons we make use of the equation for the ions

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} = 0 \quad (3.12)$$

and the difference of these equations; thus, when quasineutrality $n_i = n_e = n$ is introduced, we find

$$\operatorname{div}\mathbf{j} = 0. \quad (3.13)$$

The energy equations will be taken in the following form:

$$\frac{\partial T_i}{\partial t} + \frac{c}{H^2}[\mathbf{H}\nabla\varphi]\nabla T_i = 0;$$

$$\frac{\partial T_e}{\partial t} + \frac{c}{H^2}[\mathbf{H}\nabla\varphi]\nabla T_e = \chi_{\|}^e \Delta_{\|} T_e + \chi_{\perp}^e \Delta_{\perp} T_e. \quad (3.14)$$

The system of equations (3.9), (3.11)-(3.14) is equivalent to the system that has been treated in [19].

As in §2, we write the total current $\mathbf{j}$ in the form

$$\mathbf{j} = \mathbf{j}_{\perp} + \alpha\mathbf{H},$$

where

$$\alpha = \mathbf{jH}/H^2. \quad (3.15)$$

From Eq. (3.9) we find

$$\mathbf{j}_{\perp} = \frac{c}{H^2}\left[\mathbf{H}, \nabla p + m_i n \frac{d\mathbf{v}}{dt}\right]. \quad (3.16)$$

The expression for α can be obtained by using the longitudinal component of

Ohm's law (3.11):

$$\alpha = \sigma(\mathbf{HE})/H^2. \qquad (3.16a)$$

Friction can be neglected in the transverse components [cf. Eq. (3.11)] and we find

$$\mathbf{v}_\perp = -\frac{c}{H^2}[\mathbf{HE}]. \qquad (3.17)$$

Substituting this expression in Eq. (3.16), and carrying out the operation $\operatorname{div} \mathbf{j} = 0$, making use of the relation (3.15), we have

$$\mathbf{H}\nabla\alpha + 2\frac{[\mathbf{H}\nabla H]}{H^3}\nabla p + m_i\frac{c^2}{H^2}\operatorname{div} n\frac{\partial \mathbf{E}}{\partial t} = 0, \qquad (3.18)$$

which differs from Eq. (2.5) in that ion inertia appears [last term in Eq. (3.18)].

Now, taking the projection of the first equation in (3.8) along **H** we have

$$\alpha = \frac{c}{4\pi}\frac{\mathbf{H}\operatorname{curl}\mathbf{H}}{H^2}. \qquad (3.19)$$

The systems of equations (3.18), (3.19), together with (3.12) and (3.13) and the relation $\operatorname{div}\mathbf{H} = 0$ is more convenient for the subsequent calculations than the original equations, (3.7), (3.9)-(3.13).

Let us consider the stability of a rarefied plasma ($\beta \ll 1$) which forms a pinch of length L in a helical magnetic field $\mathbf{H}_0 = \{0, H^0_\vartheta, H^0_z\}$. The perturbation will be expressed in the form $\exp\left\{-i\omega t + im\vartheta - 2\pi\frac{inz}{L}\right\}$. In what follows we shall make use of the more convenient quantities $k_\parallel = \left(\frac{m}{r}H^0_\vartheta - \frac{n}{L}H^0_z\right)H_0^{-1}$ and $k_y = \frac{H^0_\vartheta}{H_0}\frac{n}{L} + \frac{H^0_z}{H_0}\frac{m}{r} \approx \frac{m}{r}$. In this case the perturbation of the longitudinal component of the magnetic field $H'_\parallel$ can be neglected (it is of order β compared with the other components), so that we are saying that the lines of force are only curved, and not compressed or expanded. This distorting curvature can be described by one component of the vector potential $A_\parallel$. Actually, neglecting $k_\parallel H$, from the relation

$$\operatorname{div}\mathbf{H} = 0$$

we find

$$\mathbf{H}' = -\frac{[\mathbf{H}_0\nabla A'_\parallel]}{H_0}; \quad \mathbf{E}' = -\nabla\varphi' - \frac{1}{c}\frac{\partial A'_\parallel}{\partial t}\mathbf{h}_0. \qquad (3.20)$$

Thus, as follows from Eq. (3.20), the transverse components of the electric field **E'** are derivable from a potential. This is the feature that allows us to write the energy equation in the form in (3.4) and (3.5).

Linearizing Eq. (3.18) and using Eq. (3.20), we find

$$\mathbf{H}_0\nabla\alpha' + ik_yA'_\parallel\frac{d\alpha_0}{dr} - 2ik_yp'\frac{1}{H_0R} + i\omega\frac{m_ic^2}{H_0^2}\operatorname{div} n\nabla\varphi = 0. \qquad (3.21)$$

The pressure perturbation p' can be found from the temperature equation and the equation of continuity. In the expression for p' we can neglect the longitudinal electron thermal conductivity, writing as an approximation

$$p' = \frac{\omega^*_{pi}}{\omega}en_0\varphi, \qquad (3.22)$$

where

$$\omega^*_{pi} = -\frac{c}{H_0}\cdot\frac{k_y}{en_0}\cdot\frac{dp_0}{dr}.$$

We now consider Eq. (3.19). Since

$$(\mathbf{H}\operatorname{curl}\mathbf{H})' \approx H^0_z\operatorname{curl}_z\mathbf{H}' = -H^0_z\Delta A'_\parallel,$$

Eq. (3.19) can be written in the form

$$-\Delta A'_\parallel = \frac{4\pi H_0}{c}\alpha'. \qquad (3.23)$$

Finally, linearizing the expression for the longitudinal current (3.16), we have

$$\alpha' = \frac{\sigma'}{\sigma_0}\cdot\frac{(\mathbf{H}_0\mathbf{j}_0)}{H_0^2} + \sigma_0\left(-\mathbf{H}_0\nabla\varphi + \frac{i\omega}{c}A'_\parallel H_0\right)\frac{1}{H_0^2}.$$

The conductivity depends primarily on the electron temperature, so that

$$\sigma' = \frac{d\sigma_0}{dT_0}T'_e = \frac{3}{2}\sigma_0\frac{T'_e}{T_e}; \qquad (3.24)$$

on the other hand, using Eq. (3.14), we can find the perturbation in the electron temperature T'_e:

$$\frac{T'_e}{T_0} = \frac{\omega^*_{Te}}{\omega + i\chi_\parallel k^2_\parallel + i\chi_\perp k^2_\perp}\frac{e\varphi}{T_{0e}}, \qquad (3.25)$$

where $\omega^*_{Te} = -(ck_y/eH)(dT_{0e}/dr)$. Thus, if the electron thermal conductivity is neglected ($\omega \gg \chi^e_\parallel k^2_\parallel, \chi^e_\perp k^2_\perp$),

$$\alpha' = \frac{3}{2}\cdot\frac{\omega^*_{Te}}{\omega}\cdot\frac{\mathbf{H}_0\mathbf{j}_0}{H_0^2}\cdot\frac{e\varphi}{T_{0e}} + \frac{\sigma_0}{H_0}\left(-ik_\parallel\varphi + \frac{i\omega}{c}A'_\parallel\right). \qquad (3.26)$$

Expressing $\mathbf{j}_0$ in terms of the directed electron velocity $u = -\mathbf{j}_0/en$, we can now write α' in the form

$$\alpha' = -\mathrm{i}\,\frac{e^2 n_0 k_\parallel}{m_e \nu_e H_0}\left[\left(1 - \mathrm{i}\,\frac{3}{2}\cdot\frac{\omega_{Te}^*}{\omega}\cdot\frac{u}{v_e}\cdot\frac{\nu_e}{k_\parallel v_e}\right)\varphi - \frac{\omega}{k_\parallel c}A_\parallel'\right]. \quad (3.27)$$

Equations (3.21), (3.23), and (3.27), together with Eq. (3.22), represent the system of equations that we need to describe the slow hydromagnetic dissipative instabilities. Substituting Eq. (3.27) for α' in Eqs. (3.21) and (3.23), we obtain the two following coupled equations:

$$\left.\begin{aligned} &\rho^2 \operatorname{div}(n_0\nabla\varphi) - 2\,\frac{a_p n_0}{R}\left(\frac{\omega_p^*}{\omega}\right)^2\varphi + \frac{k_y T_0}{\omega e^2 H_0}\cdot\frac{dj_\parallel}{dr}A_\parallel' + \\ &+\frac{n_0 k_\parallel^2 v_e^2}{i\nu_e\omega}\left\{\left(1-\mathrm{i}\,\frac{3}{2}\,\frac{\omega_{Te}^*}{\omega}\cdot\frac{u}{v_e}\cdot\frac{\nu_e}{k_\parallel v_e}\right)\varphi - \frac{\omega}{k_\parallel c}A_\parallel'\right\} = 0; \\ &\Delta_\perp A_\parallel' = -\frac{4\pi e^2 n_0}{m_e}\cdot\frac{k_\parallel}{i\nu_e c}\left[\left(1-\mathrm{i}\,\frac{3}{2}\cdot\frac{\omega_{Te}^*}{\omega}\cdot\frac{u}{v_e}\times\right.\right. \\ &\left.\left.\times\frac{\nu_e}{k_\parallel v_e}\right)\varphi - \frac{\omega}{k_\parallel c}A_\parallel'\right]. \end{aligned}\right\} \quad (3.28)$$

Here, ρ is the mean Larmor radius: $\rho^2 = T_e/m_i\Omega_i^2$; $\Omega_i = eH/m_i c$; $a_p^{-1} = -\frac{1}{p_0}\cdot\frac{dp_0}{dr}\cdot\frac{T_i + T_e}{T_e}$.

It is easy to show that these equations are equivalent to the equations in [19] for the case of a plane plasma layer. If the longitudinal field is strong, $H_z^0 \gg H_\vartheta^0$, the quantity $F' = \frac{dF}{dx} = \frac{d}{dx}\left(\frac{k_y H_y^0 + k_z H_z^0}{kH_0}\right)$ that appears in [19] can be written in the form $F' \approx \frac{k_y}{kH_0}\cdot\frac{dH_y}{dx}$. Taking account of this feature, and also the fact that the conversion to planar geometry proceeds by the transformations

$$r \to \infty,\quad \frac{d}{dr} \to \frac{d}{dx},\quad \frac{m}{r} \to k_y = \text{const},\quad k_z = n/R,$$

we can write the system in (3.28) in the form

$$\frac{\psi''}{\alpha^2} = \psi(1 + \tilde{p}/\tilde{\eta}\alpha^2) + \frac{W}{\alpha^2}\left(F/\tilde{\eta} + \frac{\tilde{\eta}'F'}{\tilde{\eta}\tilde{p}}\right);$$

$$\left.\begin{aligned} \frac{(\tilde{n}W')'}{\alpha^2} &= W\left[\tilde{n} - \frac{S_0^2 G}{\tilde{p}^2} + \frac{FS_0^2}{\tilde{p}}\left(\frac{F}{\tilde{\eta}} + \frac{\tilde{\eta}'F'}{\tilde{\eta}\tilde{p}}\right)\right] + \\ &+ \psi S_0^2\left(\frac{F}{\tilde{\eta}} - \frac{F''}{\tilde{p}}\right). \end{aligned}\right| \quad (3.29)$$

Here, we have introduced the notation of [19]: $\psi = A_\parallel'$; $W = \mathrm{i}c\tau_R k_y\varphi$;

$$\alpha = k_y a;\ \tau_H = a/c_A;\ c_A = \frac{H_0}{\sqrt{4\pi n_0 m_i}};\ \tau_R = \frac{4\pi a^2}{c^2\langle\eta\rangle};\ \eta = \frac{m_e\nu_e}{e^2 n_0};$$

$$\tilde{\eta} = \eta/\langle\eta\rangle;\ F = k_\parallel/k_y;\ \tilde{n} = n/\langle n\rangle;\ S_0 = \tau_R/\tau_H;\ \tilde{p} = \gamma\tau_R = -\mathrm{i}\omega\tau_R.$$

The primes denote differentiation with respect to the dimensionless variable $\tilde{\mu} = x/a$, where a is the thickness of the layer. The quantity G is a dimensionless quantity that takes account of the curvature of the lines of force. In a planar plasma layer the lines of force are straight; thus, in order to carry out an analogy with the curvature effect in a cylindrical system, we introduce a simulated gravitational force $m_i g$, in which case $G = \tau_H^2 g\,\frac{1}{n_0}\cdot\frac{dn_0}{dx}$.

If we do not convert to planar geometry in the system in (3.28), in the variables in which we have written the system (3.29), Eq. (3.28) can be written in the form

$$\left.\begin{aligned} &\frac{1}{\mu m^2}(\mu\psi')' = \psi\left(\frac{1}{\mu^2} + \frac{\tilde{p}}{\tilde{\eta}m^2}\right) + \frac{W}{m^2}\left(\frac{F}{\tilde{\eta}} + \frac{\tilde{\eta}'F_1}{\tilde{\eta}\tilde{p}}\right); \\ &\frac{1}{\mu m^2}(\mu\tilde{n}W')' = W\left[\frac{\tilde{n}}{\mu^2} - \frac{S_0^2 G}{\tilde{p}^2} + \frac{FS_0^2}{\tilde{p}}\left(\frac{F}{\tilde{\eta}} + \frac{\tilde{\eta}'F_1}{\tilde{\eta}\tilde{p}}\right)\right] + \\ &+ \psi S^2\left[\frac{F}{\tilde{\eta}} - \frac{1}{\mu\tilde{p}}(\mu F_1)'\right]. \end{aligned}\right\} \quad (3.30)$$

Here, $\mu = r/r_0$, r_0 is the point at which

$$F(r_0) = 0;.\quad F = \frac{H_\vartheta\frac{m}{r} - \frac{n}{R}H_z}{H_0 m/r_0};\quad G = \frac{r_0^2\beta}{Ra_p};$$

$$F_1 = \frac{c}{4\pi}\left(\frac{m}{r} j_z + \frac{n}{R} j_\vartheta\right)\frac{r_0}{\frac{m}{r_0}H_0} = \frac{r_0^2}{mH_0}\left(\frac{m}{r^2}\cdot\frac{d}{dr}(rH_\vartheta) - \frac{n}{R_0}\cdot\frac{dH_z}{dr}\right);$$

the remaining quantities are defined, as in Eq. (3.29), only as $a \to r_0$. In the planar case, $F_1 = F'$; the condition that replaces Eq. (15) of reference [19], which expresses the stationarity of the unperturbed state, becomes

$$\frac{\tilde{\eta}' F_1}{\tilde{\eta}} = -\frac{1}{\mu}(\mu F_1)'.$$

Since the dissipative instabilities usually have a very narrow range of localization (in which the change in quantities, such as n_0, r, etc., can be neglected) Eqs. (3.29) and (3.30) are essentially the same. (The only small difference is the fact that $F_1 \neq F'$.) The most important difference from the planar case arises because of G. We have considered the real curvature, so that G can contain the pressure gradient (as it should) and not the density gradient, as is the case when the curvature is simulated by the introduction of a gravitational force.

Equation (3.30) contains a large parameter $S_0 = \tau_R/\tau_H$ which is the ratio of the skin penetration time to the characteristic hydrodynamic time. The transition to ideal hydrodynamic occurs as $S_0 = \infty$, and in this case the frequencies are found to be of order τ_H^{-1} (i.e., $p \sim S_0$). However, if $S_0 \neq 0$ but $S_0 \gg 1$, then, in addition to the hydrodynamic oscillations there can be slower dissipative oscillations. For the latter case, $\tilde{p} \sim S_0^\gamma$, where $0 \leq \gamma \leq 1$. The case $\gamma = 0$ corresponds to a growth rate of the same order as the collisional diffusion time and is thus not of interest; $\gamma = 1$ corresponds to the usual hydrodynamic case. For this reason, in what follows primary attention will be given to the case $0 < \gamma < 1$.

For a plasma characterized by high conductivity ($S_0 \gg 1$) dissipation is important only in a narrow region near the point at which $F(r_0) = 0$. Actually, when $S_0 \to 0$ from the second equation in (3.30) we find

$$\tilde{p}\psi = -FW\left(\text{ i.e. } \varphi - \frac{\omega}{k_\parallel c} A'_\parallel = 0\right) \tag{3.31}$$

everywhere except for the point F = 0. The condition in (3.31) is simply the condition that the magnetic field be frozen into the plasma; in accordance with the relation $\mathbf{E} = -\frac{1}{c}[\mathbf{vH}]$, this condition can be written $E_\parallel = (\mathbf{HE})/H = 0$. If we take account of the next term of order S_0/p in Eq. (3.31),

and substitute in the first equation of (3.30), we obtain an equation (which describes the oscillations of an ideally conducting plasma for $\tilde{p} = 0$, since $\tilde{p}/S_0 \to 0$) which holds everywhere except for a small region around the point $r = r_0$ [cf. Eq. (2.25)]:

$$\frac{1}{\mu}(\mu\psi')' = \left(m^2 - \frac{G}{F^2} + \frac{1}{\mu F}(\mu F_1)'\right)\psi. \tag{3.32}$$

Thus, if it is necessary to find the exact characteristic functions and the exact characteristic values of a problem, the general procedure for solution of Eq. (3.30) (for the case $S_0 \gg 1$, $0 < \gamma < 1$) is essentially to find those solutions ψ and W for which W falls from the point $r = r_0$ and ψ goes over to the solution in (3.32) with the appropriate boundary conditions for $\mu \to \infty$.

Equations (3.28) and (3.30) have been obtained under certain special assumptions. We now wish to see under what conditions these assumptions actually hold. If the exact equations (1.25) and (1.29) are used instead of the simplified equations (3.9) and (3.13), for example we obtain the following expression for α' (for simplicity we shall confine ourselves to the electrostatic case $A'_\parallel = 0$):

$$\alpha' = -i\frac{e^2 n_0 k_\parallel}{m_e \nu_e H_0} \times$$

$$\times \frac{1 - \frac{\omega^*_{ne}}{\omega} - \frac{\omega^*_{Te}}{\omega}\cdot\frac{1 + i\frac{3}{2}\frac{u}{v_e}\cdot\frac{\nu_e}{k_\parallel v_e}}{1 - k_\parallel u/\omega + 2ik_\parallel^2 v_e^2/\omega\nu_e + i\chi_\perp k_\perp^2}\left(1 - \frac{2}{3}\cdot\frac{k_\parallel u}{\omega}\cdot\frac{d\ln n_0}{d\ln T_0}\right)}{1 + i\frac{k_\parallel^2 v_e^2}{\omega\nu_e} + \frac{i\frac{2}{3}\cdot\frac{k_\parallel^2 v_e^2}{\omega\nu_e}\left(1 + i\frac{2}{3}\cdot\frac{u}{v_e}\cdot\frac{\nu_e}{k_\parallel v_e}\right)}{1 - \frac{k_\parallel u}{\omega} + 2ik_\parallel^2 v_e^2/\omega\nu_e + i\chi_\perp k_\perp^2}\left(1 - \frac{k_\parallel u}{\omega}\right)}\varphi, \tag{3.33}$$

where

$$\omega^*_{ne} = -\frac{ck_y T_0}{Hen_0}\cdot\frac{dn_0}{dr};\quad \omega^*_{Te} = -\frac{ck_y}{He}\cdot\frac{dT_0}{dr}.$$

Comparing this expression with Eq. (3.27), we can draw the following conclusions. The Hall current $\frac{\sigma}{c}[\mathbf{jH}]$ can be neglected if

$$\omega \gg \omega^*_{pe}, \tag{3.34}$$

where $\omega^*_{pe} = -\frac{ck_y}{Hen_0}\cdot\frac{dp_{0e}}{dr}$ is the electron drift frequency.

The term containing the gradient of electron pressure in Ohm's law is small if (3.34) is satisfied and if the frictional contribution is large, specifically,

$$k_{\parallel}^2 v_e^2/\omega\nu_e \ll 1. \tag{3.35}$$

In this case we can also omit the $T_0 \frac{\partial}{\partial z} v_z^e$ term in the temperature equation. Furthermore, if $\omega \gg k_z u$, we can omit the $u \frac{\partial}{\partial z} T_e'$ term.

In what follows we shall generally assume that the condition in (3.35) is satisfied. However, in order to have the possibility of treating frequencies of the order of the drift frequency, we must generalize Eqs. (3.28) and (3.30). Retaining the ion viscosity tensor in the equation of motion and using Ohm's law in the form in (3.8), we obtain the following system of equations that take account of the ion Larmor radius (ω^*) and the ion viscosity:

$$\rho^2\left(1-\frac{\omega_i^*}{\omega}\right)\left(1-\mathrm{i}\frac{\nu_{ii}}{\omega}\rho_i^2\Delta_\perp\right)\Delta_\perp\varphi-2\frac{a_p}{R}\left(\frac{\omega_p^*}{\omega}\right)^2\varphi+$$

$$+\frac{k_y}{\omega}\cdot\frac{T_{0e}}{e^2n_0H_0}\cdot\frac{dj_\parallel}{dr}A_\parallel'+\frac{k_\parallel^2 v_e^2}{\mathrm{i}\omega\nu_e}\left[\left(1-\frac{\omega_{pe}^*}{\omega}-\frac{\omega_{Te}^*}{\omega}\,\mathrm{i}\,\frac{3}{2}\cdot\frac{u}{v_e}\cdot\frac{\nu_e}{k_\parallel v_e}\right)\varphi-\right.$$

$$\left.-\frac{\omega}{k_\parallel c}\left(1-\frac{\omega_{pe}^*}{\omega}\right)A_\parallel'\right]=0; \tag{3.36}$$

$$\Delta_\perp A_\parallel'=-\frac{4\pi e^2 n_0}{m_e}\cdot\frac{k_\parallel}{\mathrm{i}\nu_e c}\left[\left(1-\frac{\omega_{pe}^*}{\omega}-\frac{\omega_{Te}^*}{\omega}\,\mathrm{i}\,\frac{3}{2}\cdot\frac{u}{v_e}\,\frac{\nu_e}{k_\parallel v_e}\right)\varphi-\right.$$

$$\left.-\frac{\omega}{k_\parallel c}\left(1-\frac{\omega_{pe}^*}{\omega}\right)A_\parallel'\right].$$

Here, $\rho^2 = T_e/m_i\Omega_i^2$ is the ion Larmor radius;

$$\omega_i^*=\frac{ck_y}{H_0 c n_0}\times\frac{dp_i^0}{dr};$$

$$\omega_{pe}^*=-\frac{ck_y}{eH_0n_0}\cdot\frac{dp_{0e}}{dr},\quad a_p^{-1}=-\frac{1}{p_0}\cdot\frac{dp_0}{dr}\cdot\frac{T_i+T_e}{T_e}.$$

In the notation of Furth et al. [19],

$$\left(\frac{\tilde{p}+\mathrm{i}\tilde{\omega}_i^*}{\tilde{p}}\right)\left(1-\frac{3}{2}\sqrt{\frac{m_i}{m_e}}\cdot\frac{\beta_i}{\tilde{p}}\tilde{\Delta}_\perp\right)\left[\frac{1}{\mu m^2}(\mu\tilde{n}W')'-\frac{\tilde{n}}{\mu^2}W\right]=$$

$$=W\left[-\frac{S_0^2G}{\tilde{p}}+\frac{FS_0^2}{\tilde{p}}\left(\frac{F}{\tilde{\eta}}\cdot\frac{\tilde{p}+\mathrm{i}\tilde{\omega}_e^*}{\tilde{p}}+\frac{\tilde{\eta}'F_1}{\tilde{\eta}\tilde{p}}\right)\right]+$$

$$+\psi S_0^2\left[\frac{F}{\tilde{\eta}}\cdot\frac{\tilde{p}+\mathrm{i}\tilde{\omega}_e^*}{\tilde{p}}-\frac{1}{\mu\tilde{p}}(\mu F_1)'\right];$$

$$\frac{1}{m^2}\cdot\frac{1}{\mu}(\mu\psi')'=\psi\left(\frac{1}{\mu^2}+\frac{\tilde{p}+\mathrm{i}\tilde{\omega}_e^*}{\tilde{m}^2\tilde{\eta}}\right)+\frac{W}{m^2}\left(\frac{F}{\tilde{\eta}}\cdot\frac{\tilde{p}+\mathrm{i}\tilde{\omega}_e^*}{\tilde{p}}+\frac{\tilde{\eta}'F_1}{\tilde{\eta}\tilde{p}}\right), \tag{3.37}$$

where $\tilde{\omega}^*=\omega^*\tau_R$; $\beta_i=8\pi p_i/H^2$; $\tilde{\Delta}_\perp=r_0^2\Delta_\perp$ is the dimensionless transverse Laplacian.

We now wish to investigate the system in (3.28). These equations contain a number of terms that can lead to instabilities. These terms include the following: the longitudinal electron current (terms containing u); the curvature of the lines of force, which provides an effective gravitational force (the term containing a/R); the term containing the gradient of the current dj_0/dr; and, finally, the pressure gradient. The importance of the various instabilities depends on which specific term plays the dominant role. Below we shall consider small-scale oscillations, in which case it is possible to use the semiclassical approximation, in which $A_\parallel'$ and φ, which characterize the deviation from the equilibrium configuration, vary as $\exp\left\{-\mathrm{i}\omega t+\mathrm{i}\int k_r dr+\mathrm{i}m\vartheta-\mathrm{i}\frac{nz}{R}\right\}$. It is reasonable to assume that these localized perturbations will be primarily electrostatic oscillations, which do not distort the lines of force. For this reason we shall first take $A_\parallel' = 0$; nonelectrostatic waves will be considered later.

Taking $A_\parallel' = 0$, and using the first equation in (3.28), we obtain the following semiclassical dispersion equation:

$$1-\mathrm{i}\frac{\omega_T^*}{\omega}\cdot\frac{3}{2}\frac{u}{v_e}\cdot\frac{1}{k_\parallel\lambda_e}-\mathrm{i}\frac{\omega^2+\omega_g^2}{\omega\omega_s}=0 \tag{3.38}$$

or, using (3.36) and taking account of the finite Larmor radius:

$$1-\frac{\omega_{pe}^{*}}{\omega}-\frac{\omega_{T}^{*}}{\omega+i\chi_{\parallel}^{2}k_{\parallel}^{2}}\left(1+i\frac{3}{2}\frac{u}{v_e}\frac{1}{k_{\parallel}\lambda_e}\right)+i\left[\frac{\omega^2}{\omega_s}\left(1-\frac{\omega_{pi}^{*}}{\omega}\right)+\right.$$
$$\left.+\frac{\omega_g^2}{\omega_s}\right]\left(1+i\frac{k_{\parallel}^2 v_e^2}{\omega\nu_e}\right)=0. \tag{3.38a}$$

Here, we have introduced the notation

$$\omega_g^2=\frac{2a}{R}(\omega_p^*)^2/k_{\perp}^2\rho^2;\ \ \omega_s=\frac{k_{\parallel}^2}{k_{\perp}^2}\frac{\Omega_e\Omega_i}{\nu_e}.$$

We shall classify the instabilities in terms of the region of localization, going from instabilities characterized by a large localization region to smaller scale instabilities.

Before actually studying the instabilities, it is instructive to make some remarks with regard to the structure of these equations and to the method of classification.

The dispersion relation (3.38a) contains the drift frequency ω^*, which is a characteristic parameter in studies of the stability of an inhomogeneous plasma. When the dissipative effects are sufficiently large, the characteristic frequencies of the instabilities that derive from Eq. (3.38) are found to be larger than ω^*. Hence, everywhere that the term $1-\omega^*/\omega$ appears the second term can be neglected compared with the first. In this approximation, all of the instabilities are aperiodic. In this case, as we have noted in the Introduction, it is not meaningful to seek asymptotic ($t\to\infty$) characteristic functions of the linear approximation. It is found to be completely sufficient to use the semiclassical (local) approach because, for large growth rates $\gamma>\omega$, the localization (spatial structure of the perturbation) is determined by the nonlinearity.

As the temperature increases, dissipation is reduced and the characteristic frequency ω approaches the drift frequency. Under these conditions the growth rate of the instability is diminished. As the temperature increases still further, the frequency becomes the drift frequency and the growth rate becomes smaller than the frequency $\gamma<\omega^*$. When this occurs, the nonlinear effects no longer localize the perturbation and it becomes meaningful to investigate the differential equations of the linear stability problem in order to find the characteristic functions, the characteristic values, and the stability criteria. Even under these conditions it is possible to obtain an answer from the semiclassical analysis, which is accurate to within a factor of order unity. In this case (cf. below) the localization is usually determined by the ion inertia. Hence, instabilities with high growth rates can be classified as inertialess instabilities, whereas instabilities with small growth rates can be classified as inertial instabilities.

3. Current— Convective Instability

When $k_{\parallel}\to\infty$, Eq. (3.38) yields the following expression for the frequency:

$$\omega=i\omega_T^*\frac{3}{2}\cdot\frac{u}{v_e}\cdot\frac{1}{k_{\parallel}\lambda_e}. \tag{3.39}$$

This is the so-called current-convective instability, which develops in the presence of a longitudinal current [20]. At small values of $k_{\parallel}\sim x$ and small x, the current-convective instability becomes the gravitational instability:

$$\omega=i\omega_g^2/\omega_s, \tag{3.40}$$

which arises from the curvature of the lines of force.

We shall consider the first of these two instabilities in greater detail.

It follows from Eq. (3.39) that the current-convective instability can be obtained from the condition $j_{\parallel}'=0$, i.e., the ion inertia is not important for this instability. If account is taken of the finite Larmor radius, as well as effects due to thermal conductivity, which we have neglected for reasons of simplicity up to this point, the dispersion equation for this instability is written

$$1-\frac{\omega_{ne}^*}{\omega}\left(1+i\mu\frac{u}{v_e}\cdot\frac{1}{k_{\parallel}\lambda_e}\right)-\frac{\omega_{Te}^*}{\omega+i\chi_{\parallel}k_{\parallel}^2+i\chi_{\perp}k_{\perp}^2}\times$$
$$\times\left(1+i\alpha\frac{u}{v_e}\frac{1}{k_{\parallel}\lambda_e}\right)=0, \tag{3.41}$$

where $\alpha=\frac{d\ln\sigma}{d\ln T}$; $\mu=\frac{d\ln\sigma}{d\ln n}$; $\chi_{\parallel}$ is the longitudinal (electron) thermal conductivity and $\chi_{\perp}$ is the transverse thermal conductivity.

In a fully ionized plasma, $\alpha=3/2$, while μ is a small quantity ($\sim 1/20$) which arises as a consequence of the density dependence of the Coulomb logarithm. Hence, we first investigate the instability associated with temperature perturbations:

$$\omega=\omega_{pe}^*+i\omega_{Te}^*\frac{\alpha u}{v_e k_{\parallel}\lambda_e}+i\omega_{ne}^*\frac{\mu u}{v_e k_{\parallel}\lambda_e}-i\chi_{\parallel}k_{\parallel}^2. \tag{3.42}$$

In this expression we must assume that $k_{\|}\lambda_e < 1$; if this condition is not satisfied, the hydrodynamic approximation cannot be used.

The longitudinal thermal conductivity allows the development of a temperature perturbation for each pair of numbers m and n, but only over a narrow range of order

$$x \leqslant x_0 = r_0 \xi^{1/3} m^{-2/3}, \qquad (3.43)$$

where $\xi = c^2/\omega_0^2 \lambda_e^2 \theta^2$, $\omega_0^2 = 4\pi e^2 n_0/m_e$. If the density n_0 is reasonably large, the quantity $\xi \ll 1$, so that $x_0 \ll r_0$. At larger values of x the temperature perturbations are stabilized. In this case, it is necessary to take account of the weaker instability associated with the density perturbation. Because the localization range $x \sim \rho_i/\theta$ for this instability can exceed the value given in (3.43), it can be more important than the temperature instability. As the quantity $k_{\|}\lambda_e$ increases, the longitudinal electron viscosity begins to play a role; this factor can be introduced in Eq. (3.42) by adding to μ the quantity $(k_{\|}\lambda_e)^2 > 1$. When $k_{\|}\lambda_e \sim 1$, we find the transition to the collisionless damping, i.e., Landau damping. If (3.42) is compared with the corresponding expression for a collisionless plasma [(3.103)], we find that there is a continuous transition from collisional dissipation to collisionless damping when $k_{\|}\lambda_e \sim 1$.

In order to obtain an expression for the characteristic growth rate in terms of the system parameters we shall use the results of the nonlinear analysis $\gamma = \omega_T^* \dfrac{\alpha u}{v_e k_{\|} \lambda_e}$ from which it follows that the characteristic number m is of the following order of magnitude: $m \sim \sqrt{a/x_0}$. Substituting in the expression for ω, $x \sim x_0$, and $m \sim \sqrt{a/x_0}$, we find the following expression for γ:

$$\gamma \sim \frac{v_i}{a}\left[\frac{u}{v_e}\,\theta\sqrt{\frac{m_i}{m_e}}\,\frac{\rho}{a}\right]^{1/2}, \qquad (3.44)$$

and the corresponding value for the localization region is found to be

$$\frac{x_0}{\rho} \approx \left[\frac{u}{v_e}\cdot\frac{1}{\theta^3}\sqrt{\frac{m_e}{m_i}}\,\rho/a\right]^{1/2}\frac{a^2}{\lambda_e\rho}.$$

The drift frequency for this localization ω^* is given by

$$\omega^* \sim \frac{v_i}{a}\left[\frac{v_e}{u}\,\theta^3\sqrt{\frac{m_i}{m_e}}\left(\frac{\lambda_e\rho}{a_2}\right)^2\frac{\rho}{a}\right]^{1/4}. \qquad (3.45)$$

Hence, the characteristic frequency $\omega \sim \gamma$ becomes of the order of the drift

frequency, or smaller, when

$$\left[\frac{v_e^3}{u^3}\,\theta\sqrt{\frac{m_e}{m_i}\cdot\frac{a}{\rho}}\right]^{1/2}\frac{\lambda_e\rho}{a^2} > 1. \qquad (3.46)$$

The current–convective instability being considered can be regarded as inertia-free, since the transverse ion inertia does not play any role. The localization region, as we have noted above, is determined by the longitudinal thermal conductivity and by nonlinear effects, i.e., essentially by the enhanced transverse thermal conductivity.

As the temperature is increased, the dissipation is reduced and the scale size of the instability becomes smaller and smaller; the localization is then determined by the ion inertia rather than the thermal conductivity. Under these conditions, all three terms in Eq. (3.38) are of the same order of magnitude, so that

$$\omega \sim i\omega_T^* \frac{u}{v_e k_{\|} \lambda_e} \sim \omega_s. \qquad (3.47)$$

This is the so-called inertia current–convective instability (or, in the terminology of [19], the rippling mode). It is evident from Eq. (3.47) and the definition of ω_s that, in this case, the thermal conductivity does not play a role up to values such that $k_\perp^2 \rho^2 \sim 1$, since

$$\omega \gg \chi_{\|} k_{\|}^2 \sim v_e^2 k_{\|}^2/\nu_e.$$

When the finite Larmor radius is taken into account, it follows from Eq. (3.36) that the differential equation for this instability is

$$\rho^2\left(1 - \frac{\omega_i^*}{\omega}\right)\Delta_\perp\varphi + i\,\frac{k_{\|}^2 v_e^2}{\omega\nu_e}\left(1 - \frac{\omega_{pe}^*}{\omega} - \frac{\omega_{Te}^*}{\omega}\,i\,\frac{3}{2}\cdot\frac{u\nu_e}{k_{\|} v_e^2}\right)\varphi = 0. \qquad (3.48)$$

Here we have omitted the ion viscosity terms and have assumed that the oscillations are electrostatic: $A_{\|} \approx 0$.

A simple substitution of variables in this equation leads to the familiar equation for the quantum-mechanical oscillator (taking account of the fact that $k_{\|} = k_y \theta x/a$).

The fastest-growing characteristic value corresponds to the function for the ground state of the oscillator:

$$\varphi = \exp(-\alpha x^2/\rho^2). \qquad (3.49)$$

The localization condition on the perturbation implies that

$$\operatorname{Re} a > 0. \tag{3.50}$$

Substituting the function in Eq. (3.49) in Eq. (3.48), we obtain the following equations for a and ω:

$$\left.\begin{aligned} &2a + k_y^2\rho^2 - i\frac{\nu_e}{\omega}\cdot\frac{9}{16}\left(\frac{u}{v_e}\right)^2\frac{\gamma_0^2\,\omega_{pe}^{*2}}{\omega_{pe}^{*2}-\omega^2} = 0;\\ &4a - i\frac{\omega_{pe}^{*2}}{\nu_e\omega}\cdot\frac{m_i}{m_e}\,\theta^2\,\frac{\omega_{pe}^*-\omega}{\omega_{pe}^*+\omega} = 0, \end{aligned}\right\} \tag{3.51}$$

where we have assumed that $T_e = T_i$; $\omega_{pi}^* = -\omega_{pe}^*$; $\gamma_0 = \frac{d\ln T_0}{d\ln p_0}$. Equation (3.51) and the condition in (3.50) determine the characteristic function and the characteristic value ω.

If it is assumed that $\omega \gg \omega_{pe}^*$, then Eq. (3.51) yields the result obtained in [19] for the rippling mode:

$$\omega = i\,\omega_{pe}^*\left\{\left(\frac{3}{4}\right)^4\frac{m_e}{m_i}\left(\frac{\nu_e}{\omega_{pe}^*}\right)^3\frac{\gamma_0^4}{\theta^2}\left(\frac{u}{v_e}\right)^4\right\}^{1/5}. \tag{3.52}$$

As the temperature increases, the frequency is reduced, and when $\omega < \omega_{pe}^*$, this instability is stabilized by the finite Larmor radius of the ions [21].

Thus, in addition to the fact that this instability has a small range of localization as compared with the inertialess instability [this result can be established from (3.47), or directly from (3.49) and (3.51)], it is stabilized by the finite Larmor radius when $\omega < \omega_{pe}^*$. By substituting in this inequality the expression for the frequency (3.52), it is easy to show that the instability is essentially stabilized for the values of temperature and density taken above.

When ion inertia is taken into account, we find an additional short-wave instability, the so-called drift-dissipative instability.

4. Drift-Dissipative Instability

Using the dispersion relation (3.38a) and neglecting terms with ω_g^2, we obtain the following expression for the frequency when $\omega_s \gg \omega_{pe}^*$:

$$\omega = \omega_{pe}^*\left(1 - k_\perp^2\rho^2\left(1 - \frac{T_i}{T_e}\right)\right) - i\,\chi_\parallel^e k_\parallel^2 + \\ + i\,\omega_{Te}^*\frac{3}{2}\frac{u}{v_e k_\parallel \lambda_e} + i\frac{(\omega_{pe}^*)^2}{\omega_s}\left(1 + \frac{T_i}{T_e}\right). \tag{3.53}$$

In the first term on the right side we have taken account of the small frequency correction due to the finite Larmor radius. It follows from Eq. (3.53) that the current–convective instability becomes the drift–dissipative instability when

$$\frac{(\omega_{pe}^*)^2}{\omega_s} > \frac{\omega_T^* u}{v_e k_\parallel \lambda_e}, \tag{3.54}$$

The condition in (3.54) can be rewritten in the form

$$x < \rho\left(\frac{v_e}{u\theta}\sqrt{\frac{m_e}{m_i}}\right)^{1/3}. \tag{3.55}$$

Let us consider a wave packet which is approximately a plane wave with wave vector **k**. In a time $t \sim \gamma^{-1}$ this packet is displaced by a distance

$$x \sim \frac{1}{\gamma}\cdot\frac{\partial\omega_k}{\partial k_x}. \tag{3.56}$$

This quantity can be regarded as the characteristic localization region for the instability when $\gamma < \omega$ even if there are no localized solutions in the linear approximation.

Using Eq. (3.56) and the expression for the frequency (3.53), we find

$$x < \rho\sqrt{\delta}, \tag{3.57}$$

where

$$\delta = \frac{1}{m\theta^2}\sqrt{\frac{m_e}{m_i}}\cdot\frac{a^2}{\lambda_e\rho},$$

while the growth rate [found by substitution of the localization region (3.57) in Eq. (3.53)] is

$$\gamma \sim \omega^*/\delta. \tag{3.58}$$

It is easy to show that solution for the characteristic values leads to precisely the same expressions [(3.57) and (3.58)].

If m is replaced by the quantity $m_0 = \sqrt{a/x}$, which corresponds to cell overlapping, the width of the localization region (3.57) is found to be

$$x_0 = \rho\left[\sqrt{\frac{m_e}{m_i}}\cdot\frac{1}{\theta^2}\cdot\frac{a^2}{\lambda_e\rho}\sqrt{\frac{\rho}{a}}\right]^{2/3}. \qquad (3.59)$$

It follows from Eq. (3.59) that as the temperature increases the perturbations fall in the region $x < \rho$, where Eq. (3.53) does not hold. In this case, the condition $k_\perp^2\rho^2 < 1$ is violated and the transverse motion of the ions must be described kinetically. As before, the longitudinal motion of the electrons can still be described by the hydrodynamic equations, since the condition $k_\parallel\lambda_e < 1$ improves when $x \to 0$. Making use of kinetic theory, we can obtain the following integral equation, which holds for an arbitrary ratio of x/ρ [22]:

$$(\omega - \omega_{pi}^*)\int e^{ik_x x}\varphi(k_x)\,dk_x\left(1 - \frac{\omega e^{-z}I_0(z)}{\omega + i\nu_{ii}z}\right) + i\frac{(\omega - \omega_{pe}^*)k_\parallel^2 v_e^2}{\omega\nu_e + ik_\parallel^2 v_e^2}\varphi = 0. \qquad (3.60)$$

Since the oscillations turn out to be aperiodic, it is sufficient to consider the semiclassical approximation.

Assuming that $\omega < \omega^*$, $z = k^2\rho_i^2 \gg 1$, we obtain the following relation from Eq. (3.60) [23]:

$$1 - i\frac{k_\parallel^2 v_e^2}{\omega\nu_e}\left(\frac{\omega}{\omega_{pe}^*} + \frac{1}{\sqrt{2\pi z}}\cdot\frac{1}{1 + i\nu_{ii}z/\omega}\right) = 0. \qquad (3.61)$$

The effect of viscosity [the $i\gamma_{ii}z$ term in Eq. (3.60)] has been introduced by means of the kinetic equation with a Coulomb collision term. It is well known that the Coulomb collision term is in the form of a differential operator in velocity space. For this reason, for shortwave perturbations $z = k_\perp^2\rho_i^2 > 1$ the effective collision frequency in the viscosity coefficient is of order $\nu_i k_\perp^2\rho_i^2$, i.e., this frequency is larger than ν_i by a factor of z [24] This feature leads to a strong stabilization of shortwave perturbations.

From Eq. (3.61) we find that

$$\omega = -i\nu_{ii}z + i\frac{k_\parallel^2 v_e^2}{\nu_e\sqrt{2\pi z}}\left(1 - 2i\frac{k_\parallel^2 v_e^2}{\nu_e\omega_{pe}^*}\right)^{-1}. \qquad (3.62)$$

It follows from this expression that the frequency ω is of the order of the growth rate γ:

$$\gamma \sim \omega \sim \omega_{pe}^*\sqrt{\delta}, \qquad (3.63)$$

while, as before, the localization region is determined by (3.57), i.e., $x \sim \rho\sqrt{\delta}$. This same result can be obtained to within a factor of order unity by solving Eq. (3.60). When $z \gg 1$, this procedure leads to a differential equation for the degenerate hypergeometric function in k space.

We have neglected ion viscosity in the above. It is reasonable that viscosity will play an important role for inertia-type shortwave oscillations $x \sim \rho_i$. If the viscosity is introduced in the hydrodynamic region $x > \rho$ $(\delta > 1)$, we find that these oscillations are stabilized when

$$\theta^2 \gtrsim \sqrt{\frac{m_e}{m_i}} \quad (\delta > 1). \qquad (3.64)$$

In the kinetic region $x < \rho_i$, as follows from Eq. (3.62), the stabilization criterion becomes

$$\theta^2 \gtrsim \sqrt{\frac{m_e}{m_i}\delta} \quad (\delta < 1). \qquad (3.65)$$

Thus, viscosity inhibits the shortwave drift–dissipative oscillation; when $\theta \gtrsim (m_e/m_i)^{1/4}$, the instability is stabilized completely.

5. Gravitational Dissipative Instability

We now wish to consider the effect of curvature of the lines of force on plasma stability. In the hydrodynamic approximation the effect of curvature is equivalent, in some sense, to a gravitational force $g \sim T/m_iR$, which is frequently used to simulate curvature. The instability due to the curvature of the lines of force is then, for brevity, frequently called the gravitational (or flute, or convective) instability.

Neglecting u in Eq. (3.38), we find

$$\omega^2 + \omega(i\omega_s - \omega_{pi}^*) - i\omega_s\omega_{pe}^* + \omega_g^2 = 0, \qquad (3.66)$$

where

$$\omega_g^2 = 2\frac{k_y^2}{k_\perp^2}\cdot\frac{T_i + T_e}{Rm_i}\cdot\frac{d\ln p_0}{dr}.$$

From Eq. (3.66) we find

$$\omega = \omega^*_{pe} + i\,\frac{\omega^2 + \omega_g^2}{\omega_s} - i\,\frac{\omega\omega^*_{pi}}{\omega_s}. \qquad (3.67)$$

There are two important variations of the gravitational instability. When ($\omega_s \gg \omega^*_{pe}$) we are concerned with an inertialess gravitational instability

$$\omega \approx \omega^*_{pe};\ \gamma \sim \frac{\omega_g^2}{\omega_s}, \qquad (3.68)$$

whose region of localization is given by

$$k_{\parallel}^2 v_e^2/\nu_e < \omega^*_{pe}. \qquad (3.69)$$

The meaning of this inequality is a simple one. When the inequality is not satisfied, the electrons can successfully produce a Boltzmann equilibrium along the lines of force and, in accordance with the more complete equation (3.38a), we find that the instability is stabilized under these conditions. It is easy to show, from Eqs. (3.68) and (3.69), that, for the inertialess instability $\omega \ll \omega_s$ ($\omega \sim k_\perp^2 \rho_i^2 \omega_s$). Since the region of localization of this instability is also determined by (3.69), as is the case for the drift–dissipative instability, it will play an important role if its growth rate is greater than the growth rate of the drift–dissipative instability, i.e., for $T_e \sim T_i$, if

$$\omega_g^2 > 2\omega^{*2}_{pe}, \qquad (3.70)$$

Since $\omega_g^2/\omega^{*2} \sim (a/R)(k_\perp \rho)^{-2}$, the effect of the gravitational force is found to be unimportant when $k_\perp^2 \rho_i^2 > a/R$. As is well known, the gravitational instability does not develop in a system characterized by a minimum in B, i.e., in a system in which, on the average, the magnetic field increases in all directions going outward. In our terminology, this corresponds to the case $R < 0$. As is evident from the foregoing, shortwave perturbations characterized by $k_\perp^2 \rho_i^2 > a/R$ are not stabilized in minimum-B configurations.

The characteristic region of localization and the frequency of the inertialess gravitational instability can be expressed in terms of the parameter δ [cf. Eq. (3.57)]:

$$x \sim \rho\sqrt{\delta};\quad \omega \sim \omega^*_{pe};\ \gamma \sim \frac{a}{R}\,\omega^*_{pe}, \qquad (3.71)$$

while the condition that must be satisfied if the gravitational instability is to be more important than the drift instability (3.70) is of the form $(a/R)\delta > 1$.

If the dissipation is small, so that nonlinear effects are not in a position to localize the perturbation, the localization is determined by the ion inertia and the inertialess instability becomes the gravitational inertial instability (the G mode in the terminology of [19]). In this case all three terms in Eq. (3.67) are of the same order of magnitude, i.e.,

$$\omega \sim \omega_g \sim \omega_s. \qquad (3.72)$$

Under these conditions, $k_\parallel^2 v_e^2/\nu_e < \omega$, and the differential equation for the inertial gravitational instability is of the following form, as follows from Eq. (3.67):

$$\rho^2\left(1 - \frac{\omega^*_{pi}}{\omega}\right)\left(1 - i\,\frac{\nu_{ii}}{\omega}\,\rho^2\Delta_\perp\right)\Delta_\perp\varphi - 2\,\frac{a}{R}\left(\frac{\omega^*_{pe}}{\omega}\right)^2\varphi + \frac{k_\parallel^2 v_e^2}{\omega(\omega + i\nu_e)}\left(1 - \frac{\omega^*_{pe}}{\omega}\right)\varphi = 0. \qquad (3.73)$$

Here, we have taken account of the collisional ion viscosity (the ν_{ii} term) as well as the longitudinal electron inertia. It is easy to show from the electron equation of motion that the longitudinal inertia can be introduced in Eq. (3.67) by making the substitution $i\nu_e \to \omega + i\nu_e$.

If viscosity is neglected in Eq. (3.73), then we obtain an equation similar to that for the quantum-mechanical oscillator. Taking the solution in the form

$$\varphi = \exp(-a_0 x^2/\rho^2), \qquad (3.74)$$

which corresponds to the largest growth rate, we obtain the following dispersion relation:

$$\bar\omega\left(\bar\omega + \frac{T_i}{T_e}\right)(\bar\omega - 1) = -\frac{(\bar\omega + i\bar\nu_e)\,m_i}{\theta^2 m_e} \times \left[2\,\frac{a}{R} + k_y^2\rho^2\bar\omega\left(\bar\omega + \frac{T_i}{T_e}\right)\right]^2, \qquad (3.75)$$

where $\bar\omega = \omega/\omega^*_{pe}$, $\bar\nu_e = \nu_e/\omega^*_{pe}$, and the localization condition for the perturbation is of the form

$$\mathrm{Re}\,2a_0 = -\mathrm{Re}\left[k_y^2\rho^2 + 2\,\frac{a}{R}\cdot\frac{1}{\bar\omega\left(\bar\omega + \frac{T_i}{T_e}\right)}\right] > 0. \qquad (3.76)$$

We first consider the collisionless case $\nu_e \to 0$, in which the role of the finite conductivity is played by the electron inertia. Then, as follows from Eq. (3.75), the plasma is stable when

$$\theta > \frac{2a}{R}\sqrt{\frac{m_e}{m_i}}. \tag{3.77}$$

We note that this instability is an extension of the flute instability to the electrostatic case. If the Suydam criterion is satisfied, this criterion being written in the following form in the planar-layer approximation:

$$\theta^2 > \frac{a}{R}\beta, \tag{3.78}$$

there develops a gravitational instability in which the electrons are no longer "frozen," because of their inertia [25]. In order to stabilize this instability we require that (3.77) be satisfied; as is evident from comparison with (3.78), the former is a weaker condition. However, even if (3.77) is satisfied, the introduction of dissipation ($\nu_e \neq 0$) leads to the appearance of unstable solutions. The following cases are possible [26]:

a. $\nu_e \gg \omega \gg \omega^*$, in which case

$$\omega = \omega^*_{pe}\left\{-\mathrm{i}\,\frac{2a}{R}\cdot\frac{1}{\theta}\sqrt{\frac{m_e}{m_i}}\,\nu_e^{-1/2}\right\}^{2/3}; \tag{3.79}$$

b. With increasing temperature we find $\nu_e \to 0$ and, at some point, the inequalities $\omega \ll \omega^*$ and $\omega \ll \nu_e$ are satisfied. In this case,

$$\omega = \omega^*_{pe}\,\frac{T_e}{T_i}\cdot\frac{\mathrm{i}\,\bar{\nu}_e m_i}{\theta^2 m_e}\cdot\frac{4a^2}{R^2}. \tag{3.80}$$

In other words, the growth rate starts to drop rapidly with ν_V. Hence we can assume essential stabilization when $\omega \ll \omega^*$, the stabilization criterion being

$$\frac{1}{m}\sqrt{\frac{m_i}{m_e}}\cdot\frac{a^2}{\lambda_e\rho} < \frac{\theta^2 m_e}{m_i}\cdot\frac{4a^2}{R^2}\left(\frac{T_i}{T_e}\right)^{5/2}. \tag{3.81}$$

Let us now consider the order of magnitude of the localization region for case (a) ($\omega > \omega^*$):

$$x/\rho \sim \frac{R}{a}\left(\frac{4a^2 m_i \bar{\nu}_e}{R^2\theta^2 m_e}\right)^{1/3} > R/a. \tag{3.82}$$

This same result can be obtained from the condition in (3.72). The inertial instability exhibits a localization region which is smaller than the inertialess instability.

6. Temperature Drift Instability

As k_y increases, we find that collisionless instabilities appear. The characteristic region is given by $k_x \sim k_y$ ($k_y x \sim 1$) [27]. Dissipation does not play a role in this case. As a matter of fact, we find $k_\parallel \lambda_e \sim k_y x \theta \lambda_e / a \sim \theta \lambda_e / a > 1$.

The introduction of a large shear $\theta \gg \sqrt{m_e/m_i}$ (to be definite, below in the calculations we will take $\theta \sim 10^{-1}$) simplifies the classification of these instabilities greatly. For reasons of simplicity we shall first neglect effects associated with curvature and inhomogeneity of the magnetic field along the lines of force (trapped particles), as well as the longitudinal current. In this case there are two mechanisms for excitation: (a) collisionless dissipation (Landau damping) on the electrons, the maximum of which is reached at $\omega/k_\parallel v_e \sim 1$, and (b) dissipation due to the ions, which is a maximum when $\omega/k_\parallel v_i \sim 1$.

We note that if the electrons and ions exhibit Boltzmann distributions (in the formulation of the problem given here this corresponds to the condition $\omega \ll k_\parallel v_j$, where v_j is the thermal velocity for the appropriate particle species) then no instability will arise. We assume that the existence of the instability is associated with the departure from electron equilibrium, i.e.,

$$\frac{\omega}{k_\parallel v_e} \gtrsim 1. \tag{3.83}$$

Then, taking account of the fact that $k_\parallel = k_y \theta x / a$, and also that $\omega < \omega^*$, where ω^* is the drift frequency, and using Eq. (3.83), we obtain the characteristic localization region for these oscillations:

$$x_e \sim \sqrt{\frac{m_e}{m_i}}\,\rho_{ie}/\theta. \tag{3.84}$$

The subscript e means that this localization region is characteristic of instabilities excited by electrons; $\rho_{ie} = \sqrt{\frac{T_e}{m_i}}\cdot\frac{1}{\Omega_i}$ is the ion Larmor radius for the specified electron temperature. From (3.84) and the inequality $\theta \gg \sqrt{m_e/m_i}$ it follows that $x_e/\rho_i \ll \sqrt{T_e/T_i}$, where ρ_i is the ion Larmor radius, i.e., when $T_e \sim T_i$, the wavelength of the oscillations is smaller than the ion Larmor radius. In these shortwave oscillations the ions can assume a Boltzmann distribution. We are then led to the following conclusion: instabilities that develop in a plasma with large shear, and which are due to a departure from electron equilibrium, have a region of localization (3.84), and the ions can be regarded as exhibiting a Boltzmann distribution for these instabilities. Only one instability of this kind is known, the electron temperature instability [28].

It is reasonable to ask whether there are oscillations characterized by a larger localization region. If $x \gg x_e$ ($\omega/k_\parallel v_e \gg 1$) the electrons can successfully set up a Boltzmann distribution and the only remaining mechanism for departure from equilibrium must be exhibited by the ions because for these particles it is always true that $\omega/k_\parallel v_i \gg 1$ (§7). This condition gives another possibility for the region of localization:

$$x_i \sim \rho_i/\theta. \qquad (3.85)$$

When $x > x_i$, the ions also can successfully achieve equilibrium [if there are no other mechanisms which can prevent them, for example, particle trapping (§4)] and the oscillations become damped [27]. Thus, for this formulation of the problem the localization (3.85) is the largest possible one and it remains to be seen whether there is an instability in this range. It will be shown below that the only instability of this kind is the electrostatic drift-temperature instability.

In studying the instabilities we will apply the semiclassical local approximation, and since the growth rates are found to be of the order of the frequency, this approximation can be regarded as sufficient.

We start with the drift-temperature instability, which has the largest region of localization, and which is thus most important. The electron temperature instability is similar to the drift-temperature instability in many respects.

For reasons of simplicity, we first consider the case $\omega \gg k_\parallel v_i$. The ion longitudinal motion can be described by the hydrodynamic equation. Assuming that the wavelength transverse to the magnetic field is larger than the ion Larmor radius, we can neglect the transverse inertia of the ions. Then, using the equation of continuity, we find

$$\frac{n_i'}{n_0} = -\frac{k_y c}{H\omega n_0}\cdot\frac{dn_0}{dr}\varphi + \frac{k_\parallel v_z'}{\omega}. \qquad (3.86)$$

Using the expression for the perturbation pressure

$$p_i' = -\frac{k_y c}{H\omega}\cdot\frac{dp_{0i}}{dr}\varphi,$$

we find the longitudinal ion velocity

$$v_z' = \frac{k_\parallel}{m_i\omega}\left(1 - \frac{\omega_{pi}^*}{\omega}\right)\varphi \qquad (3.87)$$

and substitute this expression in Eq. (3.86). Finally, we obtain an expression for the perturbation of the ion density:

$$\frac{n_i'}{n_0} = -\frac{k_y c}{H\omega n_0}\cdot\frac{dn_0}{dx}\varphi + \frac{k_\parallel^2 e}{m_i\omega^2}\left(1 - \frac{\omega_{pi}^*}{\omega}\right)\varphi, \qquad (3.88)$$

where $\omega \gg k_\parallel v_i$; $\omega_{pi}^* = \frac{k_y c}{Hn_0}\,\frac{dp_{0i}}{dr}$ is the ion drift frequency.

Equating (3.88) to the electron density, which exhibits a Boltzmann distribution, $n_e'/n_0 = e\varphi/T_e$, we obtain the following dispersion relations [29]:

$$1 - \frac{\omega_{en}^*}{\omega} - \frac{k_\parallel^2 T_e}{m_i\omega^2}\left(1 - \frac{\omega_{pi}^*}{\omega}\right) = 0. \qquad (3.89)$$

Here, $\omega_{en}^* = -\frac{k_y c T_e}{eHn_0}\cdot\frac{dn_0}{dr}$ is the electron drift frequency. It follows from this relation that Eq. (3.89) is valid only when $k_\parallel^2 T_e \ll m_i\omega^2$, so that it can have solutions that differ from $\omega = \omega_e^*$ only when $\omega \ll \omega^*$. Taking account of this fact, we obtain the following expression for the frequency from (3.89):

$$\omega^2 = -\frac{k_\parallel^2 T_e}{m_i}(1 + \eta), \qquad (3.90)$$

where

$$\eta = \frac{d\ln T_0}{d\ln n_0}.$$

One of the roots indicates excitation. The condition on the applicability $\omega \gg k_\parallel v_i$ indicates that this result holds only when $\eta \gg 1$. Equation (3.90) refers to the region of small $k_\parallel \sim x$. As $k_\parallel$ increases, the second term in Eq. (3.89) becomes unimportant and we obtain a cubic equation for the frequency [17]

$$\omega^3 = -\frac{k_\parallel^2 T_e}{m_i}\,\omega_{Ti}^*\ (\eta \gg 1), \qquad (3.91)$$

from which it is possible to find the frequency and growth rate of the unstable oscillations.

Equation (3.91) applies in the region of small $k_{\|}$, i.e., in the immediate vicinity of the point at which $k_{\|} = 0$. As x increases, a point is reached at which the condition assumed above ($\omega/k_{\|}v_i \gg 1$) is violated and the oscillations fall in a region characterized by $\omega \sim k_{\|}v_i$. The growth rate is reduced, vanishes, and finally becomes negative at sufficiently large values of x. Evidently the frequency region $\omega \sim k_{\|}v_i$ must be treated by the kinetic equation. A general integral equation for electrostatic oscillations can be written [29] in the following form, which is symmetric with respect to the ions and electrons:

$$\int e^{ik_x x}\varphi(k_x)\left\{\frac{1}{T_i}\left[1-\frac{\omega^*_{Ti}}{\omega}e^{-z_i}I_0(z_i)\,y_i^2+i\sqrt{\pi}\,e^{-z_i}I_0(z_i)y_iW(y_i)\times\right.\right.$$
$$\left.\times\left(1-\frac{\omega^*_{ni}}{\omega}+\frac{\omega^*_{Ti}}{\omega}\left(z_i-\frac{I_1(z_i)z_i}{I_0(z_i)}+\frac{1}{2}-y_i^2\right)\right)\right]+$$
$$+\frac{1}{T_e}\left[1-\frac{\omega^*_{Te}}{\omega}e^{-z_e}I_0(z_e)\,y_e^2+i\sqrt{\pi}\,e^{-z_e}I_0(z_e)\,y_eW(y_e)\times\right.$$
$$\left.\times\left(1-\frac{\omega^*_{ne}}{\omega}+\frac{\omega^*_{Te}}{\omega}\left(z_e-\frac{I_1(z_e)z_e}{I_0(z_e)}+\frac{1}{2}-y_e^2\right)\right)\right]+$$
$$\left.+\frac{k^2}{4\pi e^2 n_0}\right\}dk_x=0. \qquad (3.92)$$

Here we have introduced the following notation:

$$\omega^*_{Tj}=\frac{k_y}{m_j\Omega_j}\cdot\frac{dT_j}{dr}\ (j=e\ \ i);\ \ \omega_{nj}=\frac{k_yT_j}{m_j\Omega_j n_0}\cdot\frac{dn_0}{dr};$$

$$e_i=-e_e=e;\ \ z_j=\frac{k_\perp^2T_j}{m_j\Omega_j^2};\ \ k_\perp^2=k_x^2+\left(\frac{m}{r}\right)^2;\ \ k_y=m/r;$$

$$y_j=\frac{\omega}{k_{\|}v_j};\ \ k_{\|}=k_y\theta x/a;\ \ v_j=\sqrt{\frac{2T_j}{m_j}};$$

I_0 and I_1 are the modified Bessel functions:

$$W(y)=e^{-y^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^y e^{t^2}dt\right).$$

For the shortwave oscillations treated below, which are characterized by a localization region $x \sim \rho_i/\theta \ll a$, the macroscopic quantities n_0, dT_0/dr, etc., can be regarded as being constant. Assuming that the localization region

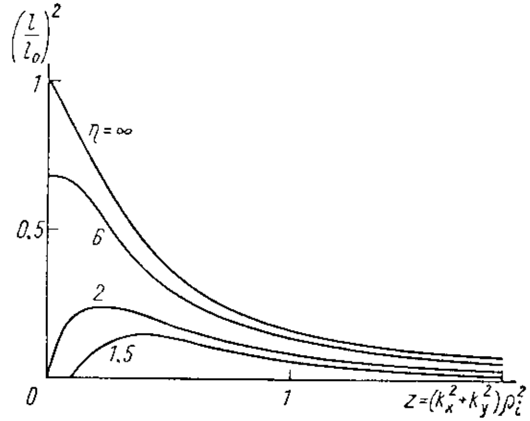

Fig. 6. The width of the localization region as a function of wave number.

contains several nodes, we can express φ in the form of a semiclassical wave function $\varphi(x) = \exp\left(i\int k_x(x)\,dx\right)$. Then, using Eq. (3.92), taking account of the fact that $\omega \ll k_{\|}v_e$ and $z_e \ll 1$, we obtain the following dispersion equation:

$$1+\frac{T_i}{T_e}-\frac{\omega^*_T}{\omega}e^{-z}I_0(z)\,y^2+i\sqrt{\pi}\,e^{-z}I_0(z)\,yW(y)\left\{1-\frac{\omega_n}{\omega}+\right.$$
$$\left.+\frac{\omega^*_T}{\omega}\left[z\left(1-\frac{I_1}{I_0}\right)+\frac{1}{2}-y^2\right]\right\}+i\sqrt{\pi}\sqrt{\frac{m_e}{m_i}}\times$$
$$\times\left(\frac{T_i}{T_e}\right)^{3/2}y\left\{1+\frac{T_e}{T_i}\cdot\frac{\omega^*_n}{\omega}-\frac{T_e}{T_i}\cdot\frac{\omega^*_T}{\omega}\frac{1}{2}\right\}=0. \qquad (3.93)$$

Here, all quantities without subscript refer to the ions.

An important parameter that characterizes the drift-temperature instability is the region of localization l, which can be taken to be the distance from the point $x = r - r_0 = 0$ to the point at which the local growth rate vanishes. Actually, when $x > l$, the growth rate becomes negative and the waves are damped in this region, i.e., the waves are primarily concentrated in the region in which $\gamma > 0$. Taking account of the fact that where $x = l$, $\gamma = \operatorname{Im}\omega = 0$, and equating the real and imaginary parts of Eq. (3.93) to zero

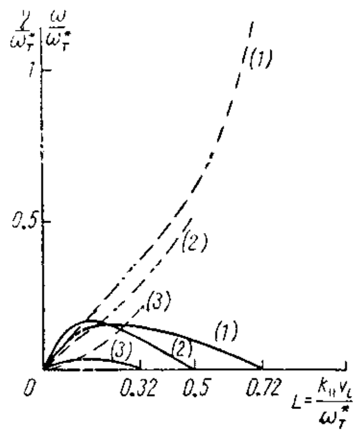

Fig. 7. The local growth rate and frequency as functions of x. 1) $a = 0.5$; $b = 1.6$; 2) $a = 0.5$; $b = 2$; 3) $a = 0.2$; $b = 2$.

separately and then eliminating ω from these two equations, we obtain the following expression for l:

$$\left(\frac{l}{l_0}\right)^2 = \frac{2}{1 + T_i/T_e}\,(e^{-z} I_0(z))^2 \cdot$$

$$\cdot \frac{1 - 2/\eta + 2z(1 - I_1/I_0)}{1 + T_i/T_e - e^{-z} I_0}, \tag{3.94}$$

where $l_0 = \frac{\rho_i}{2\theta} \cdot \frac{d \ln T}{d \ln r}$ is the characteristic region of localization of the drift-temperature instability $\eta = \frac{d \ln T}{d \ln n}$.

It is evident from Eq. (3.94) that the instability being considered can exist ($l^2 > 0$) when [30]

$$\eta \equiv \frac{d \ln T}{d \ln n} > \frac{2}{1 + 2z(1 - I_1/I_0)} \quad \text{for} \quad \eta < 0. \tag{3.95}$$

In the region of relatively large-scale perturbations $x \sim \rho_i/\theta$ ($z \ll 1$), the criterion in (3.95) leads to the condition $\eta > 2$. The minimum value $\eta \approx 0.95$ obtained when $z \approx 1$.

The quantity l^2/l_0^2 is shown as a function of $z = (k_x^2 = k_y^2)\rho^2$ in Fig. 6 for various values of the parameter η for the case $T_i = T_e$. As the temperature ratio T_e/T_i is reduced, the region of localization is reduced, and vice versa. If $T_e = T_i$, $z \ll 1$, and $\eta \gg 1$, i.e., if the case is realized that corresponds to the hydrodynamic analysis for small x at the beginning of this subsection, then $l = l_0$.

Knowing the region of localization l, we can determine how many nodes the solution has for a specified value of θ. As an approximation we find that the number of nodes is given by

$$n \lesssim \frac{k_x l}{\pi}\,\alpha \quad (n = 1, 2 \ldots), \tag{3.96}$$

where n is the closest integer smaller than $k_x l \alpha/\pi$; α is a numerical factor of order unity which need not be determined in the semiclassical approximation. If $k_x l \alpha/\pi < 1$, then, in general, not even one level will exist. This relation

then serves as the shear stabilization condition. A critical value of θ_c can be found by substituting l in the relation $k_x l \alpha/\pi = 1$ from Eq. (3.94). As a result, we obtain θ_c as a function of k_y. The maximum of this quantity with respect to k_y is denoted by θ_{mc}. An approximate expression for this quantity is

$$\theta_{mc} \approx \frac{0.3}{\pi}\,\alpha\left(1 - \frac{0.95}{\eta}\right)^{1/2} (T_e = T_i). \tag{3.97}$$

If $T_e \ll T_i$, then

$$\theta_{mc} \approx \frac{0.5}{\pi}\,\alpha\,\frac{T_e}{T_i}\left(1 - \frac{0.95}{\eta}\right)^{1/2}. \tag{3.98}$$

Another important characteristic of the drift-temperature instability is the growth rate. In the general case, Eq. (3.98) does not contain small parameters, so that numerical calculations are required. In Fig. 7, we show the local growth rate and the frequency as functions of x for various values of the parameters:

$$a = \left[\frac{1}{2} - \frac{1}{\eta} + z\left(1 - \frac{I_1(z)}{I_0(z)}\right)\right]; \quad b = \left(1 + \frac{T_i}{T_e}\right) e^z I_0^{-1}(z).$$

7. Collisionless Instabilities Excited by Electrons

In the transition to oscillations characterized by localization smaller than ρ_i/θ, the dissipation associated with the ions becomes exponentially small and the excitation of the instabilities is then due to the electrons. These are the so-called electron instabilities, and we shall begin our study with the instabilities having the largest growth rate in this class, specifically the electron temperature instability [28].

This instability is characterized by a localization $x_e = \sqrt{\frac{m_e}{m_i}}\,\frac{\rho_i}{\theta}$ [cf. Eq. (3.84)]. The argument of the ion Bessel function is much greater than unity $z_i = x_\perp^2 \rho^2 \sim m_i/m_e$, $\theta^2 \gg 1$ (since we take $\theta \gg \sqrt{\frac{m_e}{m_i}}$) and the ions exhibit a Boltzmann distribution). The dispersion relation for these oscillations can be obtained from Eq. (3.92):

$$1 + \frac{T_e}{T_i} - \frac{\omega_{Te}^*}{\omega} e^{-z_e} I_0(z_e)\, y_e^2 + i\sqrt{\pi}\, e^{-z_e} I_0(z_e)\, y_e W(y_e) \times$$

$$\times\left\{1-\frac{\omega_{ne}^{*}}{\omega}+\frac{\omega_{Te}^{*}}{\omega}\left(z_e-\frac{z_e I_1(z_e)}{I_0(z_e)}+\frac{1}{2}-y_e^2\right)\right\}=0. \quad (3.99)$$

Comparing this relation with the expression for the ion instability (3.93), we see that the two equations are completely identical if the subscript i is replaced by the subscript e, except for the last term in Eq. (3.93). But in many cases this term is unimportant, so that we can carry over the appropriate results for the ion instability to the electron instability simply by replacing i by e. Thus, Eqs. (3.90) and (3.91) remain valid for the width of the localization region, Eq. (3.94), for the form of the growth rate shown in Fig. 7, etc. (The effect of departures from electrostatic oscillations on this instability are discussed in subsection 8.)

In all of the foregoing analyses of collisionless instabilities we have tacitly assumed that when $\omega \ll k_{\|} v_e$ the electrons exhibit a Boltzmann distribution, i.e., $n_e = (e\varphi/T_e)n_0$. Strictly speaking, the expression for the density n_e contains a small imaginary correction of order $\omega/k_{\|}v_e$, which is due to resonance electrons. Since this term falls off as $1/k$, i.e., very slowly in going away from the point $k_{\|} = 0$, then, in principle, it would appear possible to have electron oscillations far from the characteristic region of localization (3.84). In order to examine this question we consider oscillations characterized by $x > \rho_i$. It is a simple matter to obtain the following expression for the frequency of the drift oscillations excited by the electrons [31]:

$$\omega=\omega_e^{*}\left(1-k_{\perp}^2\rho^2-\frac{a}{R}+\frac{k_{\|}^2 T_e}{m_i\omega_e^{*2}}\right)+$$
$$+i\sqrt{\pi}\,\frac{(\omega_e^{*})^2}{|k_{\|}|v_e}\left(k_{\perp}^2\rho^2+\frac{a}{R}-\frac{1}{2}\frac{d\ln T_e}{d\ln n}-\frac{k_{\|}^2 T}{m_i\omega_e^{*2}}\right). \quad (3.100)$$

Here we have taken account of small corrections associated with ion inertia and curvature of the magnetic field. The maximum growth rate in Eq. (3.100) is reached when $x \sim \rho_i$, and at this point the growth rate is a quantity of order $\sqrt{m_e/m_i}$ multiplied by the frequency.

In the instability being considered, the growth rate is small compared with the frequency, and there can be a stabilizing effect due to the finite Larmor radius. The origin of this effect can be understood as follows. Consider a wave packet which is approximately a plane wave with wave vector **k**. Taking account of the finite Larmor radius of the ions leads to a modification of the real part of the frequency by a small quantity of order $\omega^* k_{\perp}^2\rho_i^2$. As a consequence, the group velocity of the drift wave $\mathbf{u}_g = \partial\omega/\partial\mathbf{k}$ acquires a small component $\sim k_x\rho_i^2\omega^*$ in the radial direction and in a time $t \sim x/u_g \sim x/k_x\rho_i^2\omega^*$ the wave packet propagates from the region of localization into a region of absorption. Since $k_x \gg x^{-1}$, the perturbation cannot increase its amplitude significantly if

$$\gamma/\omega < \rho_i^2/x^2 \lesssim k_x^2\rho_i^2. \quad (3.101)$$

Actually, if γ/ω is independent of k_x, then the stability condition can be even weaker. The point is that the oscillation frequency $\omega = \omega_e^*$ is not a characteristic frequency, since it is a function of x. When account is taken of the longitudinal ion inertia, this dependence is still stronger, since the term ω_e^* is supplemented by a term of the form $k_{\|}^2 T/m_i\omega^*$. As a consequence, the component k_x increases in time as $(\partial\omega^*/\partial x)t$. When $x \sim \rho_i/\theta$ and $t_i \sim 1/\gamma$, we find $k_x \sim \omega^*\theta/\rho_i\gamma$, so that the stabilization condition becomes

$$\gamma/\omega < 0. \quad (3.102)$$

The part of the growth rate in Eq. (3.100) which corresponds to the first term $k_{\perp}^2\rho^2$ in the curved brackets corresponds to the usual drift instability. The condition for stabilization of this part can be obtained from (3.101) by taking $x \sim \rho$. This condition assumes the form $\theta \sim \sqrt{m_e/m_i}$ [32].

The second term for the imaginary part in the curved brackets describes the gravitational excitation of drift waves. The stabilization for this part is

$$\theta > \frac{a}{R}\sqrt{\frac{m_e}{m_i}}.$$

As we have seen above, this instability is stabilized when $\theta \gg \sqrt{m_e/m_i}$.

Finally, the third term takes account of excitation of drift waves by virtue of a gradient in electron temperature, in which case an instability is possible only when $\eta_e = \frac{d\ln T_e}{d\ln n} < 0$. The stabilization condition then becomes $\theta > \eta_e\sqrt{m_e/m_i}$.

Thus, all the instabilities that are excited by electrons far beyond the region of characteristic localization (3.84) can be stabilized relatively easily by shear. A single exception might be the current excitation of drift waves. In this case, we obtain the following expression for the drift frequency:

$$\omega=\omega_e^{*}+i\sqrt{\pi}\,\omega_e^{*}\frac{u}{v_e}. \quad (3.103)$$

Using the criterion in (3.102), we find the stabilization condition

$$0 > u/v_e. \tag{3.104}$$

In a plasma with cold ions $T_i \ll T_e$ it is also possible to have electron excitation of ion-acoustic waves. When $T_i \ll T_e$, we obtain the following dispersion relation from Eq. (3.92):

$$1 - \frac{\omega^*_{ne}}{\omega} + \frac{T_e k_\perp^2}{m_i \Omega_i^2} - \frac{k_\parallel^2 T_e}{m_i \omega^2} + i\sqrt{\pi}\,\frac{\omega}{k_\parallel v_e}\left(1 - \frac{\omega^*_{ne}}{\omega} + \frac{1}{2}\,\frac{\omega^*_{Te}}{\omega}\right) = 0. \tag{3.105}$$

Assuming that ω is real, and equating the real and imaginary parts to zero individually, we can obtain the following instability condition:

$$-(1 - \eta_e/2)\,\eta_e > 0, \tag{3.106}$$

where $\eta_e = \frac{d \ln T_e}{d \ln n}$.

The condition $\eta_e < 0$ corresponds to the case given above. Ion-acoustic waves are excited when $\eta_e > 2$. In order not to complicate the calculations unduly, we shall consider the case that is most unfavorable from the point of view of stability $\eta_e \gg 1$ (i.e., $\nabla n_0 \approx 0$). Using Eq. (3.105) we find

$$\omega^2 = \frac{k_\parallel^2 T_e}{m_i}\left\{1 + i\,\frac{\sqrt{\pi}}{2}\sqrt{\frac{m_e}{m_i}}\cdot\frac{1 + \frac{1}{2}\frac{\omega^*_{Te}}{k_\parallel}\sqrt{\frac{m_i}{T_e}}}{1 + k_\perp^2 T_e/m_i\Omega_i^2}\right\}. \tag{3.107}$$

This relationship shows that the maximum growth rate obtains when $x \sim \rho_{ie} \sim \sqrt{T_e/m_i\Omega_i^2}$ (ρ_{ie} is the ion radius given by the electron temperature) and $\gamma \sim \omega\sqrt{m_e/m_i}$. We then find from Eq. (3.102) that these oscillations are stabilized when $\theta > \sqrt{m_e/m_i}$.

Thus, if $\theta > \sqrt{m_e/m_i}$, essentially all of the instabilities excited by electrons are found to be relatively unimportant.

8. Nonelectrostatic Instabilities

We now wish to consider the effect of nonelectrostatic small oscillations, an effect which has been neglected up to now. When $\beta = 8\pi p/H^2 \ll 1$, the plasma does not contain enough energy to cause compression or rarefaction of the lines of force (more precisely, $B_\parallel \sim \beta B_\perp$) so that the magnetic lines of force can be distorted only slightly, if at all. This distortion of the curvature corresponds to propagation of Alfvèn waves and mathematically can be described, as was done in §3, by the introduction of a single component of the

vector potential $A_\parallel$. Thus, Eq. (3.37), which takes account of nonelectrostatic effects, contains only one branch of the oscillations, the Alfvèn wave. The electrostatic approximation adopted in subsections 3-7 of §3, $A_\parallel \cong 0$, corresponds to the condition $\omega \ll k_\parallel c_A$. When $\omega \gtrsim k_\parallel c_A$, it is necessary to consider the entire system. Using the condition $\omega \sim k_\parallel c_A$ we can find the characteristic region in which it is important to take account of nonelectrostatic effects. Actually, when $\omega \gtrsim k_\parallel c_A$ one cannot neglect the rigidity of the lines of force in their distortion or curvature; on the other hand, since all of the considered instabilities are characterized by $\omega \sim \omega^*$, then $\omega^* \sim \omega > k_\parallel c_A$. It then follows that nonelectrostatic effects are important when

$$x < \frac{\rho_i}{\theta}\sqrt{\beta}\,. \tag{3.108}$$

From the equilibrium condition (cf. subsection 1 of §2), $\beta \leq (R/a)\theta^2$. Hence, nonelectrostatic effects become evident when $x \sim \rho_i$, but in this case the hydrodynamic approximation itself is violated and the transverse motion of the ions must be described by a kinetic equation, as in subsection 4 of §3.

We note that the only dissipative instability that is important when $x \sim \rho_i$ is the drift-dissipative instability. A semiclassical equation that generalizes Eq. (3.60) to the nonelectrostatic case can be obtained by taking account of the equation for the vector potential [the second equation in (3.36)]. As a result, we find

$$\frac{T_e}{T_i}\left(1 - e^{-z}I_0\right)\left(1 - \frac{\omega_i^*}{\omega}\right) + i\,\frac{k_\parallel^2 v_e^2/\omega\nu_e}{1 + i\,\frac{k_\parallel^2 v_e^2}{\omega\nu_e}}\left(1 - \frac{\omega_e^*}{\omega}\right) \times$$

$$\times\left[1 - \frac{\omega^2}{k_\parallel^2 c_A^2}\left(1 - \frac{\omega_i^*}{\omega}\right)\frac{1 - e^{-z}I_0}{z}\right] = 0. \tag{3.109}$$

Assuming that $x < \rho_i$, for simplicity we shall first consider the case $x \ll \rho_i$; from Eq. (3.109) we find

$$\omega = i\,\frac{k_\parallel^2 v_e^2}{\nu_e\sqrt{2\pi z}}\cdot\frac{1}{1 - i\,\frac{k_\parallel^2 v_e^2}{\omega_e^* \nu_e}\left(1 + \frac{T_e}{T_i} - \frac{T_i}{T_e}\beta/\theta^2\right)}. \tag{3.110}$$

Comparing Eqs. (3.110) and (3.62), we see that the nonelectrostatic features lead to an unimportant change in the region of localization.

For the collisionless drift instabilities one expects that the nonelectrostatic features will be associated with the electron temperature instability, which has the smallest localization. However, if $\beta \ll \theta^2$, it can be shown that the electrostatic approximation is adequate.

The condition in (3.108) is not satisfied for the temperature drift instability with a localization range $x \sim \rho_i/\theta$; hence, in considering this instability we must also take account of the electrostatic oscillations. More precisely, the nonelectrostatic feature encompasses only the heart of the perturbation, about a singular point of width $x \sim \rho_i/\theta\sqrt{\beta}$.

Thus, the effect of nonelectrostatic features is found to be unimportant for shortwave drift oscillations ($\lambda \ll a$) in systems with sufficiently high shear $\theta^2 > \beta$.

However, it is possible to have nonelectrostatic instabilities in the lowest modes ($m \sim 1$). In ideal hydrodynamics this corresponds to the screw instability. In §2 we have shown that this hydrodynamic instability does not develop for a sufficiently smooth current distribution and a large stability margin q. But if account is taken of dissipation (or electron inertia), which destroys the "frozen-in" properties of the plasma, it is possible that under certain conditions the instabilities will develop in the lowest modes, although with rather small growth rates. This results in the so-called tearing mode [20]. This mode can encompass an appreciable part of the pinch and thus depends sensitively on the distribution of current over the cross section and on the presence of the chamber wall.

We now wish to obtain a dispersion equation for the tearing mode. As we have shown in §3, dissipative processes and inertia appear only in the narrow region $k_{\|} = 0$ ($F = 0$). The ideal hydrodynamics approximation holds elsewhere, and the vector potential satisfies Eq. (3.32):

$$\frac{1}{\mu}(\mu\psi')' = \left(m^2 - \frac{G}{F^2} + \frac{1}{\mu F}(\mu F_1)'\right)\psi,$$

where

$$\frac{1}{\mu}(\mu F_1)' \sim \frac{dj_{\|}}{dr}. \tag{3.111}$$

Here and below we will use the dimensionless variables introduced in §3.

The solution in Eq. (3.111) that satisfies the required boundary conditions [that the vector potential vanish at the center ($r = 0$) and at the chamber wall ($r = b$)] must be joined to the solution of Eq. (3.30) that holds in the vicinity of the singular point $F(\mu_0) = 0$. We shall write the solutions here, retaining only those terms which are important in what follows:

$$\frac{1}{m^2}\psi'' = \psi\left(\frac{1}{\mu_0^2} + \frac{\tilde{p}}{\tilde{\eta} m^2}\right) + \frac{1}{m^2} W \frac{F}{\tilde{\eta}};$$

$$\frac{1}{m^2} W'' = \left(\frac{1}{\mu_0^2} + \frac{F^2 S^2}{\tilde{p}\tilde{\eta}}\right) + \psi S^2 \frac{F}{\tilde{\eta}}. \tag{3.112}$$

Since the width of the singular region in which Eq. (3.111) is inapplicable is much smaller than the characteristic dimensions of the plasma, within the limits of this region we can regard the macroscopic quantities as being constant. Furthermore, for reasons of simplicity, we shall assume $\eta' = 0$, and also that $\omega \gg \omega^*$. The effect of the drift frequency will be considered later. Furthermore, in Eq. (3.112) we omit the derivative of the longitudinal current $dj_{\|}/dr$, since it can be shown that this term does not make any contribution to the subsequent expressions [21].

The exact form of $\psi(\mu)$ depends on the concrete form of the functions H_0 and H_ϑ. In what follows we shall only require an expression for the difference in the logarithmic derivatives

$$\Delta^e = \frac{1}{\psi_2}\cdot\frac{d\psi_2}{d\mu} - \frac{1}{\psi_1}\cdot\frac{d\psi_1}{d\mu}\bigg|_{\mu=\mu_0}, \tag{3.113}$$

which will be assumed to be specified. This quantity can be found if the quantities H_0 and H_ϑ are known. Although the derivatives $(1/\psi)(d\psi/d\mu)$ diverge logarithmically when considered individually, the difference between the derivatives remains finite as $\mu \to \mu_0$.

We shall also find it necessary to solve Eq. (3.112) under the condition that $W \to 0$ when $\mu \to \pm\infty$, while ψ becomes the solution of Eq. (3.111). Actually, we join Δ^i for the inner region with Δ^e (3.113). The expression for Δ^i can be obtained from the first equation in (3.112):

$$\Delta^i = \frac{\psi'(\infty) - \psi'(-\infty)}{\psi} = \frac{1}{\psi}\int_{-\infty}^{\infty}\frac{1}{\tilde{\eta}}(\tilde{p}\psi + WF)\,d\mu. \tag{3.114}$$

In the vicinity of the point $\mu = \mu_0$, the quantity ψ can be regarded as a constant as an approximation. Then the second of the equations can be solved quite easily, for example, by expansion in Hermite functions or by converting to the k representation [21]. The expression obtained for W is then substituted in Eq. (3.114). Carrying out the integration [24], we find

$$\Delta^i = 3\frac{\tilde{p}}{\tilde{\eta}}\,[\tilde{p}\tilde{\eta}/4m^2S^2(F')^2]^{1/4}. \qquad (3.115)$$

Writing $\Delta^i = \Delta^e$, we obtain a dispersion equation, from which it follows that

$$\tilde{p} = \left(\frac{\Delta^e}{3}\right)^{4/5}(2mSF')^{2/5}. \qquad (3.116)$$

The characteristic localization range for this instability is found to be of order

$$\Delta x \sim a\left(\frac{\tilde{p}}{4m^2S^2(F')^2}\right)^{1/4}. \qquad (3.117)$$

It follows from the frequency expression (3.116) that this instability arises only when $\Delta^e > 0$.

This result holds for sufficiently small $m \sim 1$ because as m increases the terms that have been dropped earlier in Eq. (3.12) become important.

When the effect of the finite Larmor radius is introduced, the dispersion equation (3.116) is complicated somewhat [21]:

$$\Delta^i = \frac{3}{\tilde{\eta}}\left[\frac{\tilde{p}(\tilde{p}+i\tilde{\omega}_i^*)(\tilde{p}+i\tilde{\omega}_e^*)^5\tilde{\eta}}{4m^2S^2(F')^2}\right]^{1/4}. \qquad (3.118)$$

Hence, when $\omega^* \to \infty$, we have

$$\tilde{p} \cong -i\tilde{\omega}_e^* + \frac{1}{(2\tilde{\omega}_e^{*2})^{1/3}}\left(\frac{\tilde{\eta}}{3}\,\Delta^i\right)^{4/3}(4m^2S^2(F')^2)^{1/3}, \qquad (3.119)$$

i.e., the oscillations become oscillatory with $\gamma < \omega^*$.

9. Instability in a Dense Plasma

In a dense plasma, i.e., one in which the mean free path λ_e is sufficiently small, the width of the region in which the electron temperature is not equalized along the magnetic field is increased and the perturbation in electron temperature T_e must be taken into account in considering the drift oscillations. As shown in [33], this effect leads to the possibility of an instability in an inhomogeneous plasma, the instability appearing in the formation of filaments that are stretched along the magnetic field. These filaments are regions of increased and reduced temperatures. We shall see below (cf. §10) that this instability is not important, but in order to obtain a complete picture we shall consider it here.

We assume for simplicity that there is no temperature gradient in the equilibrium state (the presence of a temperature gradient does not lead to any basically different results). Since it does not play any significant role in the instability being considered, we shall neglect the ion inertia. It then follows from the condition $\operatorname{div} \mathbf{j}' = 0$ and the relation $\mathbf{j}'_\perp = 0$ that the longitudinal velocities of the ions $v_\parallel^i$ and electrons $v_\parallel^e$ are the same ($j'_\parallel = 0$), i.e., the term containing the frictional force in the longitudinal component of the electron equation of motion is zero. Thus, neglecting the electron inertia, we can write this equation in the form

$$\mathbf{h}\nabla p_e = -enE_\parallel - sn_0\mathbf{h}\nabla T_e, \qquad (3.120)$$

where the second term on the right represents the thermal force, and $s = 0.71$ [18].

For electrostatic waves $E_\parallel = -(\mathbf{h}\nabla)\varphi$. Integrating Eq. (3.120) along the line of force, in the linear approximation we find

$$T_0n' + (1+s)T_e'n_0 - en_0\varphi = 0, \qquad (3.121)$$

where n' is the density perturbation and T_e' is the perturbation in electron temperature.

The localization width for this instability will generally not exceed the maximum width of any of the drift instabilities ρ_i/θ; in practice, as we shall see below, it is actually much smaller. Furthermore, the growth rate and frequency are always smaller than $k_\parallel v_i \ll \omega^*$. Hence, in the linearized equation of continuity for either the electrons or ions (these coincide in the present case) we find

$$-i\omega\frac{n'}{n_0} - i\omega^*\frac{e\varphi}{T_0} + ik_\parallel v_\parallel = 0 \qquad (3.122)$$

where the first and second terms are negligibly small; in the zeroth approximation in ω/ω^* we find $\varphi = 0$, i.e., in accordance with Eq. (3.121):

$$(1+s)n_0T_e' + T_0n' = 0. \qquad (3.123)$$

In order to obtain a complete description of the oscillation it is necessary to consider the ion equation of motion

$$-i\omega m_in_0v_\parallel = -ik_\parallel(n_0T_e' + n_0T_i' + 2T_0n') \qquad (3.124)$$

and the heat-balance equations for the electrons and the ions

$$-i\omega T_e' = -\frac{2}{3} T_0 i k_\parallel v_\parallel - \chi_e k_\parallel^2 T_e' - \nu (T_e' - T_i'); \qquad (3.125)$$

$$-i\omega T_i' = -\frac{2}{3} T_0 i k_\parallel v_\parallel - \nu (T_i' - T_e'), \qquad (3.126)$$

where the term containing $\nu \sim \nu_e(m_e/m_i)$ takes account of the heat exchange between the electrons and ions. We neglect the longitudinal thermal conductivity of the ions, since $\chi_i \ll \chi_e$.

Using Eqs. (3.123)-(3.126), we have

$$\omega^2 = -\frac{4s}{3} k_\parallel^2 v_i^2 \quad \text{for } \chi_e k_\parallel^2 \ll k_\parallel v_i; \qquad (3.127)$$

$$\omega = \pm\sqrt{\frac{2}{3}} k_\parallel v_i + i\frac{1}{2}\left(\frac{2(2s+1)}{3}\frac{v_i^2}{\chi_e} - \nu\right) \quad \text{for } \chi_e k_\parallel^2 \gg k_\parallel v_i. \qquad (3.128)$$

Substituting the numerical values for χ_e and ν from [18], we find that the frequency (3.128) corresponds to damped oscillations, so that the instability can occur only in the region characterized by $\chi_e k_\parallel^2 < k_\parallel v_i$, i.e.,

$$\lambda_e k_\parallel < \sqrt{\frac{m_e}{m_i}}. \qquad (3.129)$$

The localization width for this region of x can be found from (3.129):

$$x = \frac{\rho_i}{\theta}\frac{1}{mS}\sqrt{\frac{m_e}{m_i}}, \qquad (3.130)$$

where m is the azimuthal mode number; $S = \lambda_e \rho_i / a^2$. It is evident that when $S \gg \sqrt{m_e/m_i}$ the localization region for this instability is much smaller than ρ_i/θ; thus our assumption that $\omega \ll \omega^*$ is justified.

According to Eq. (3.127), the growth rate for small perturbations is less than $v_i^2/\chi_e \sim (m_e/m_i)\nu_e$.

We note further that the temperature drift instability considered in subsection 6 of §3 exists, but it is somewhat modified in a dense plasma, in which the mean free path is small ($k_\parallel \lambda < 1$) [34].

In obtaining the dispersion equation (for the case $k_\parallel \lambda < 1$) we start with the equations for two-fluid hydrodynamics [18]. Perturbations characterized by $k_\parallel v_i \ll \omega^*$ (i.e., $x \ll \rho_i/\theta$) in cases in which the shear is not too large ($\theta \ll 1$) will exhibit a characteristic localization region x, which is appreciably greater than ρ_i. In this case, we can neglect the ion inertia in the equations for the transverse velocity components; as a result, we obtain the following equations:

$$\mathbf{v}_\perp^i = \frac{c}{H}\left[\mathbf{h}, \left(\nabla\varphi + \frac{\nabla p_i}{en}\right)\right]; \quad \mathbf{v}_\perp^e = \frac{c}{H}\left[\mathbf{h}, \left(\nabla\varphi - \frac{\nabla p_e}{en}\right)\right], \qquad (3.131)$$

where

$$\mathbf{h} = \frac{\mathbf{H}}{H}.$$

Now, substituting the expression for $v_\perp^i$ in the ion equation of continuity and in the equations for longitudinal motion and thermal conductivity, we finally have

$$\frac{\partial n}{\partial t} + \frac{c}{H}[\mathbf{h}\nabla\varphi]\nabla n + \mathbf{h}\nabla n v_\parallel^i = 0; \qquad (3.132)$$

$$Mn\left(\frac{\partial v_\parallel^i}{\partial t} + v_\parallel^i \mathbf{h}\nabla v_\parallel^i + \frac{c}{H}[\mathbf{h}\nabla\varphi]\nabla v_\parallel^i\right) = -\mathbf{h}\nabla p_i - en\mathbf{h}\nabla\varphi; \qquad (3.133)$$

$$\frac{\partial T_i}{\partial t} + \frac{c}{H}[\mathbf{h}\nabla\varphi]\nabla T_i + v_\parallel^i \mathbf{h}\nabla T_i + \frac{2}{3}T_i \mathbf{h}\nabla v_\parallel^i = 0. \qquad (3.134)$$

The term containing the friction $\sigma j_\parallel$ does not appear in Eq. (3.133). This result follows because in using the neutrality condition div j = 0, Eq. (3.131) and the assumption that we have made earlier that there is no equilibrium current, we have automatically assumed that $j_\parallel = 0$. Furthermore, Eq. (3.133) does not contain a term corresponding to the longitudinal ion viscosity, while Eq. (3.134) does not contain a term with the longitudinal thermal conductivity. These relations are valid if $\omega \gg k_\parallel^2 \chi_\parallel^i$ (ω is the oscillation frequency, and $\chi_\parallel^i$ is the longitudinal ion thermal conductivity [23]). We note that in the cases of interest here, where $\omega \sim k_\parallel v_i$, this inequality coincides with the condition for applicability of collisional hydrodynamics $k_\parallel \lambda < 1$. The system in Eqs. (3.132)-(3.134) must be supplemented by a further equation for the electrons, for which we take the following:

$$-\mathbf{h}\nabla p_e + en\mathbf{h}\nabla\varphi = 0; \quad T_e = \text{const}. \qquad (3.135)$$

Here it is assumed that the electrons can set up a Boltzmann distribution along the lines of force. This condition is indicated by the inequality $\omega \ll$

$k_{\parallel}^2 v_e^2/\nu_e$; when $\omega \sim k_{\parallel} v_i$, this inequality can be written in the form $k\lambda > (m_e/m_i)^{1/2}$. Thus, the analysis given here is valid when $1 > k_{\parallel}\lambda > (m_e/m_i)^{1/2}$.

Linearizing the system in (3.132)-(3.134), and writing the perturbation quantities in the form $\varphi = \varphi(x)e^{-i\omega t + ik_y y + ik_z z}$, we obtain the dispersion relation

$$1 - \frac{\omega_{ne}^*}{\omega} - \frac{k_{\parallel}^2 T_e}{m_i\omega^2}\left(1 - \frac{\omega_{pi}^*}{\omega}\right) - \frac{5}{3}\,\frac{k_{\parallel}^2 T_i}{m_i\omega^2}\left(1 - \frac{\omega_{ne}^*}{\omega}\right) = 0, \quad (3.136)$$

where

$$\omega_{ne}^* = -\frac{ck_yT_e}{eHn}\,\frac{dn}{dx}\,;\ \omega_{pi}^* = \frac{ck_y}{eHn}\,\frac{dp_i}{dx}\,;\ k_{\parallel} = (\mathbf{h}\,\mathbf{k}).$$

It follows from this equation that the instability can arise only when $k_{\parallel} v_i \lesssim \omega_{pi}^*$, with $\omega \lesssim \omega_{pi}^*$. For simplicity we consider the case $k_{\parallel} v_i \ll \omega_{pi}^*$, $\omega \ll \omega_{pi}^*$; using Eq. (3.136), we obtain the following expression for the frequency:

$$\omega^2 = \frac{k_{\parallel}^2 T_i}{m_i}\left(\frac{2}{3} - \eta\right),$$

where

$$\eta = \frac{d\ln T_i}{d\ln n}. \quad (3.137)$$

Thus, the oscillations are excited if $\eta > 2/3$. It will be found that the maximum growth rate is of order

$$\gamma \sim \omega^* \sim k_{\parallel} v_i. \quad (3.138)$$

It then follows, as in the case of the drift-temperature instability for a collision-free plasma, that the characteristic localization region is of order

$$x \sim \rho_i/\theta. \quad (3.139)$$

Using the estimate in (3.138), we find the condition for the applicability of the preceding analysis $k_{\parallel}\lambda < 1$, which can be written in terms of the collision parameter $S = \frac{\lambda\rho_i}{a^2}$: $S \ll 1$.

§4. TRAPPED-PARTICLE INSTABILITY

1. Collision-Free Instability

Everywhere above in our investigation of dissipative instabilities the toroidal geometry of the system has been found to be unimportant in practice. We now wish to investigate instabilities that derive specifically from the toroidal geometry. We first consider the case of a collision-free plasma and assume that the oscillations are electrostatic, so that the distortion of the magnetic field can be neglected. In this case, the kinetic equation (for small oscillations) is written

$$\frac{\partial f'}{\partial t} + \mathbf{v}\nabla f' - \frac{e}{m}\nabla\varphi_0\frac{\partial f'}{\partial \mathbf{v}} + \frac{e}{mc}[\mathbf{v}\,\mathbf{H}]\frac{\partial f'}{\partial \mathbf{v}} = \frac{e}{m}\nabla\varphi'\frac{\partial f}{\partial \mathbf{v}}. \quad (4.1)$$

Here, f is the equilibrium (unperturbed) distribution function, which satisfies the equation

$$\mathbf{v}\nabla f - \frac{e}{m}\nabla\varphi_0\frac{\partial f}{\partial \mathbf{v}} + \frac{e}{mc}[\mathbf{v}\,\mathbf{H}]\frac{\partial f}{\partial \mathbf{v}} = 0, \quad (4.2)$$

where φ_0 is the unperturbed potential for the electric field and φ' is the perturbation; $\mathbf{H}$ is the unperturbed magnetic field; f' is the perturbation in the distribution function. In the equilibrium state φ_0 is a function of the magnetic surface, i.e., of the variable r. We now convert to a coordinate system that moves along ξ and assume that $\nabla\varphi_0$ vanishes. For the localized perturbations being treated here it can be assumed that the transport velocity is independent of r, so that the term containing $\nabla\varphi_0$ in Eqs. (4.1) and (4.2) can be omitted.

We assume further that the equilibrium distribution function f is approximately a Maxwellian f_0 in which the density n and temperature depend on r. Then, to first-order accuracy in Ω_H^{-1}, using Eq. (4.2) we find

$$f = f_0 - \frac{mc}{eH}[\mathbf{h}\mathbf{v}]\nabla f_0, \quad (4.3)$$

where $\mathbf{h} = \mathbf{H}/H$.

From Eq. (4.1) we have

$$f' = \frac{e}{m}\int_{-\infty}^{t}\nabla\varphi'\,\frac{\partial f}{\partial \mathbf{v}}\,dt', \quad (4.4)$$

where the integration is carried out over the unperturbed trajectory $\mathbf{r}' = \mathbf{r}(t)$ which, at time t, passes through the point of observation, $\mathbf{r}$ and $\mathbf{v}$. The motion of a charged particle along the trajectory is specified by the expression

$$\mathbf{r}' = \mathbf{r}_0(t') + \mathbf{n}\frac{v_\perp}{\Omega_H}\cos\alpha' + \mathbf{b}\frac{v_\perp}{\Omega_H}\sin\alpha', \tag{4.5}$$

where $\mathbf{r}_0(t)$ is the trajectory of the guiding center; $\mathbf{n}$ is the normal; $\mathbf{b}$ is the binormal to the line of force of the magnetic field; $v_\perp$ is the transverse velocity component; $\alpha' = \alpha_0 - \Omega_H(t' - t_0)$ is the azimuthal angle in velocity space. Since we are interested in oscillations characterized by frequencies much below the cyclotron frequency, it is necessary to carry out an averaging process over the fast oscillations at frequency Ω_H under the integral sign in Eq. (4.4). Thus,

$$f' = \frac{e}{m}\int_{-\infty}^{t} F'\nabla\varphi'\frac{\partial f}{\partial \mathbf{v}}\,dt'. \tag{4.4a}$$

Here, F' is an operator that acts on φ'; in the $\mathbf{k}$ representation this operator is given simply by $F' = J_0(k'_\perp v_\perp/\Omega n)$, where J_0 is the Bessel function of zero order while the integration in Eq. (4.4a) is carried out over the trajectory of the guiding center.

Since we have assumed that the function f_0 is approximately a Maxwellian, in accordance with Eq. (4.3) we have

$$\frac{\partial f}{\partial v} = -\frac{m\mathbf{v}}{T}f + \frac{mc}{eH}[\mathbf{h}\nabla f_0]. \tag{4.6}$$

Actually, because of the magnetic drift of the particles, the function f_0 is somewhat different from a Maxwellian; however, this difference can be neglected when $\rho_i/a \ll 1$.

Assuming that f is independent of t', and that $a\mathbf{v}'\nabla\varphi' = \frac{d\varphi'}{dt'} - \frac{\partial\varphi'}{\partial t'}$ and writing the dependence of φ' on t' in the form $\exp(-i\omega t')$, we have

$$f' = -\frac{e\varphi'}{T}f + \frac{e}{m}\int_{-\infty}^{t} F'\left\{-\frac{i\omega m}{T}f\varphi' + \frac{mc}{eH}[\mathbf{h}\nabla f_0]\nabla\varphi'\right\}dt'. \tag{4.7}$$

In the first term we have neglected the weak dependence of T on t' due to the deviation of the guiding center from the magnetic surface in the drift motion (taking account of this effect would mean replacing T by some mean value as obtained for a nearby magnetic surface).

Since ζ and ϑ are both periodic, the functions φ' and f' can be written in the form

$$\varphi' = \exp(-i\omega t + im\vartheta - il\zeta)\,\varphi(r, \vartheta), \tag{4.8}$$

where $\varphi(r, \vartheta)$ is a periodic function of ϑ upon which we can impose the requirement that it have a minimum number of nodes in ϑ. Isolating the exponential factor analogous to (4.8) in f', and proceeding in the same way, we obtain an expression for the density perturbations by integrating over $\mathbf{v}$:

$$n' = -\frac{e\varphi}{T}n + \int_{-\infty}^{0} FF'\exp[-i\omega t' + im(\vartheta' - \vartheta) - il(\zeta' - \zeta)] \times$$
$$\times\left\{-\frac{i\omega e}{T}f_0\varphi + \frac{c}{\sqrt{g}H}\left[i(mh_3 + lh_2)\varphi + h_3\frac{\partial\varphi}{\partial\vartheta'}\right]\frac{\partial f_0}{\partial r}\right\}dt'd\mathbf{v}. \tag{4.9}$$

Here, n is the unperturbed density; the second operator under the integral sign F arises in taking the average over azimuth α in velocity space; h_2 and h_3 are covariant components of the vector $\mathbf{h} = \mathbf{H}/H$. In the integration over trajectories, to accuracy of order ρ_i/a we can assume that f_0 and T are constant. Thus, the density expression (4.9) differs from the corresponding expression in cylindrical geometry only in the more complicated nature of the particle motion along the unperturbed trajectory.

Using the neutrality condition, i.e., the condition that the perturbations in the electrons and ion densities be the same, we can obtain a dispersion equation for the frequency ω:

$$\left(\frac{1}{T_e} + \frac{1}{T_i}\right)n\varphi = \sum_{j=i,e}\int\int_{-\infty}^{0} F_jF'_j e^{-i\omega t' + im(\vartheta'_j - \vartheta) - il(\zeta_j - \zeta)} \times$$
$$\times\left\{\frac{i\omega}{T_j}f_{0j}\varphi' - \frac{c}{H\sqrt{g}\,e_j}\left[i(mh_3 + lh_2)\varphi + h_3\frac{\partial\varphi}{\partial\vartheta'}\right]\frac{\partial f_0}{\partial r}\right\}dt'd\mathbf{v}. \tag{4.10}$$

Here, the summation on the right side is carried out over the electrons and ion, in which case $e_i = e$, $e_e = -e$. The relation in (4.10) is a homogeneous integral equation in which ω appears as the characteristic value. In order to write Eq. (4.10) in explicit form we must now make use of the expressions for the particle drift trajectories in a torus that have been developed in §1.

As we have indicated in §1, the most important characteristic feature of particle motion in a torus is the fact that particles can be trapped. The presence of trapped particles leads to the possible development of flute instabilities associated with the trapped particles. The point here is that the trapped particles in a force tube between magnetic mirrors can be completely isolated from other regions in the plasma; consequently, they are completely analogous to trapped particles in a conventional mirror system. In general, these particles will execute an unfavorable magnetic drift in the magnetic field that falls off to the periphery, as in a conventional magnetic trap. Thus, a small flute-type perturbation leads to a polarization or separation of the charges, which in turn amplifies the initial perturbation. The only distinction from a conventional open-ended mirror device is the fact that in the toroidal geometry the flutes associated with the trapped particles are immersed in a plasma containing the transiting particles which, because of their own high longitudinal dielectric constant $\varepsilon = 1 + 8\pi nc^2/k_{\|}^2 T \gg 1$, will, to a considerable degree, neutralize the charges of the trapped particles. However, since $\varepsilon \neq \infty$, complete neutralization cannot be achieved. Thus, the plasma will always be subject to highly retarded flute instabilities associated with the trapped particles.

In order to investigate this instability we now turn to the equation that describes the potential (4.10). In terms of macroscopic effects, the most dangerous perturbations are the large-scale perturbations. Hence, we shall assume that the range of localization of perturbations in r is appreciably greater than Δr, the excursion of the particles in the radial direction in their unperturbed drift. In this approximation, we can neglect the dependence of φ on r on the right side of Eq. (4.10). Speaking more precisely, by $\varphi(r)$ we are to understand a function that has a narrow range of localization compared with a, so that $\omega = \omega(r)$ represent some local value for the characteristic frequency; however, the localization with $\varphi(r)$ exceeds Δr appreciably. In this approximation, $F_i = F_j' = 1$.

Furthermore, in Eq. (4.10), in the factor under the integral sign that multiplies df_0/dr we neglect small terms of order a/R_0. It is also assumed that $m \gg 1$, so that we can neglect the derivative $\partial\varphi/\partial\vartheta'$ compared with $m\varphi$. In this approximation, under the assumption that $T_i = T_e$, we find that Eq. (4.10) can be written in the form

$$2n\varphi = -\sum_j \int_{-\infty}^{0} e^{-i\omega t' + im(\vartheta_j' - \vartheta) - il(\zeta_j' - \zeta)} \left(i\omega f_{0j} - i\frac{cTm}{e_j Hr} \cdot \frac{df_{0j}}{dr} \right) \varphi' dt' d\mathbf{v}. \qquad (4.11)$$

We first consider the integral over t' for the transiting particles. For these particles, $\vartheta' - \vartheta \cong v_{\|}t'/Rq$, $\zeta' - \zeta \cong q(\vartheta' - \vartheta)$ and, consequently, the integral over t' gives a factor of the form $i\{\omega - (m - lq)v_{\|}/qR\}^{-1}$. For the oscillations being treated here (drift oscillations characterized by frequency $\omega \sim \omega_* \sim m\rho_i v_i/a^2$), when $m - lq \sim 1$ this factor is appreciably smaller than ω, even for the ions. Consequently, the contribution of the transiting particles in the integral in Eq. (4.11) can be neglected compared with the factor 2φ on the left side.

In computing the integral over t' for the trapped particles, we shall assume $\zeta_j' - \zeta = q(\vartheta_j' - \vartheta) + \xi'$. Then, the exponential that multiplies $\vartheta_j' - \vartheta$ will contain a factor $(m - lq)$. The number m is as yet unspecified. It is evident that for any given value of l we can always choose a value of m such that the difference $m - lq$ is much smaller than one-half. This choice of m means that φ is a function with a minimum number of nodes in ϑ. We shall first consider the simpler case $m - lq \ll 1$. Then, the term $(m - lq) \cdot (\vartheta' - \vartheta)$, which is a periodic function of t' of order $m - lq$, is to be neglected for the trapped particles. Furthermore, neglecting the quantity $l\xi$ over one period of the oscillation, we can carry out the averaging of ξ_j' over the period of oscillation τ_j and take $\xi_j' = v_j t'$. Under these conditions, Eq. (4.11) is simplified to

$$2n\varphi = \sum_j \int \frac{1}{\omega + v_{\xi j} l} \left(\omega f_{0j} - \frac{mcT}{re_j H} \cdot \frac{df_{0j}}{dr} \right) \int_{-\tau}^{0} \varphi \frac{dt'}{\tau} d\mathbf{v}. \qquad (4.12)$$

Here, the integration over t' is replaced by an integration over the angle ϑ', it being assumed that $\sqrt{\varepsilon} = \sqrt{r/R}\; dt'/\tau = (2\chi^2 - 1 - \cos\vartheta')^{-1/2} d\vartheta'$. Furthermore, $d\mathbf{v}$ for trapped particles and $\varepsilon \ll 1$ is of the form $d\mathbf{v} = 2\pi v^2 dv \cdot d\gamma_{\vartheta}$, where $\gamma_{\vartheta} = \pi/2 - \psi$ is the angle in velocity space at the point ϑ. By conservation of the transverse adiabatic invariant we find

$$v_{\perp}^2/v^2 = \cos^2\gamma_{\vartheta} = \cos^2\gamma\,(1 + \varepsilon\,(1 + \cos\vartheta)), \qquad (4.13)$$

where γ is an angle we have introduced earlier in velocity space at the point $\vartheta = \pi$, i.e., $\gamma = \gamma_{\pi}$. In accordance with Eq. (4.13), for small ε we find $\gamma_{\vartheta}^2 = \gamma^2 - \varepsilon(1 + \cos\vartheta)$. Thus,

$$d\gamma_{\vartheta} = \frac{\gamma d\gamma}{\gamma_{\vartheta}} = \frac{\sqrt{\varepsilon}\, d\chi^2}{\sqrt{2\chi^2 - 1 - \cos\vartheta}}. \qquad (4.14)$$

Substituting the expression obtained for dt' and $d\gamma_{\vartheta}$ and the expression (1.50) for τ in Eq. (4.12), we obtain an integral equation for the potential φ:

$$\varphi(\vartheta) = \sum_j \int \frac{2\pi\sqrt{\varepsilon}\, v^2 dv}{\sqrt{2}\, n(\omega + v_{\zeta j} l)} \left(\omega f_{0j} - \frac{mcT}{re_j H} \cdot \frac{df_{0j}}{dr}\right) \times$$

$$\times \int\limits_{\frac{1+\cos\vartheta}{2}}^{1} \frac{d\chi^2}{K(\chi)\sqrt{2\chi^2 - 1 - \cos\vartheta}} \int\limits_{\vartheta_0}^{\pi} \frac{\varphi(\vartheta')\, d\vartheta'}{\sqrt{2\chi^2 - 1 - \cos\vartheta'}}, \qquad (4.15)$$

where $\vartheta_0 = \arccos(2\chi^2 - 1)$.

In the case being considered here (equal temperatures) the drift velocities of the electrons and ions $v_{\zeta e}$ and $v_{\zeta i}$ are opposite in sign but equal in absolute magnitude for a given energy $m_j v^2/2$. This feature leads to a considerable simplification of the equation. Assuming that T = const, we can write the equation in the form

$$\varphi(\vartheta) = \int \frac{2\pi\sqrt{\varepsilon}\, v^2 dv f_{0i}(\omega^2 - \omega^* l v_{\zeta i})}{\sqrt{2}\, n(\omega^2 - l^2 v^2_{\zeta i})} \int\limits_{\frac{1+\cos\vartheta}{2}}^{1} \frac{d\chi^2}{K(\chi)\sqrt{2\chi^2 - 1\cos\vartheta}} \times$$

$$\times \int\limits_{\vartheta_0}^{\pi} \frac{\varphi(\vartheta')\, d\vartheta'}{\sqrt{2\chi^2 - 1 - \cos\vartheta'}}, \qquad (4.16)$$

where

$$\omega^* = -\frac{mcT}{reHn} \cdot \frac{dn}{dr}.$$

Since Eq. (4.16) depends on the square of ω, in the presence of an instability, it is natural to take ω^2 as being a real quantity, i.e., $\omega = i\gamma$. In this case the denominator $\omega^2 - l^2 v^2_{\zeta i} = -(\gamma^2 + l^2 v^2_{\zeta i})$ becomes a monotonically increasing function of γ. We now recall that v_ζ is a small quantity of order ε. Since Eq. (4.16) contains the small factor $\sqrt{\varepsilon}$ on the right, it can be of the order unity only if the denominator $\gamma^2 + l^2 v^2_{\zeta i}$ becomes sufficiently small. In this case we neglect γ^2 in the numerator compared with $\omega^* l v_{\zeta i}$. Thus, in order-of-magnitude terms, from Eq. (4.16) we have

$$1 \sim \sqrt{\varepsilon}\, \varepsilon\omega^{*2}(\gamma^2 + \varepsilon^2\omega^{*2})^{-1}.$$

It is then evident that $\gamma^2 \sim \omega^{*2}\varepsilon^{3/2}$. Neglecting the quantity $l^2 v^2_\zeta \sim \varepsilon^2\omega^{*2}$ compared with γ^2 in the denominator and carrying out the integration over v^2, we reduce Eq. (4.16) to the form

$$\lambda\varphi(\vartheta) = \frac{\pi}{4} \int\limits_{\frac{1+\cos\vartheta}{2}}^{1} \frac{d\chi^2 G(\chi)}{K(\chi)\sqrt{2\chi^2 - 1 - \cos\vartheta}} \int\limits_{\vartheta_0}^{\pi} \frac{\varphi(\vartheta')\, d\vartheta'}{\sqrt{2\chi^2 - 1 - \cos\vartheta'}}, \qquad (4.17)$$

where $G(\chi)$ is a function that is given by Eq. (1.57) and arises as a result of the substitution of the explicit value for v_ζ, $\lambda = \gamma^2\sqrt{2}\pi/3\sqrt{\varepsilon}\,\omega_m\omega^*$; $\omega_m = 2\, lq\varepsilon cT/eHr^2 \simeq 2mcT/eHrR_0$. Since v_ζ is proportional to v^2, for the case $\nabla T \neq 0$ by ω^* we are to understand $\omega^*_p = \omega^* \frac{d \ln p}{d \ln n}$. In solving Eq. (4.17) it is convenient to make use of a Fourier representation, expanding the quantity $\varphi(\vartheta)$ in a series in $\cos\vartheta$. For this purpose we write

$$\varphi(\vartheta) = \sum_s e^{-is\vartheta}\varphi_s, \qquad (4.18)$$

where φ_s and φ_{-s} are real coefficients. Substituting this expression in Eq. (4.17), multiplying it by $e^{is\vartheta}$, and integrating over ϑ, we have

$$\lambda(1 + \delta_{s0})\varphi_s = \sum_{s'} F_{ss'}\varphi_{s'}, \qquad (4.19)$$

where

$$F_{ss'} = F^1_{ss'} + \frac{2q'r}{q} F^2_{ss'};$$

$$F^j_{ss'} = \int\limits_0^1 \frac{G_j \Pi_s \Pi_{s'}}{K(\chi)}\, d\chi; \qquad (4.20)$$

$$\Pi_s(\chi) = \int\limits_{\vartheta_0}^{\pi} \frac{\cos s\vartheta\, d\vartheta}{\sqrt{2\chi^2 - 1 - \cos\vartheta}}. \qquad (4.21)$$

The characteristic value λ is found by setting the determinant of the matrix corresponding to equations (4.19) equal to zero. The values of the matrix elements $F^j_{ss'}$, for $ss' \leq 2$ are given in the table.

It is evident that $F^j_{ss'}$ diminishes rapidly as s increases. Hence, in computing the largest value of λ, which corresponds to the most unstable oscillation, it will be sufficient to bound the system to finite order, assuming that $\varphi_s = 0$ for some value of s larger than the critical value s_0. Confining our-

$F^1_{ss'}$			
s	$s'=0$	$s'=1$	$s'=2$
0	0.681	−0.543	0.011
1	−0.543	0.380	−0.151
2	0.011	−0.151	0.161

$F^2_{ss'}$			
s	$s'=0$	$s'=1$	$s'=2$
0	0.888	−0.145	−0.189
1	−0.145	0.128	−0.029
2	−0.189	0.029	0.091

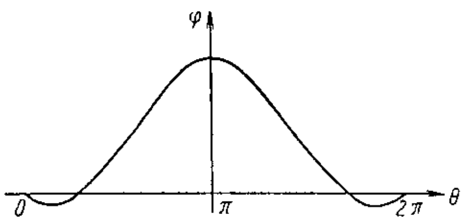

Fig. 8. The amplitude of the oscillations of the potential φ as a function of the angle ϑ.

selves to a system of equations of second order, for the case $q' = 0$, we find $\lambda = 0.74$. The second root is approximately equal to zero and lies at essentially the limits of applicability of the present analysis. Recalling the definition of λ, and making use of the numerical value that has been obtained, we find

$$\gamma^2 \cong \frac{\sqrt{\varepsilon}}{2}\,\omega_m\omega_p^*. \tag{4.22}$$

Taking account of the third equation in (4.19) we can find the first three terms in the Fourier expansion of $\varphi(\vartheta)$. When $q' = 0$, the solution for $\varphi(\vartheta)$ is of the form

$$\varphi(\vartheta) \cong 1 - 1.4\cos\vartheta + 0.4\cos 2\vartheta. \tag{4.23}$$

The function $\varphi(\vartheta)$ is shown graphically in Fig. 8, and is localized in the region $\vartheta \approx \pi$, in which the particles execute unfavorable drift motion. In other words, the oscillations develop primarily in the external region of the torus; the oscillation amplitude is very small in the region $|\vartheta| < \pi/2$.

We now wish to consider the effect of changing q' on the instability. We have already noted that increasing q' leads to a stronger instability, and that when $q' < 0$ there is some stabilization associated with the reduction of the length of the line of force between the turning points in the outward motion. By writing $\lambda = 0$ we can easily find the critical value for the parameter $q'r/q$ (below which stability obtains). This value is found to be −1.5, so that the following condition is the stabilization criterion for the trapped-particle instabilities:

$$\frac{d\ln q}{d\ln r} < -{}^3/_2. \tag{4.24}$$

We now wish to consider Eq (4.16) We recall the transition to the simpler equation (4.17) means neglecting the quantity $l^2 v^2_{\zeta i}$, which is of order ε compared with $\gamma^2 \sim \varepsilon^{3/2}$. However, since γ^2 is proportional to dn/dr, while v_ζ is independent of density, this procedure is not always valid even for small values of ε. Since γ^2/v^2_ζ at best $\sim\sqrt{\varepsilon}$, the conversion from Eq. (4.16) to (4.17) can be questionable even if the density gradient is reasonably large. It will be evident that a reduction in $|dn/dr|$ leads to a reduction in the growth rate γ, and that for some critical value dn/dr the quantity γ vanishes altogether. In order to find this critical value of the gradient we write $\omega = 0$ in Eq. (4.16) and then carry out the Fourier analysis. As a result, for the case $T = \text{const}$ we find

$$\mu(1+\delta_{s0})\varphi_s = \sum_{s'} P_{ss'}\varphi_{s'}. \tag{4.25}$$

Here,

$$\mu = \frac{\pi\omega_m}{\sqrt{2\varepsilon}\,\omega^*}\ ;\quad P_{ss'} = \int_0^1 \frac{\Pi_s\Pi_{s'}\,d\chi^2}{K(\chi)\,G(\chi)}, \tag{4.26}$$

where the integral over χ^2 is to be understood in the sense of the principal value.

The minimum value of ω^* corresponds to the maximum root for μ. When $rq/q \to \infty$ the matrix elements $P_{ss'}$ (and consequently the roots of μ) are both reduced. On the other hand, when $q' < 0$, the quantity $G(\chi)$ can change sign and the matrix elements $P_{ss'}$ become smaller for this reason than for the case $q' > 0$. Thus, the largest value of μ corresponds to the value $rq/q' \sim 1$. We have carried out a numerical calculation of the roots of μ for $rq'/q = 1/2$. For these values, $P_{00} = 11.9$, $P_{12} = 2.02$, $P_{22} = 2.24$, whence we find that μ_1 6.5 and $\mu_2 = 1.7$. Taking the larger value of μ_1 and substituting in Eq. (4.26) for μ, we find the stability condition

$$-\frac{d\ln n}{d\ln r} < \frac{1}{3}\sqrt{\frac{r}{R_0}}\,. \tag{4.27}$$

(We recall that in this case the temperature T is assumed to be constant over the cross section.)

In practice this condition can only be realized in isolated narrow ranges of r.

It will be recalled that in deriving the integral equation for φ we have assumed that $m - lq \ll 1$. It would appear that this inequality can be satisfied for values of q close to an integer. However, as an approximation it can be assumed that $m \cong lq$ even when lq is not equal to an integer. This feature follows because because we have neglected the transiting particles entirely. As a result, as is evident from Eq. (4.16), when $\vartheta \to 0$ the function $\varphi(\vartheta)$ also tends to vanish (this can be seen in Fig. 8 for the particular case $q' = 0$). Hence, in a complete solution of the form $\varphi = \exp(im\vartheta - il\zeta)\varphi(\vartheta)$, it is admissible to have an arbitrary phrase discontinuity through the point $\vartheta = 0$, so that the number m must necessarily be an integer, and we can set it equal to lq. It will be evident that m cannot be equal to lq precisely because it would then be impossible to neglect the contribution to the density given by the transiting particles, which is of order $\omega Rq/v_\parallel (m - lq)$. But when $\omega Rq/v_\parallel \sim \frac{qm\rho_i}{r} \ll 1$ the difference $m - lq$ can be taken as equal to zero with a high degree of accuracy. Thus, by assuming that $m \cong lg$, where l is an integer in the solution obtained above, we can extend the solution to the case of arbitrary r. By taking account of the small terms due to the displacement of particles in the radial direction Δr, in principle we could consider the problem of localization of the characteristic functions of the linear approximation in the radial direction. However, in practice, this is not required, as we shall see in our investigation of nonlinear oscillations.

In concluding this subsection, we shall consider briefly the instability of a nonisothermal plasma ($T_e \neq T_i$). In the Fourier representation the appropriate equation is

$$\varphi_s = \sum_{s'} \int \frac{2\sqrt{2\varepsilon} f_{0i} v^2 dv}{n} \int \frac{\Pi_s(\chi)\Pi_{s'}(\chi)}{K(\chi)} \varphi_{s'} \times$$

$$\times \frac{\omega^2 + \omega\omega_m G(\chi)\dfrac{v^2}{v_i^2}(\Delta - 1) - \Delta\omega_m\hat{\omega}^*\dfrac{v^2}{v_i^2}G(\chi)}{\omega^2 + \omega\omega_m G(\chi)\dfrac{v^2}{v_i^2}(\Delta - 1) - \Delta\left(\omega_m G(\chi)\dfrac{v^2}{v_i^2}\right)^2} d\chi^2, \tag{4.28}$$

where

$$\hat{\omega}^* = -\frac{T_i cm}{eHrf_{0i}} \cdot \frac{df_{0i}}{dr};\ \Delta = T_e/T_i;\ v_i^2 = 2T/m_i,$$

while the remaining notation is the same as given above.

When $\Delta \neq 0$ the right side of Eq. (4.28) contains terms that are linear in ω and these can lead to a stabilization of the instability. In order to examine this feature we average Eq. (4.28) over ϑ, and write

$$1 + \frac{\tilde{\omega}_m}{\omega}(\Delta - 1) + \sqrt{\varepsilon}\,\frac{\tilde{\omega}_m\tilde{\omega}^*}{\omega^2}\Delta = 0, \tag{4.29}$$

where by $\tilde{\omega}^*$ and $\tilde{\omega}_m$ we are to understand certain effective values which can differ by numerical factors of order unity from the values computed earlier for ω^* and ω_m. It is evident from Eq. (4.29) that the instability can occur only within the interval defined by the expression

$$\frac{\tilde{\omega}_m}{\tilde{\omega}^*} \cdot \frac{1}{4\sqrt{\varepsilon}} < \frac{T_e}{T_i} < 4\sqrt{\varepsilon}\,\frac{\tilde{\omega}^*}{\tilde{\omega}_m}\,. \tag{4.30}$$

The plasma is stable outside this interval. Since $\tilde{\omega}_m/\tilde{\omega}^* \sim \varepsilon$, the instability range (4.30) is extremely broad.

2. Dissipative Instabilities Due to Trapped Particles

Since instabilities due to trapped particles in a finite system are associated with the marked differences in particle motion, one expects that these instabilities will be very sensitive to collisions. The point here is that by virtue of collisions trapped particles can be expelled into the "transit cone" in velocity space, i.e., the perturbation can be damped at a frequency ν_{eff}. When $\varepsilon \ll 1$, the angle γ_0 in velocity space, which separates the region of trapped particles from transiting particles, is of order $\sqrt{\varepsilon}$. The fraction of trapped particles is correspondingly small. Hence, in the Landau collision term for the trapped particles we need retain only the term with the second derivative of the distribution function. Thus, as an approximation we can write the diffusion form

$$St \cong \nu_j v_j^2 \Delta_v f', \tag{4.31}$$

where Δ_v is the Laplacian in velocity space, ν_j is the collision frequency for particles of species j, $v_j^2 = 2T_j/m_j$. In the collision term the largest factor is the one that contains the second derivative in γ and is of order st $\sim \nu_i/\gamma_0^2 \sim \nu_i/\varepsilon$. In other words, $\nu_{eff} = \nu/\varepsilon$. If we consider the effect of collisions on f' in (4.1) and write a collision term of the form $\nu_{eff} f'$, it will be evident from the earlier calculations that collisions can be introduced into the dispersion equation (4.12) by replacing ω by $\omega = i\nu_{eff}$ in the denominators of the integrands. In a reasonably dense plasma the transiting particles can be assumed to exhibit a Boltzmann distribution when collisions are introduced. Hence, as an approximation the dispersion relation can be written in the form

$$2 = \sqrt{\varepsilon}\,\frac{\omega - \omega^*}{\omega + i\nu_i/\varepsilon - \omega_m} + \sqrt{\varepsilon}\,\frac{\omega + \omega^*}{\omega + i\nu_e/\varepsilon + \omega_m}. \tag{4.32}$$

Here, the terms on the right, which take account of the contribution of trapped particles, contain the factor $\sqrt{\varepsilon}$, which is equal to the fraction of trapped particles; ω_m is the frequency associated with the magnetic drift and the factor $1/\varepsilon$ that multiplies the frequencies ν_i and ν_e takes account of the diffusional nature of Coulomb collisions.

For reasons of simplicity we shall limit ourselves to the case $T_e = T_i$. In Eq. (4.32) we can neglect ω as compared with ω^* in the numerator; furthermore, in general we can neglect the magnetic drift ω_m. In the last term we then have only the term with $i\nu_e/\varepsilon$ in the denominator. With these approximations, we find

$$\omega = -\frac{\sqrt{\varepsilon}}{2}\omega^* + i\,\frac{\varepsilon^2}{4}\cdot\frac{\omega^{*2}}{\nu_e} - \frac{i\,\nu_i}{\varepsilon}. \tag{4.33}$$

It will be evident that an instability appears when $\nu_e\nu_i < (\varepsilon^3/4)\omega^{*2}$; it is also evident that this instability does not depend on the sign of the curvature of the lines of force, i.e., ω_m. It is natural to call this a dissipative trapped-particle instability. As ν_e is reduced, the growth rate increases, reaching a maximum value $\gamma_{max} \sim \sqrt{\varepsilon}\,\omega^*$ when $\nu_e \sim \varepsilon^{3/2}\omega^*$, beyond which it is reduced. The qualitative dependence of the growth rate on frequency ν_e is shown in Fig. 9, from which it follows that the dissipative trapped-particle instability is stabilized by ion–ion collisions when ν_e increases.

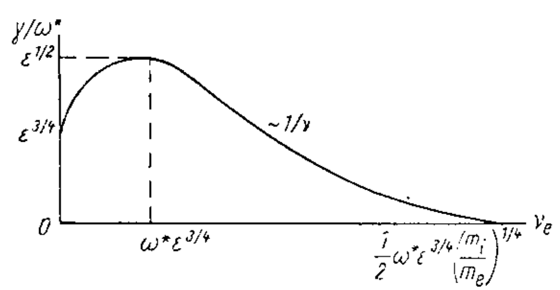

Fig. 9. The growth rate as a function of the collision frequency.

3. Instability Associated with Finite Orbits

The excitation of drift waves by electrons is formally due to the fact that the oscillation frequency ω is smaller than the drift frequency ω^*. Using the expression for the energy absorbed per unit volume of plasma $\dot{W} = \mathbf{Ej}$ and the expression for the longitudinal electron current $\mathbf{j}$ (3.27), we find

$$\dot{W} = -\frac{e^2 n}{m_e \nu_e} k_z^2 |\varphi|^2 \left(1 - \frac{\omega_e^*}{\omega}\right). \tag{4.34}$$

It then follows that $\dot{W}$ changes sign when $\omega = \omega_e^*$, i.e., the wave damping becomes wave excitation. Equation (4.34) is valid for a collisional plasma, but a similar result obtains when the dissipation is associated with resonant particles. Thus, any effect that reduces ω can lead to the excitation of oscillations. One such effect is the transverse ion inertia; when the transverse ion inertia is introduced, the drift frequency becomes $\omega = \omega^*(1 - k_\perp^2\rho_i^2) < \omega^*$, where ρ_i is the ion Larmor radius. The reduction in ω is a result of the fact that the ions gyrate over a finite orbit in the magnetic field. As a result, the particles do not "see" the field at a given point; rather they see an average of the field over an orbit, which is smaller than the true field, and this means a reduction in frequency. It follows from Eq. (4.34) and the expression for ω that this effect becomes larger, the larger the dimensions of the orbit, i.e., the larger the value of ρ_i.

In the inhomogeneous magnetic field in a trapping system the particles not only gyrate in cyclotron orbits, but also traverse a rather complicated trajectory (cf. Fig. 3) with characteristic dimensions Λ. In a circular torus the dimensions of the orbit Λ are found to be of order $\rho_i\sqrt{R/a}$ (cf. §1) and in the stellarator this dimension can be equal to the transverse dimensions of the system [36]. In these oscillations a particle can traverse an orbit several times and the possibility exists that an averaging process occurs in this gyration (in precisely the same way as the averaging over the cyclotron gyration). As a result, the frequency of the drift oscillations is now found to be given by the expression $\omega = \omega^*(1 - k_\perp^2\rho^2 - k_\perp^2\Lambda^2) \simeq \omega^*(1 - k_\perp^2\Lambda^2)$, since $\Lambda \gg \rho$. Thus, in the case being considered, the particle drift in the inhomogeneous magnetic field plays a role analogous to that of the transverse ion inertia. For this reason, the instability that arises might be called the finite-orbit instability. Qualitatively the appropriate dispersion relation for this instability can be obtained from the dispersion equation for the inertial drift instability by making the substitution $\rho_i \to \Lambda$ in the inertia term [37] (the same applies for the more complete equation). For example, in the drift dissipative instability, to take account of finite-orbits we can write the equation

$$\omega^2 + \omega\left(i\tilde{\omega}_s - \omega_i'\right) - i\tilde{\omega}_s\omega_e^* = 0, \qquad (4.35)$$

where $\tilde{\omega}_s = \omega_s(\Lambda/\rho_i)^2$. It follows from this equation that taking account of finite-orbits means that the maximum growth rate $\gamma \sim \omega^* \sim \tilde{\omega}_s$ is now reached for longer wavelengths $\tilde{\lambda} \sim \lambda(\Lambda/\rho_i)^2$. It will be shown in §10 that this effect can lead to a significant increase in the effect of diffusion.

It now remains to compute the characteristic dimension of the orbit Λ.

The basic mechanism that causes the drift of guiding centers in toroidal systems is the inhomogeneity of the magnetic field. For example, for a circular torus such as Tokomak, the magnetic field at the inner region of the chamber is stronger than at the outer surface because of the toroidal geometry. As a result, in the motion along the line of force the absolute magnitude of the total field is found to be a variable quantity. This is equivalent to an effective bumpiness of the longitudinal magnetic field. Similarly, in toroidal systems such as the stellarator the bumpiness arises by virtue of the existence of windings which provide the rotational transform. In order to consider these effects qualitatively the longitudinal magnetic field in systems of this kind is usually written in the form [cf. Eq. (1.3)]:

$$H_\zeta = H_0\left(1 - \frac{r}{R_{0c}}\cos\frac{2}{L_R}\zeta\right). \qquad (4.36)$$

Here, ζ is a coordinate measured along the line of force, R_{0c} is the characteristic scale size, and L_R is the period of corrugation of the field. Using the drift equations we can show quite easily that, in the field described by Eq. (4.36) the maximum displacement of a particle from a given magnetic surface is of order $\Lambda_0 \approx L_R v_i/R_{0c}\Omega_i$. However, it is not this quantity that appears in the dispersion relation, but rather the displacement over the period of the oscillation $t = 2\pi/\omega$, which is equal to $\Lambda = \Lambda_0(\omega_R/\omega)$, where $\omega_R = 2v_i/L_R$ is the inverse transit time for the distance between "bumps."

An exact calculation carried out by means of the kinetic equation [37] leads to the value

$$\Lambda = \left(\frac{1}{2}\sqrt{\frac{7}{8}}\,\frac{L_R}{R_{0c}}\,\frac{v_i}{\Omega_i}\right)\left(\frac{\omega_R}{\omega}\right). \qquad (4.37)$$

This expression holds when $\omega > \omega_R$.

§5. HIGH-FREQUENCY INSTABILITIES

1. Drift Cyclotron Instabilities

When the density gradient is increased the development of a drift instability at harmonics of the cyclotron frequency $\omega \sim n\Omega_i$ becomes possible. We first consider the collisionless case and then take account of collision.

A general dispersion relation that describes the electrostatic oscillations of a plasma at arbitrary frequencies is given in [29]. For frequencies close to $n\Omega_i$ (and the contribution in the density perturbation from other harmonics can be neglected), we find

$$k^2d_i^2 + 1 + \frac{T_i}{T_e} + \left(1 - \frac{\omega_i^*}{\omega}\right) i\sqrt{\pi}\,\frac{\omega}{k_\parallel v_i}\,W\left(\frac{\omega - n\Omega_i}{k_\parallel v_i}\right)e^{-z}I_n(z) +$$
$$+ \frac{T_i}{T_e}\left(1 - \frac{\omega_e^*}{\omega}\right) i\sqrt{\pi}\,\frac{\omega}{k_\parallel v_e}\,W\left(\frac{\omega}{k_\parallel v_e}\right)e^{-z_e}I_0(z_e) = 0, \qquad (5.1)$$

where

$$\omega_i^* = \frac{k_y T_i c}{eHn}\cdot\frac{dn}{dr};\quad \omega_e^* = -\frac{k_y T_e c}{eHn}\cdot\frac{dn}{dr};$$
$$z = \frac{k_\perp^2 T_i}{m_i\Omega_i^2};\quad z_e = \frac{k_\perp^2 T_e}{m_e\Omega_e^2};\quad d_i^2 = \frac{T_i}{4\pi e^2 n}.$$

Since the drift cyclotron oscillations are characterized by extremely short wavelengths ($k_\perp\rho_i \gg 1$), it can be assumed that shear will not have an important effect on these oscillations.

We first consider waves characterized by $k_\parallel \cong 0$ ($\omega/k_\parallel v_e \gg 1$). An estimate of the argument of the electron W function shows that this case can actually be realized. For small $k_\parallel \cong 0$ we can use the earlier expansion of $k_y = k_y\theta x/r_0$, in which case

$$\frac{\omega}{k_\parallel v_e} \sim \frac{\Omega_i}{k_y\theta x/r_0 v_e} \sim \frac{r_0}{\rho\theta}\left(\frac{m_e}{m_i}\right)^{1/2}$$

(here we assume that $\omega \sim \Omega_i$ and $k_y \sim x^{-1}$), but this quantity remains large. For this reason, we can neglect exponentially small imaginary terms in the electron part of the Eq. (5.1); thus, $k_\parallel = 0$. The argument of the ion W-function can be assumed to be rather large, $\omega - n\Omega_i \gg k_\parallel v_i$ (this assumption is verified by the results).

We also assume that the instability requires the condition $\omega^* > \omega - n\Omega_i$ or $k_y \rho_i > r_0/\rho_i$; we can then make use of the asymptotic expansion for the Bessel function $I_n(z)$. As a result of all of these simplifications, Eq. (5.1) now becomes

$$k^2 d_i^2 + 1 + \frac{T_i}{T_e} - \left(1 - \frac{\omega_i^*}{\omega}\right) \frac{\omega}{\omega - n\Omega_i} \cdot \frac{1}{\sqrt{2\pi z}} -$$

$$- \frac{T_i}{T_e} \left(1 - \frac{\omega_e^*}{\omega}\right) e^{-z_e} I_0(z_e) = 0. \quad (5.2)$$

It is a simple matter to find the following instability criterion from Eq. (5.2) [15]:

$$\rho_i / r_0 > 2n \left(\frac{m_e}{m_i}\right)^{1/2} \quad (5.3)$$

However, since we have assumed that the inverse inequality $\rho_i/r_0 < (m_e/m_i)^{1/2}$ is satisfied, cyclotron oscillations characterized by $k_\parallel = 0$ cannot be supported.

The situation is somewhat different for waves characterized by $k_\parallel \neq 0$ $(\omega/k_\parallel v_e \lessgtr 1)$. We first consider the case $\omega/k_\parallel v_e \ll 1$. In this case, the only change is in the electron part of Eq. (5.2): $\frac{T_i}{T_e}\left(1 - \frac{\omega_e^*}{\omega}\right) i \sqrt{\pi} \frac{\omega}{k_\parallel v_e} \cdot e^{-z_e} I_0(z_e)$. It is a simple matter to find an expression for the frequency and growth rate from the equation that has been obtained. We note that the instability with $k_\parallel \neq 0$ is also a threshold type, i.e., it can develop only if the Larmor radius is sufficiently large. On the other hand, for this instability we have satisfied the inequalities $\omega - n\Omega_i \sim \frac{\Omega_i}{k_\perp \rho \sqrt{2\pi}} > k_\parallel v_i$, $k_\parallel v_e > \omega \sim \Omega_i$, combining these we find $k_\perp \rho_i < v_e/v_i$; on the other hand, $\omega \sim \Omega < \omega^*$, i.e., $k_\perp \rho > r_0/\rho_i$. Thus, oscillations characterized by $k_\parallel \neq 0$ are unstable if $\rho_i r_0 > (m_e/m_i)^{1/2}$, and since it is assumed that the inverse inequality holds, this instability will not develop.

The oscillations being treated are characterized by extremely short wavelengths. For example, as soon as the inequality in (5.3) is satisfied, the wavelength turns out to be of the order of the electron Larmor radius. It is then natural to assume that these shortwave instabilities can be highly sensitive to the shortwave ion viscosity [22, 27]. Qualitatively (for the case $\nabla T = 0$) this can be shown by making the substitution $\omega \to \omega \quad i\nu_{ii}z$ in Eq. (5.2). If the viscosity is small, this leads to an instability for the cyclotron waves [38]. However, if $\frac{1}{\omega - n\Omega_i + i\nu_{ii}z} \cdot \frac{1}{\sqrt{2\pi z}} \sim \frac{1}{i\nu_{ii}z} \frac{1}{\sqrt{2\pi z}} < 1$, then, as follows from Eq. (5.2), the cyclotron waves are not unstable. Assuming $k_\perp \rho > r_0/\rho$, we conclude that the cyclotron waves will be stabilized by collisions if S is not very large [22, 27]:

$$S = \lambda_e \rho_i / a^2 < a/\rho_i. \quad (5.4)$$

It is clear that this criterion is independent of the assumption $\nabla T = 0$.

2. Ion Acoustic Instability

If the ion temperature T_i is appreciably lower than the electron temperature T_e, the presence of a longitudinal current in the plasma can lead to the development of a shortwave ion-acoustic instability. In order for the instability to be excited it is necessary that the longitudinal (current) velocity of the electrons $u = i/en$ exceed the acoustic velocity $c_s = \sqrt{T_e/m_i}$. Since the Joule heating per electron $j^2/\sigma n = (u^2/v_e^2)(T_e/\tau_e)$ is greater than the energy transferred from the electron to the ion $(m_e/m_i)(T_e - T_i)/\tau_e$, the condition $u > c_s$ is self-consistent. The electron temperature must "run away" from the ion temperature (i.e., propagation of the ion-acoustic wave must be possible) and the damping due to the electrons must be smaller than the electron excitation.

In the absence of a magnetic field, or in the presence of a weak magnetic field, the ion-acoustic instability leads to the development of oscillations of relatively small amplitude which propagates at various angles with respect to the current [31]. In the case being considered (strong magnetic field) all of the waves characterized by $kd_e \ll 1$ ($d_e = T_e/4\pi e^2 n$ is the Debye radius at the electron temperature) have the same phase velocity along the magnetic field, this velocity being c_s; hence, the nature of the development of the instability is modified to some extent.

Let us consider the electron distribution function that characterizes the longitudinal velocities $f(v_\parallel)$ (Fig. 10). In the presence of a longitudinal current, the peak of this function is shifted with respect to the origin by an amount u_0 of order u (for Coulomb collisions $u_0 \cong 0.5u$). For reasons of simplicity we shall assume that u, and, consequently u_0, are considerably greater than c_s. As is well known, the ion-acoustic instability develops as a result of a resonance interaction between the waves and electrons that move with velocity equal to the phase velocity of the wave. In the case of the ion-acoustic wave the phase velocity v_p diminishes with k from c_s when $kd_e \ll 1$, to zero when $k \to \infty$. Hence, the resonance interaction with low-amplitude waves occurs in the region from 0 to c_s (cf. Fig. 10) and all the waves in this region start to increase with time when $u_0 > c_s$, the maximum growth rate corresponding to the condi-

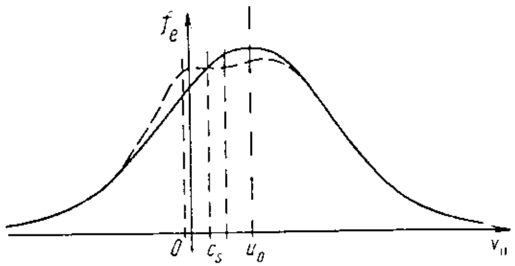

Fig. 10. The electron distribution function in the presence of a current flow.

tion $kd_e \sim 1$. As the perturbation grows, the nature of the interaction with the electrons can be modified. Specifically, if the perturbation in the potential reaches a finite amplitude φ, then all the electrons whose longitudinal velocities differ from v_p by an amount $\leq\sqrt{2e\varphi/m_e}$ will be trapped by the wave and the distribution function will exhibit a plateau (shown in Fig. 10 by the dashed line). The width of this plateau is of order $2\sqrt{2e\varphi/m_e}$. The presence of a plateau means that all waves (aside from the one being considered) that have phase velocities within the plateau will be damped. The perturbation of finite amplitude continues to grow, since the collisions are continually deforming the distribution function in such a way that the particles trapped by the wave appear to have a mean directed velocity. The growth rate is determined by the rate at which the distribution function is restored, specifically by the relation $\gamma\sim\frac{u^2}{v_e^2}\cdot\frac{1}{\tau_e}$. This growth continues until the amplitude of the wave reaches a quantity of order $e\varphi\sim T_e$, in which case some of the ions are reflected from the "hills" associated with the ion-acoustic wave. Under these conditions there is a strong transfer of energy to the ions. When $e\varphi\sim T_e$ an appreciable fraction of the electrons (about half) are trapped by the wave, so that the effective conductivity $\sigma = e^2n\tau_e/m_e$ is reduced as a consequence of the reduction in the number of carriers n. In other words, the ion acoustic instability in a strong magnetic field leads to an anomalous resistance which is of the order of the normal resistance. The Joule heat generated in the anomalous resistance is transferred to the wave and then directly to the ions by virtue of reflection of ions from the potential hills and subsequent ion collisions. The effect can be introduced by introducing a term similar to the Joule heating term in the ion heat-balance equation.

The picture presented here refers to a plasma in a uniform magnetic field. In a toroidal geometry characterized by $\lambda_e \gg a$ electrons with small

longitudinal velocities will be trapped so that, in general, the electron function will exhibit a plateau even in the absence of oscillations. Under these conditions, the excitation of the ion-acoustic wave at low amplitude can occur only within the inner region of the torus, where there are no trapped particles, and then only for a rather large amplitude $e\varphi > rT_e/R_0$, in which case the effect of trapped particles is unimportant. It is reasonable to assume that the ion-acoustic waves will exhibit "hard" excitation because of this effect. For small perturbations to grow, the quantity u must be appreciably greater than a value $\sim c_s$ required for the excitation of waves of finite amplitude. Trapping of particles in hills in the longwave drift oscillations can lead to a similar effect.

§6. HELICAL MAGNETIC CELLS

We shall start our investigation of nonlinear effects with an analysis of the helical perturbations of the magnetic field that can develop as a result of the hydrodynamic screw instability or the dissipative screw instability. It has been established in §§2 and 3 that the screw instability is stabilized if the stability margin q is large enough. However, from the point of view of achieving the maximum shear θ and from the point of view of increasing the Joule heating j^2/σ it is sometimes necessary to increase the azimuthal magnetic field H_ϑ, i.e., it is sometimes necessary to reduce q to values at which the screw instability can be excited. Hence, an analysis of the macroscopic effects due to the screw instability is extremely important.

Since the toroidal features of the geometry are not essential in the analysis of the screw instability, we shall consider a cylinder of length $L = 2\pi R_0$ and radius a. As is well known [39], for the case of helical symmetry the magnetic field can be specified by two functions I and ψ:

$$H_r = \frac{1}{r}\frac{\partial\psi}{\partial\zeta}, \tag{6.1}$$

$$H_\vartheta = \frac{1}{k^2r^2+m^2}\left(m\frac{\partial\Psi}{\partial r}+krI\right), \tag{6.2}$$

$$H_z = \frac{1}{k^2r^2+m^2}\left(-kr\frac{\partial\Psi}{\partial r}+mI\right), \tag{6.3}$$

where $\zeta = kz - m\vartheta$; $I = I(r,\zeta)$; $\Psi = \Psi(r,\zeta)$.

It is easy to show that $\mathbf{H}\nabla\psi = 0$ and $\mathrm{curl}\,\mathbf{H}\nabla I = 0$. Thus, the relation ψ = const represents the equation of the magnetic surface, while the relation I = const represents the equation for the current surface.

In order to find the perturbation of the magnetic field, we make use of the equation

$$m_i n \frac{dv}{dt} + \nabla p = \frac{1}{c}[\mathbf{jH}], \qquad (6.4)$$

which is the sum of the equations of motion for the electrons and ions with small viscosity terms neglected.

For drift waves the transverse inertia term is appreciably smaller than the pressure gradient; specifically, it is a fraction $\sim \rho_i^2/a^2$ of the latter. Hence, the transverse inertia can be neglected and the inertia term can be written in the form of a product **FH**. Multiplying Eq. (6.4) by **H**, we find

$$FH^2 = -\frac{1}{r} D(p, \Psi), \qquad (6.5)$$

where $D(p, \Psi)$ is the Jacobian:

$$D(p, \Psi) \equiv \frac{\partial p}{\partial r} \cdot \frac{\partial \Psi}{\partial \zeta} - \frac{\partial p}{\partial \zeta} \cdot \frac{\partial \Psi}{\partial r}. \qquad (6.6)$$

Taking account of Eq. (6.5) we can write Eq. (6.4) in the form

$$-\frac{\mathbf{H}}{rH^2} D(p, \Psi) + \nabla p = \frac{1}{c}[\mathbf{jH}]. \qquad (6.7)$$

This equation differs from the usual magnetohydrodynamic equilibrium equation in that the longitudinal inertia has been taken into account. The component of the equation along **H** is automatically satisfied, so that we need only consider the two transverse components.

Multiplying Eq. (6.7) by the quantity curl H and taking account of Eqs. (6.1)-(6.3), we find

$$D(p, I) + \frac{(\text{curl}\,\mathbf{HH})}{H^2} D(p, \Psi) = 0. \qquad (6.8)$$

In similar fashion, multiplying Eq. (6.7) by $\nabla\psi$, we obtan an equation that expresses the equilibrium along $\nabla\psi$

$$\Delta^*\psi + \frac{2kmI}{(k^2r^2 + m^2)^2} + 4\pi \frac{\nabla p \nabla \psi}{(\nabla\psi)^2} + \frac{I}{k^2r^2 + m^2} \cdot \frac{\nabla I \nabla \psi}{(\nabla\psi)^2} = 0, \qquad (6.9)$$

where

$$\Delta^*\psi = \frac{1}{r^2} \cdot \frac{\partial^2 \psi}{\partial \zeta^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(\frac{r}{k^2r^2 + m^2} \cdot \frac{\partial \psi}{\partial r} \right). \qquad (6.10)$$

For perturbations characterized by $\omega \ll k_{\|}$, i.e., localization $x \gg \rho_i/\theta$, the pressure can be equilibrated along the lines of force and $D(p, \psi) = 0$, i.e., $p = p(\psi)$. Under these conditions, it follows from Eq. (6.8) that $D(p, I) = 0$ and $I = I(\psi)$ in which case Eq. (6.9) leads to the well-known equation for equilibrium in helical symmetry [39]. However, in the case of perturbations that are highly elongated along **H**, the surfaces p = const, I = const, and ψ = const no longer coincide. In order to find these surfaces, it is now necessary to introduce an additional equation, this equation being Ohm's law along **H**, i.e., the longitudinal component of the electron equation of motion. In the coordinate system in which the cells are at rest and in which $\mathbf{E} = -\nabla\varphi$, this equation is of the form

$$\mathbf{H}\nabla p_e = en\mathbf{H}\nabla\varphi + \frac{m_e}{e\tau_e}\mathbf{jH} - en\mathbf{HE}_0. \qquad (6.11)$$

It then follows that

$$\mathbf{H}\,\text{curl}\,\mathbf{H} = \frac{4\pi\sigma}{c}\left\{\mathbf{HE}_0 - \frac{1}{r}D(\varphi, \psi) + \frac{1}{rn}D(p_e, \psi)\right\}, \qquad (6.12)$$

where $\sigma = e^2 n \tau_e / m_e$ is the plasma conductivity.

On the other hand, making use of Eqs. (6.1)-(6.3), we can derive the expression

$$\mathbf{H}\,\text{curl}\,\mathbf{H} = I\left\{\Delta^*\psi + \frac{2kmI}{(k^2r^2 + m^2)^2}\right\} - \frac{1}{r^2} \cdot \frac{\partial I}{\partial \zeta} \cdot \frac{\partial \psi}{\partial \zeta} - \frac{1}{k^2r^2 + m^2} \cdot \frac{\partial \psi}{\partial r} \cdot \frac{\partial I}{\partial r}. \qquad (6.13)$$

We now write ψ in the form $\psi = \psi_0 + \psi'$, where ψ_0 is the unperturbed function, which corresponds to the unperturbed magnetic field and which is defined, in accordance with Eqs. (6.2) and (6.3), by the relation

$$\frac{d\psi_0}{dr} = mH_\vartheta - krH_z. \qquad (6.14)$$

On the cylindrical surface, $r = r_0$, where $mH_\vartheta - krH_z = 0$, i.e., the point at which the pitch of the perturbation coincides with the pitch of the unperturbed lines of force, we find $d\psi_0/dr = 0$. On this surface the linearized equation (6.9) exhibits a singularity and the nonlinear equation requires special analysis, since the third and fourth terms can, in principle, lead to singularities. However, this cannot be the case under steady-state conditions. Actually, if the sum of the third and fourth terms were very large when $\nabla\psi \to 0$ then the sum of the first and second terms would also be large, as would $\Delta^*\psi$. Comparing Eqs. (6.9) and (6.13), we see that under these conditions there must be a large perturbation in the longitudinal current density, i.e., H curl H. On the other hand, it follows from Eq. (6.12) that

$$\left\langle \frac{r}{\sigma}\, \mathbf{H}\,\mathrm{curl}\,\mathbf{H} \right\rangle_{\psi} = \frac{4\pi}{c} \left\langle r\mathbf{H}\mathbf{E}_0 + \frac{1}{n} D(p_e, \psi) \right\rangle_{\psi}, \qquad (6.15)$$

where the averaging is carried out in the r, ζ, plane along the line ψ = const.

It is then evident that the perturbation in the longitudinal current density is related to the perturbation in the temperature and electron density. All other perturbations of the longitudinal current in a hydrodynamically stable plasma must decay in a "skin time," as can be shown by Eq. (6.11), if it is not assumed that the perturbed electric field is electrostatic [cf. Eq. (6.25)].

We shall first consider small-scale perturbations.

Since the relative perturbation in conductivity $\sim x_m/a$, where x_m is the localization region, while $I = m\frac{H^2}{H_z} \approx mH$ at the point $r = r_0$, then, in order-of-magnitude terms the quantity $\Delta^*\psi$ does not exceed $\frac{1}{m} \cdot \frac{x_m}{a} \cdot \frac{H_\vartheta}{a}$. Thus, when $x_m < a/m$, we find

$$\psi' \lesssim \frac{x_m^2}{a} H_\vartheta, \qquad (6.16)$$

while the unperturbed function ψ_0 varies in the range $\sim x_m$ by an amount $\sim m\frac{x_m^2}{a} H_\vartheta \gg \psi'$. Thus, when $m \gg 1$, the perturbation of the magnetic surfaces can be neglected; consequently, in treating small-scale perturbations it is valid to carry out the analysis neglecting the distortion of the lines of force.

We now wish to consider in greater detail perturbations characterized by low values of m. In order to avoid complicating the calculations, we shall make explicit use of the condition $H_\vartheta \ll H_z$ and neglect small terms of order H_ϑ^2/H_z^2 where these are not important. It is evident from Eq. (6.3) that in the zeroth approximation $I = I_0 = mH_0$, where H_0 is the uniform magnetic field outside the plasma. Taking account of this feature, and neglecting k^2r^2 compared with m^2, we can now write Eq. (6.9) in the form

$$\Delta\psi + 2kH_0 + 4\pi m^2 \frac{\nabla p \nabla\psi}{(\nabla\psi)^2} + mH_0 \frac{\nabla\psi\nabla I}{(\nabla\psi)^2} = 0, \qquad (6.17)$$

where

$$\Delta\psi = \frac{1}{r} \cdot \frac{\partial}{\partial r} r \frac{\partial\psi}{\partial r} + \frac{m^2}{r^2} \cdot \frac{\partial^2\psi}{\partial\zeta^2}.$$

It is then evident that I can be written in the form

$$I = I_0 - \frac{4\pi mp}{H_0} + I_1, \qquad (6.18)$$

in which case Eq. (6.17) can be transformed to read

$$\Delta\psi + 2kH_0 + mH_0 \frac{\nabla\psi\nabla I_1}{(\nabla\psi)^2} = 0. \qquad (6.19)$$

Thus, in this approximation the pressure p does not appear in the equilibrium equation. If we were to retain terms $\sim k^2r^2/m^2 \sim \frac{H_\vartheta^2}{H_0^2}$, the complete compensation of the pressure gradient would not be possible when Eq. (6.18) is substituted in Eq. (6.17); in particular, this would lead to the possibility of a local convective instability when the Suydam criterion is violated.

However, if $aq'/q \sim 1$ and $\beta \lesssim \frac{H_\vartheta^2}{H_0^2}$ this criterion is well satisfied and it is completely appropriate to neglect the proper small terms. When these small terms are neglected, the exact relation

$$\mathbf{H}\,\mathrm{curl}\,\mathbf{H} = I\Delta^*\psi + \frac{2kmI^2}{(k^2r^2 + m^2)^2} - \frac{1}{r^2} \cdot \frac{\partial I}{\partial\zeta} \cdot \frac{\partial\psi}{\partial\zeta} - \frac{1}{k^2r^2 + m^2} \cdot \frac{\partial\psi}{\partial r} \cdot \frac{\partial I}{\partial r} \qquad (6.20)$$

can be replaced by the simpler relation

$$\mathbf{H}\,\mathrm{curl}\,\mathbf{H} \simeq \frac{H_0}{m}\Delta\psi + \frac{2kH_0^2}{m}. \qquad (6.21)$$

Comparing this relation with Eq. (6.19), we see that

$$\mathbf{H}\,\mathrm{curl}\,\mathbf{H} = -H_0^2 \frac{\nabla\psi\nabla I_1}{(\nabla\psi)^2}. \qquad (6.22)$$

We now consider the second equilibrium equation (6.8), in which H^2 can be replaced by H_0^2.

In the approximation being used here, the derivative $\partial/\partial\zeta$ can be replaced by $(1/m)(\partial/\partial\vartheta)$, so that when Eq. (6.22) is taken into account,

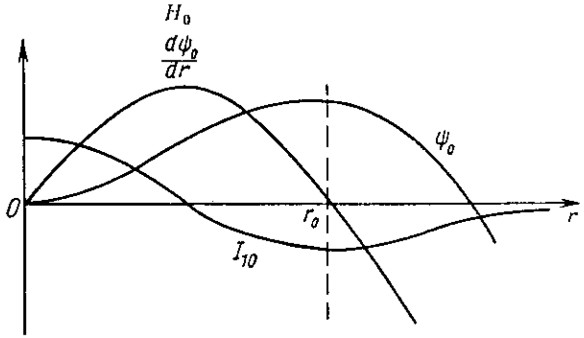

Fig. 11. The quantities ψ_0, $d\psi_0/dr$, and I_{10} as functions of r.

Eq. (6.8) can be written in the form

$$\left[\nabla p,\ \nabla I_1 - \frac{\nabla\psi\nabla I_1}{(\nabla\psi)^2}\nabla\psi\right]_z = 0. \qquad (6.23)$$

It then follows that if ∇p has a component directed along $\nabla\psi$, then the vector ∇I_1 must be directed along $\nabla\psi$, i.e., $I_1 = I_1(\psi)$. In other words, the surface I_1 = const must coincide with the magnetic surface ψ = const and the equilibrium equation (6.19) assumes the form

$$\Delta\psi + 2kH_0 + mH_0\frac{dI_1}{d\psi} = 0. \qquad (6.24)$$

It is evident from this equation that equilibrium is possible only if $\Delta\psi$ is a function of ψ. We shall assume that this condition is actually satisfied, i.e., we shall be dealing with configurations in which all oscillations associated with nonequilibrium initial conditions have already decayed (however, the possibility of reaching this state for arbitrary initial conditions is not obvious).

We now return to the longitudinal component of the electron equation of motion (6.12), which replaces Ohm's law. In order to be general, we shall not assume that the electric field is electrostatic, assuming that $E_\parallel$ can be somewhat different from $-\nabla_\parallel\varphi$. From the equation $\frac{\partial j_\parallel}{\partial t} \simeq -\frac{c^2}{4\pi}\Delta_\perp\cdot(\nabla_\parallel\varphi + E'_\parallel)$ taking account of Eq. (6.21) we find that $E'_\parallel = -\nabla_\parallel\varphi + \frac{1}{mc}\cdot\frac{\partial\psi}{\partial t}$. Thus,

$$\frac{\partial\psi}{\partial t} - \frac{cm}{rH_0}D(\varphi,\psi) + \frac{cm}{rH_0 n}D(p_e,\psi) = -mcE_0 + \frac{c^2}{4\pi\sigma}(\Delta\psi + 2kH_0). \qquad (6.25)$$

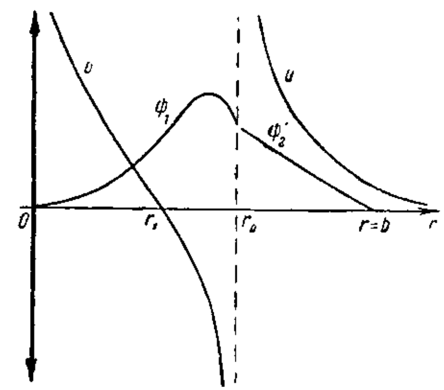

Fig. 12. The potential U and the quantity ψ' as functions of r.

We note that in the case being considered here, $H_\vartheta \ll H_0$, the term $-\frac{cm}{rH_0}D(\varphi,\psi) = \frac{c}{H_0}[\mathbf{h}\nabla\varphi]\nabla\psi$ describes the "convection" of the magnetic surfaces. It essentially derives from the condition that $\Delta\psi$ must always be a function of ψ.

We now assume that the localization region $x \gg \xi^{1/3}m^{-2/3}$. In this case, the temperature T_e can be assumed to be constant along the lines of force, i.e., $\sigma = \sigma(\psi)$, $D(p_e,\psi) = T_e D(n,\psi)$. Taking $\varphi = (T_e/e)\ln n + \varphi'$, in place of Eq. (6.25) we now find

$$\frac{\partial\psi}{\partial t} - \frac{cm}{rH_0}D(\varphi',\psi) = -mcE_0 + \frac{c^2}{4\pi\sigma}(\Delta\psi + 2kH_0). \qquad (6.26)$$

The perturbation of the magnetic field is small far from the singular point $r = r_0$, and for this reason ψ can be written in the form $\psi_0 + \psi'$, where ψ_0 is the part of ψ that is independent of ζ, while ψ' is the perturbation; where $\psi' \ll \psi_0$. Similarly, $I_1 = I_{10} + I_1'$. In the linear approximation, taking $\psi' \sim \exp(\gamma t + i\zeta)$ and using (6.24) and (6.26), we find

$$\frac{1}{r}\cdot\frac{d}{dr}r\frac{d\psi_0}{dr} + 2kH_0 = \frac{4\pi m}{c}j_0 = -mH_0\frac{dI_{10}}{dr}\left(\frac{d\psi_0}{dr}\right)^{-1}; \qquad (6.27)$$

$$\frac{1}{r}\cdot\frac{d}{dr}r\frac{d\psi'}{dr} - \frac{m^2}{r^2}\psi' = \frac{4\pi\sigma_0}{c^2}\left(\gamma\psi' + i\frac{cm}{rH_0}\varphi\frac{d\psi_0}{dr}\right) + \frac{4\pi E_0}{c}\cdot\frac{d\sigma_0}{dr}\left(\frac{d\psi_0}{dr}\right)^{-1}\psi' = \frac{4\pi m}{c}\cdot\frac{dj_0}{dr}\left(\frac{\partial\psi_0}{dr}\right)^{-1}\psi' \qquad (6.28)$$

We note that these equations also apply to the case of a current j_0 which varies slowly in time (consequently, also ψ_0), i.e., in the presence of skin effects. In this case, by E_0 we are to understand j_0/σ_0.

In the derivation of Eq. (6.28) we have taken account of the fact that σ and $dI_1/d\psi$ are functions of ψ, so that in the perturbation of ψ, σ varies by an amount $\frac{d\sigma_0}{d\psi_0}\psi' = \frac{d\sigma_0}{dr}\left(\frac{d\psi_0}{dr}\right)^{-1}\psi'$; a similar change appears in $dI_1/d\psi$. If $\sigma_0(r)$ is a function of r that diminishes monotonically, in accordance with Eq. (6.27), the qualitative dependence of ψ_0 and I_{10} on r should be of the form shown in Fig. 11 (we have chosen the arbitrary constant in ψ_0 in such a way that ψ_0 vanishes at r = 0).

It will be evident that ψ_0 and I_{10} exhibit extrema at the singular point $r = r_0$, at which

$$\frac{d\psi_0}{dr} = mH_\vartheta^0 - krH_0 = 0.$$

We now wish to consider the equations in (6.28). One of these can be conveniently written in the form

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi'}{dr} - U\psi' = 0,$$

where

$$U = \frac{m^2}{r^2} + \frac{4\pi m}{c}\frac{dj_0}{dr}\left(\frac{d\psi_0}{dr}\right)^{-1}. \qquad (6.29)$$

The dependence of the "potential" U on r when $dj_0/dr < 0$ is shown qualitatively in Fig. 12. The curve has a well near the singular point when $r < r_0$. If this well is broad enough, then the solution of (6.29), ψ_1' that satisfies the boundary condition $\psi_1'(0) = 0$ in the range $0 < r < r_0$ can have modes. In this case, the plasma is unstable even when $\sigma = \infty$ [40]. Since the width of the well diminishes as the index m increases, at large q, i.e., large longitudinal magnetic field H_0, it may turn out that the well can exist only at large values of m (we recall that at the singular point $q = m/n \leqslant m$). However, at these values an ideal plasma is stable in the magnetohydrodynamic sense. In this case, ψ_1' does not vanish in the range $0 < r < r_0$, and has the form shown in Fig. 12. In the same figure we have shown the qualitative solution ψ_2' of Eq. (6.29) for $r > r_0$ that satisfies the boundary conditions on an ideally conducting chamber of radius $\psi_2' = 0$. When $\psi_1'(r_0) = \psi_2'(r_0)$, the derivatives $d\psi_1'/dr$ and $d\psi_2'/dr$ will not generally be equal at the point $r = r_0$, so that ψ_1' and ψ_2' do not give solutions of the linearized equations over the entire range $0 < r < b$. As is evident from Eq. (6.28), the second linear equation reduces to the condition

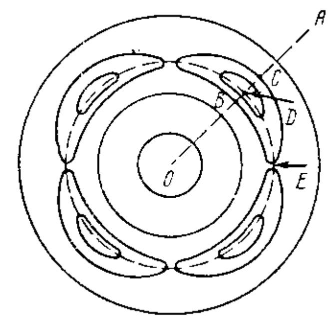

Fig. 13. Destruction of magnetic surfaces.

that the lines of force be frozen: $\gamma\psi' + i\frac{cm}{rH_0}\varphi\frac{d\psi_0}{dr} = 0$; thus, this equation cannot be used to find the complete linear solution ψ'. Within the framework of the linear approximation this paradox can be resolved by taking account of inertial effects in the vicinity of the singular point. The appropriate analysis is carried out in [20] (cf. also § 3) and shows that when

$$\left(\frac{1}{\psi_1'}\cdot\frac{d\psi_1'}{dr} - \frac{1}{\psi_2'}\cdot\frac{d\psi_2'}{dr}\right)_{r=r_0} < 0$$

the plasma can exhibit a finite-conductivity screw instability (the "tearing" mode in the terminology of [20]); when

$$\left(\frac{1}{\psi_1'}\cdot\frac{d\psi_1'}{dr} - \frac{1}{\psi_2'}\cdot\frac{d\psi_2'}{dr}\right)_{r=r_0} > 0$$

only the current-convective and gravitational (actually the drift inertial) instabilities remain.* It will be shown below that the nonlinear analysis with inertia neglected leads to the same result.

It will be evident that the nonlinearity should be taken into account primarily near the singular point. Since ψ_0 reaches a maximum at $r = r_0$, and thus varies rather slowly in the vicinity of $r = r_0$, even a small perturbation ψ' leads to a marked change in the magnetic surfaces. Specifically, the surface ψ = const is broken up near $r = r_0$, as shown for the case m = 4 in Fig. 13, where the point D corresponds to ψ = max, while the point E is a saddle point. The width of the region bounded by the separatrix, i.e., the surface passing through the saddle points, is of order $x \sim \sqrt{\psi'\left(\frac{d^2\psi_0}{dr^2}\right)^{-1}}$; in other words, it is explicitly a nonlinear function of ψ'. The linear approximation is meaningful only at distances sufficiently removed from the boundaries of the cell

*We note that the derivatives $d\psi_1'/dr$ and $d\psi_2'/dr$ diverge logarithmically at the singular point; however, the difference $\frac{1}{\psi_1'}\cdot\frac{d\psi_1'}{dr} - \frac{1}{\psi_2'}\cdot\frac{d\psi_2'}{dr}$ remains finite.

where $\psi' \ll \psi_0(r) - \psi_0(r_0)$. Since the difference $\psi_0(r) - \psi_0(r_0)$ increases quadratically with $r - r_0$, the region of applicability of the linear analysis starts at distances removed from the separatrix by an amount of the order of half the cell width.

We now consider Eq. (6.26) inside the cell. The second term on the left side of this equation describes the transport of lines of force that are frozen in the plasma. Since the corresponding flow is incompressible, by itself it cannot change the area bounded by the surfaces ψ = const; however, the shape of the area can be changed slightly, so that, in the final analysis, the width of the cell is determined by the diffusion of the magnetic field, which is described by the term $(c^2/4\pi\sigma)\Delta\psi$.

We now wish to trace the variation of ψ along the radius OA which passes through the center of an individual cell (cf. Fig. 13). In doing this we shall assume that the quantity $\Delta\psi$ is a constant on the surface ψ = const. It will be evident that if $\Delta\psi$ and the conductivity σ are both constant, then the difference in ψ at the points D and C in the presence of cells will be smaller than in the absence of cells (because of diffusion of field in the azimuthal direction). In other words, ψ will have a greater slope at the center of the cell than is the case when there is no cell. It then follows that in the stationary case the perturbation ψ' must increase somewhat in both directions going away from the center of the cell. If this increase is sufficiently large, then, at the center of the cell, ψ will increase in time, i.e., $\partial\psi/\partial t > 0$, and, consequently, the dimensions of the cell will be increased; in the opposite case the perturbation will be damped in a skin time. Thus, cells develop only so long as there is a finite-conductivity screw instability, that is, when $\left(\frac{1}{\psi_1'}\cdot\frac{d\psi_1'}{dr} - \frac{1}{\psi_2'}\cdot\frac{d\psi_2'}{dr}\right)_{r=r_0} < 0$ in the linear approximation, in which the rate of expansion of the cell is determined by the skin time $t = (4\pi\sigma/c^2)x^2$. For sufficiently large values of x this time is appreciably greater than the transit time a/v_i, and ion inertia becomes unimportant. In order-of-magnitude terms this condition leads to the relation $x > \frac{c}{\omega_0}\sqrt{\left(\frac{m_e}{m_i}\right)^{1/2}\frac{a}{\lambda_e}}$.

The width of the cell can be determined in the linear approximation since the region of applicability of the linear approximation starts very close to the separatrix. The width of the cell will obviously not exceed the quantity $r_0 = r_2$, where r_2 is the point at which $d\psi_1'/dr = 0$ (cf. Fig. 12). This width is not greater than the region $r_0 - r_1$, where $U < 0$, which, in accordance with Eq. (6.29), is of order a/m^2, and is not very sensitive to dj_0/dr because near the singular point $\frac{d\psi_0}{dr} = x\frac{d^2\psi_0}{dr^2}$, while the second derivative $\left(\frac{d^2\psi_0}{dr^2}\right)_{r=r_0}$ is also proportional to dj_0/dr.

In the case at hand it is not difficult to include the effect of excitation of the cells by external perturbations. For this purpose it is sufficient to add to the solution of the linear equation given above ψ_2', a solution which is equal to the external perturbation at the chamber (at r = b), and which falls off exponentially inside the plasma.

It is evident that an external perturbation will lead to a small spreading of the magnetic surfaces even in the case in which the screw instability is stabilized. As m increases the perturbation inside the plasma falls off exponentially, so that the corresponding macroscopic effect is small.

For sufficiently large values of m and with $\theta \neq 0$, the screw instability is stabilized and $\psi' = 0$. In this case, as is evident from Eq. (6.26), the perturbation of the conductivity is balanced by convection [the second term in the left side of Eq. (6.26)], which will be treated in the following section. There is no perturbation ψ' in this case in spite of the perturbation of the pressure p' because the pressure, in accordance with Eq. (6.18), is balanced by the perturbation of the longitudinal magnetic field in the approximation used here, $H_\vartheta^2 \ll H_0^2$. Thus, the perturbation ψ' for thermal inertialess convection is a factor H_ϑ^2/H_0^2 smaller than the quantity given in (6.16), i.e., it is negligibly small.

§7. THERMAL CONVECTION OF A CURRENT-CARRYING PLASMA

1. Basic Equations

We now wish to consider the nonlinear plasma convection that develops as a consequence of the current-convective (screw) instability. For simplicity we shall assume that the density is constant, although this limitation is not important. Under these conditions the heat-transport equation for incompressible flow can be written in the form

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi_\parallel \nabla_\parallel^2 T + \chi_\perp \Delta_\perp T, \tag{7.1}$$

where $\chi_\parallel$ and $\chi_\perp$ are the longitudinal and transverse thermal conductivities; these can be regarded as constant in the region of the highly localized convection cells that are considered below.

The flow that develops as a consequence of the current-convective instability can be regarded as inertialess, and if the inhomogeneity of the magnetic field is disregarded the flow can also be regarded as incompressible, so that

$$\mathbf{v}_\perp = \frac{c\,[\mathbf{h}\nabla\varphi]}{H_0}; \quad \operatorname{div}\mathbf{v}_\perp = 0. \tag{7.2}$$

On the other hand, the longitudinal velocity $v_\parallel$ can be neglected for a highly localized perturbation (as shown in §3, it is sufficient that the localization width of the perturbation be smaller than ρ_i/θ).

Furthermore, since the motion is inertialess, the current density in the plasma is essentially unperturbed and the longitudinal current can be regarded as constant:

$$j_\parallel = \text{const} = \sigma_0 E_0, \tag{7.3}$$

where we have provisionally introduced the unperturbed electric field E_0. Although all of the results that are obtained below are valid for a slowly varying current distribution (i.e., in the presence of relaxation effects), in this case by E_0 we shall simply understand $j_\parallel/\sigma_0$.

For reasons of simplicity, in Eqs. (7.1) and (7.3) we assume the ion and the electron temperatures to be equal. In a dense plasma, it is true that $T_i = T_e$; however, as the collision frequency is reduced the equilibrium between the electrons and the ions may not be established in convection. However, by neglecting the change in χ due to the change in the heat capacity we can still case Eq (7.1), understanding T to be the electron temperature. In this case Eq. (7.1), which describes the heat transfer by electrons, still applies for localization of a perturbation comparable with, or smaller than ρ_i, in which case the ions will generally not participate in the convection. Consequently, convection is to be associated only with an effective electron thermal conductivity.

We shall first consider the case of an individual convection cell, i.e., we assume that the plasma exhibits a flow in which all quantities are periodic functions of the variable $m\vartheta - n\zeta$, where $\zeta = z/2\pi R_0$. In this case, close to the point $r = r_0$, where $q = m/n$, the derivative along the magnetic field $\mathbf{h}\nabla = \frac{H_\vartheta}{rH_0}\frac{\partial}{\partial\vartheta} + \frac{\partial}{\partial z}$ can be written in the form

$$\mathbf{h}\nabla = -\frac{H_\vartheta q'}{qrH_0}\, x\frac{\partial}{\partial\vartheta}, \tag{7.4}$$

where $q' = dq/dr$; $x = r - r_0$. Thus, for flow close to the singular point, Eq. (7.1) assumes the form of two-dimensional convection

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = m^2\alpha x^2\frac{\partial^2 T}{\partial y^2} + \chi_\perp\frac{\partial^2 T}{\partial x^2}, \tag{7.5}$$

where

$$\alpha = \chi_\parallel\theta^2/r^4; \quad y = m\vartheta, \tag{7.6}$$

where θ is the shear, given by $\theta = rq'H_\vartheta/qH_0$. It should be recalled that y is a dimensionless quantity.

In Eq. (7.5) we have neglected the cylindrical nature of the geometry and the term $\frac{\chi_\perp}{r^2}\cdot\frac{\partial^2 T}{\partial\vartheta^2}$, assuming that the region of localization satisfies the condition $x_m \ll r$.

We now consider Eq. (7.3). For reasons of simplicity we take Ohm's law in the form $j_\parallel = \sigma E_\parallel = \sigma(E_0 - \nabla_\parallel\varphi)$. If the temperature perturbation is small, as is actually the case in highly localized cells, Eq. (7.3) can be linearized, and we find

$$E_0\frac{d\sigma_0}{dT_0}T' - \sigma_0\mathbf{h}\nabla\varphi = 0, \tag{7.7}$$

where T' is the temperature perturbation. Then

$$xv_x = AT', \tag{7.8}$$

where

$$A = \frac{qcE_0}{q'H_\vartheta\sigma_0}\cdot\frac{d\sigma_0}{dT_0}. \tag{7.9}$$

We note that taking account of the electron pressure term in Ohm's law would lead to a modification of the velocity **v** in the form of a term which, in the approximation $H_z = H_0$ = const is given by $\mathbf{v}_d = \left\{-\frac{c}{mreH_0}\cdot\frac{\partial T}{\partial y}, \frac{c}{eH_0}\cdot\frac{\partial T}{\partial x}\right\}$. It is evident, however, that $\operatorname{div}\mathbf{v}_d = 0$ and $\mathbf{v}_d\nabla T = 0$, so that this flow will have no effect on the convection; it simply leads to a slippage of the cell with respect to the material (with a velocity $\sim\frac{c}{eH_0}\cdot\frac{dT_0}{dx}$). For a high degree of localization we can also neglect the distortion of the cell which arises by virtue of the variation of dT_0/dx with the variable x. Any inhomogeneity in the plasma density would lead to precisely the same kind of simple slippage of the cell, as can be seen from the linear dispersion relation. Equation (7.8) can also be written in the form

$$\nabla p = -xv_x\mathbf{e}_r + AT\mathbf{e}_r, \tag{7.10}$$

where $\mathbf{e}_r$ is a unit vector directed along the radius, while p is an arbitrary function of r and ϑ. It follows from the azimuthal component of (7.10) that $\partial p/\partial\vartheta = 0$; averaging the radial component and taking account of the fact that $\int v_r d\vartheta = 0$, by virtue of the incompressibility we find $dp/dr = AT_0$, where $T_0 = \int T \frac{d\vartheta}{2\pi}$. Taking account of this feature, it is easy to see that Eqs. (7.8) and (7.10) are equivalent.

Equation (7.10) can be regarded as the equation of motion of an inertia-less fluid (with density $\rho_0 = 0$) which experiences a friction xv_x in the radial direction, and which is subject to a gravitational force AT proportional to the temperature. Thus, Eqs. (7.5), (7.6), and (7.10) describe the thermal convection of a peculiar fluid with an antistropic thermal conductivity and an antistropic friction force in a fixed (porous) medium.

2. Convection in an Individual Cell

We now consider convection in an individual cell, neglecting the transverse thermal conductivity for the time being ($\chi_\perp = 0$). In the linear approximation, using Eqs. (7.5) and (7.8), we find

$$\gamma = \operatorname{Im}\omega = -\frac{A}{x}\cdot\frac{dT_0}{dr} - m^2\alpha x^2. \tag{7.11}$$

When $A > 0$ and $dT_0/dr < 0$, an instability occurs if $x > 0$. The width of the localization region for the instability x_m is determined by the condition $\gamma = 0$. From Eq. (7.11) we have

$$x_m = \left(\frac{AS_0}{m^2\alpha}\right)^{1/3} = r\xi^{1/3}m^{-2/3}, \tag{7.12}$$

where $S_0 = -dT/dr$ is the unperturbed temperature gradient and $\xi = AS_0/r^3\alpha$.

Assuming that $\chi_\parallel \approx \lambda_e v_e = \tau_e v_e^2$, and $\sigma = e^2 n_0 \tau_e/m_e$, we can make an estimate for $\xi \sim \frac{c^2}{\omega_0^2\lambda_e^2}\cdot\frac{H_0^2}{H_\vartheta^2}$. We shall assume that $\xi \ll 1$ and, consequently, even when $m = 1$, we find $x_m \ll r$. (For example, in Tokomak, $c/\omega_0 \sim 0.1$, $H_0/H_\vartheta \sim 10$, $\lambda_e \sim 10^2$-10^3, so that $\xi \sim 10^{-3}$-10^{-2}).

The convection that develops as a result of the instability leads to heat transport and to a change in the profile of the average temperature T_0. We

shall first consider the stationary problem in the quasilinear approximation, assuming that the temperature perturbation is small (specifically, that $T' \ll x_m S_0$). In this approximation the temperature perturbation can be written in the form $T' = T_m \cos m\vartheta = T_m \cos y$.

Averaging Eq. (7.5) over ϑ and taking account of Eq. (7.8), we find

$$\frac{\partial T}{\partial t} = -\frac{1}{r}\cdot\frac{\partial}{\partial r} r q_T; \quad q_T = \frac{AT_m^2}{2x} - \chi_\perp\frac{dT_0}{dr}. \tag{7.13}$$

Here, q_T is the thermal flux, which is equal to the sum of the convective-flux $\langle v_x T\rangle$ and the conductivity-flux $\chi_\perp(dT_0/dr)$. The coefficient in the first term in the expression for q_T in Eq. (7.13) arises by virtue of the averaging $(\cos m\theta)^2$. If $x_m \ll r$, the thermal flux q_T within the confines of a given convective cell can be regarded as constant.

The stationary state appears when the deformation of the temperature profile T_0 reaches a magnitude such that the growth rate for the small perturbation vanishes. Using this condition and neglecting $\chi_\perp$ in the expression for γ, i.e., using Eq. (7.1), we find

$$\frac{dT_0}{dx} = -\frac{m^2\alpha}{A}x^3, \quad T_0 = T_a - \frac{S_0}{4}\cdot\frac{x^4}{x_m^3} \quad 0 < x < x_m, \tag{7.14}$$

where $T_a = T_0(x = 0)$.

Substituting the value that has been found for T_0 in Eq. (7.13), we have

$$\frac{AT_m^2}{2x} + \chi_\perp S_0\frac{x^3}{x_m^3} = q_\perp = \chi_\perp S_0,$$

whence

$$T_m^2 = \frac{2x}{A}\left(1 - \frac{x^3}{x_m^3}\right) q_T. \tag{7.15}$$

It is evident that for a small transverse thermal conductivity $\chi_\perp$ the amplitude of the temperature perturbation is also small, so that the quasilinear approximation is valid over a large portion of the range $0 < x < x_m$. However, when $x \to 0$, $v_m^2 = \frac{A^2T_m^2}{x^2} \to \infty$ and, consequently, it is necessary to introduce appropriate corrections for the nonlinear terms and the heat transport Eq. (7.5).

In order to find an approximate solution for nonlinear equation (7.5) we can exploit the following situation.

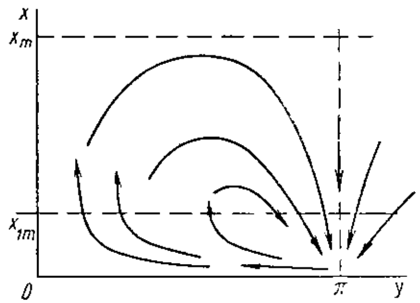

Fig. 14. Flow in a convection cell.

The perturbation in temperature develops in the basic part of the cell, where an important role is played by the longitudinal thermal conductivity, the latter being described by the first term on the right side of Eq. (7.5). This term leads to a damping of all higher harmonics, so that over most of the range of values x only the fundamental mode is important. However, at small values of x the term containing the longitudinal thermal conductivity becomes negligibly small, so that there is a possibility that the second and higher harmonics of the form sin ny can develop. The amplitudes of these harmonics will then be determined by the nonlinear term $\mathbf{v}\nabla T$, which describes the convective transport of heat without dissipation. We shall use the symbol x_{1m} to denote the boundary of the region at which the longitudinal thermal conductivity no longer dominates and the convection becomes essentially nonlinear. In this region, i.e., in the region $x < x_{1m}$, the rising flux transports the temperature $T_a = T_0(x = 0)$ essentially without dissipation, while the falling flux transports the temperature of the point $x = x_{1m}$. Since the mean temperature T_0 falls off with x, the perturbation of temperature in the falling flux T_- is equal to the difference in the temperature at the point and the mean temperature T_0 and increases as x is reduced. The velocity associated with the flux $v_- = T_-/Ax$ increases still more sharply, and by virtue of the incompressibility the transverse dimension of the jet must approach zero when $x \to 0$. In other words, the falling flux converges at the point $y = \pi$, $x = 0$ (Fig. 14) so long as the transverse thermal conductivity does not play a role; the latter leads to a reduction of T and, consequently, a reduction of v_-. Since the flow is incompressible, a thin boundary layer spreads out along the boundary $x = 0$ and absorbs heat from the region $x < 0$. This heat exchange leads to some perturbation of the temperature in the stable region $x < 0$, but to avoid complicating the analysis we shall neglect this perturbation.

In view of the considerations given above, we now seek an approximate solution in the form

$$T' = T_+ \cos\frac{y}{2l} \quad 0 < y < \frac{\pi l}{2};$$
$$T' = T_- \cos\frac{y-\pi}{2(1-l)} \quad \frac{\pi l}{2} < y < \pi. \qquad (7.16)$$

In other words, as before, we assume that the profile of T' in both the rising and the falling fluxes is a cosine function but with width proportional, respectively, to l and $1 - l$; it is thus a function of x. When $l = 1/2$, $T_- = -T_+$, we obtain the solution of the linearized equation.

Substituting Eq. (7.16) and (7.5), and taking $y = 0$ and $y = \pi$, we obtain two approximate equations for determining T_+ and T_-:

$$\frac{AT_+}{x}\cdot\frac{d}{dx}(T_0 + T_+) = -\frac{\alpha m^2}{4l^2}x^2T_+ + \chi_\perp\frac{d^2}{dx^2}(T_0 + T_+); \qquad (7.17)$$

$$\frac{AT_-}{x}\cdot\frac{d}{dx}(T_0 + T_-) = -\frac{\alpha m^2}{4(1-l)^2}x^2T_- + \chi_\perp\frac{d^2}{dx^2}(T_0 + T_-). \qquad (7.18)$$

Furthermore, it follows from the condition $\int T'dy = 0$ that

$$T_+ l + T_-(1 - l) = 0, \qquad (7.19)$$

whence

$$T_+ = (1 - l)\,T; \quad T_- = -lT, \qquad (7.20)$$

where $T = T_+ - T_-$.

In the presence of a perturbation of the form in Eq. (7.16) the expression for the heat flux (7.13) is replaced by the following:

$$q_T = -\chi_\perp\frac{dT_0}{dr} + \frac{A}{2x}\left[lT_+^2 + (1-l)\,T_-^2\right] = -\chi_T\frac{dT_0}{dr} - \frac{A}{2x}T_+T_-. \qquad (7.21)$$

Equations (7.17) and (7.18), together with the relations in Eqs. (7.19) and (7.21), can be used to find an approximate solution for the linearized problem.

We shall simplify the problem by assuming the transverse thermal conductivity to be small, writing $\chi_\perp = 0$ in Eqs. (7.17), (7.18), and (7.19). Cancelling T_+ from Eq. (7.17) and T_- from Eq. (7.18) and subtracting one from the other, we have

$$\frac{dT}{dx} = -\frac{\alpha m^2x^3}{4A}\cdot\frac{1-2l}{l^2(1-l)^2} = -\frac{\alpha m^2A}{16q_T^2}x(1-2l)\,T^4. \qquad (7.22)$$

Furthermore, from Eq. (7.21) we have $l^2 - l + 2q_Tx/AT^2 = 0$, whence, by virtue of the considerations given above, which indicate that $l \to 1$ when $x \to 0$, we have

$$l = \frac{1}{2} + \sqrt{\frac{1}{4} - 2q_Tx/AT^2} \qquad (7.23)$$

For small values of x we can make the approximation $l = 1$; it then follows from Eq. (7.22) that

$$T^{-3} + \frac{3\alpha m^2 A}{32 q_T^2} x^2 = T_c^{-3}, \qquad (7.24)$$

where T_c is the value of $T = T_+ - T_-$ at $x = 0$.

The required solution must be such that for sufficiently large $x > x_{1m}$ the quantity l assumes the value $l = 1/2$; starting at this value of x we must have

$$T_+ = -T_- = \frac{T}{2} = \sqrt{\frac{2q_T}{A} x}\,. \qquad (7.25)$$

If we replace T in Eq. (7.22) by the value given in Eq. (7.25), then

$$2l - 1 \cong \frac{\sqrt{2Aq_T}}{4\alpha m^2} x^{-7/2}. \qquad (7.26)$$

It is evident from this relation that the quantity $2l - 1$ increases rapidly as x is reduced. Hence, the position of the point at which the two solutions (7.24) and (7.25) are matched, $x = x_{1m}$, can be found with the required accuracy by making the simple assumption that $l = 1$ in Eq. (7.26):

$$x_{1m} \cong \left(\frac{Aq_T}{8\alpha^2 m^4}\right)^{1/7}. \qquad (7.27)$$

In Eq. (7.24) we now substitute this value for x_{1m} and the value of T from Eq. (7.25), so that

$$T_c \cong \left(\frac{4}{7}\right)^{1/3} T_1 \cong 0.83 \sqrt{\frac{8q_T}{A} x_{1m}}\,, \qquad (7.28)$$

where T_1 is the value of T at the point $x = x_{1m}$.

Thus, the quantity T remains essentially constant over the range $0 < x < x_{1m}$; the total variation is less than 20%. We now wish to determine T_0. Neglecting the term containing $\chi_\perp$ in Eq. (7.17), we find

$$\frac{dT_0}{dx} = -\frac{dT_\perp}{dx} - \frac{\alpha m^2}{4l^2 A} x^3. \qquad (7.29)$$

Since l varies from 1/2 to 1, where it is equal to unity only for very small values of x, as an approximation in Eq. (7.29) we can write $l = 1/2$. Thus, using Eq. (7.29), we obtain the approximate relation

$$T_0 = T_a - T_+ - \frac{\alpha m^2}{4A} x^4. \qquad (7.30)$$

Using Eqs. (7.25) and (7.29) at $x > x_{1m}$, we now find

$$\frac{dT_0}{dx} \cong -\frac{\alpha m^2}{A}\left(x^3 + 2\frac{x_{1m}^{9/2}}{x^{3/2}}\right). \qquad (7.31)$$

When $x \to 0$, by virtue of the fact that $T_+ = (1 - l)T \cong 2q_T x / AT_c$, we have

$$\left(\frac{dT_0}{dx}\right)_{x=0} = -\frac{2q_T}{AT_c} \cong -2\left(\frac{7}{4}\right)^{1/3} \frac{\alpha m^2}{A} x_{1m}^3. \qquad (7.32)$$

Thus, when $x_{1m} \ll x_m$, the total dependence of dT_0/dx on x can be approximated roughly as follows:

$$-\frac{dT_0}{dx} \approx \frac{\alpha m^2}{A}\left(x^3 + 2x_{1m}^3 - 2x_{1m}^3 \frac{x^3}{x_m^3}\right) =$$

$$= \frac{q_T}{\chi_\perp}\left(\frac{x^3}{x_m^3} + 2\frac{x_{1m}^3}{x_m^3} - 2\frac{x_{1m}^3}{x_m^3} \cdot \frac{x^3}{x_m^3}\right). \qquad (7.33)$$

The approximate solution that has been found is valid only if the conductivity is small, in which case $x_{1m} \ll x_m$, where x_m is the boundary of the convection cell, while x_{1m} is the boundary of its highly nonlinear part. Actually, however, the ratio x_{1m}/x_m, as can be seen from Eqs. (7.12) and (7.27) and the relation

$$\chi_\perp/\chi_\parallel \xi \approx \sqrt{\frac{m_e}{m_i}} \frac{1}{\Omega_i^2 \tau_i^2 \xi} \approx \sqrt{\frac{m_e}{m_i}} \beta \frac{H_\vartheta^2}{H_0^2},$$

is found to be of order

$$\frac{x_{1m}}{x_m} = \left(\frac{\eta m^{2/3}}{8}\right)^{1/7} \approx \left(\frac{\beta m^{2/3}}{8\xi^{1/3}} \sqrt{\frac{m_i}{m_e}}\right)^{1/7} \sim 1, \qquad (7.34)$$

where the parameter $\eta = \left(\dfrac{\chi_\perp^3 \alpha}{S_0^4 A^4}\right)^{1/3} \approx \beta \xi^{-1/3} \sqrt{\dfrac{m_i}{m_e}}$; $\beta = 8\pi p/H^2$ is the ratio of the plasma pressure to the pressure of the magnetic field; m_i is the ion mass; m_e is the electron mass; $\Omega_i = eH/m_i c$; τ_i is the mean ion–ion collision time.

As is evident from Eq. (7.34), the ratio x_{1m}/x_m is a very weak function of β, m, and ξ, and for this reason it is almost always a quantity of order unity. This result means that the transverse thermal conductivity must be taken into account.

We now eliminate the derivative dT_0/dx from Eqs. (7.17) and (7.18) by means of the solution (7.21), in which case the two equations become

$$\frac{AT_+}{x}\left\{\frac{dT_+}{dx} - \nu q_T - \nu\frac{A}{2x}T_+T_-\right\} = -\frac{\alpha m^2}{4l^2}x^2T_+ - \frac{d}{dx}\left(\frac{AT_+T_-}{2x}\right) + \chi_\perp\frac{d^2T_+}{dx^2}; \qquad (7.35)$$

$$\frac{AT_-}{x}\left\{\frac{dT_-}{dx} - \nu q_T - \nu\frac{A}{2x}T_+T_-\right\} = -\frac{\alpha m^2}{4(1-l)^2}x^2T_+ - \frac{d}{dx}\left(\frac{AT_+T_-}{2x}\right) + \chi_\perp\frac{d^2T_-}{dx^2}, \qquad (7.36)$$

where $l = -T_-(T_+ - T_-)^{-1}$; $(1-l) = T_+(T_+ - T_-)^{-1}$; $\nu = \chi_\perp^{-1}$.

In order to avoid complications we neglect the perturbation of the temperature due to the thermal conductivity in the region $x < 0$, taking $T_+ = T_- = 0$ when $x = 0$.

We shall first consider the linear equation, for example, for T :

$$\frac{d^2T_+}{dx^2} + \frac{Aq_T\nu^2}{x}\left(1 - \frac{x^3}{x_m^3}\right)T_+ = 0. \qquad (7.37)$$

This equation has a nontrivial solution when $g_m \equiv Aq_T\nu^2x_m = a_0$, where a_0 is the characteristic value of the equation

$$xV'' + a_0(1 - x^3)V = 0. \qquad (7.38)$$

A numerical calculation with the boundary condition $V(0) = 0$ yields the value $a_0 = 2.9$.

Using Eq. (7.12) for x_m, we write the stability condition $g_m \equiv Aq_T\nu^2x_m > a_0$ in the form*

$$g_m^{-1} = m^{2/5}\eta = 8\left(\frac{x_{1m}}{x_m}\right)^7 < a_0^{-1}, \qquad (7.39)$$

*We note that the stability condition has been written under the assumption that $\chi_\perp$ is determined by the ion thermal conductivity, i.e., under the assumption that T_i and T_e are equal. Actually, when $S = \lambda_e\rho_i/a^2 > \sqrt{m_e/m_i}$, the heat exchange between the electrons and ions is not able to provide an equilibrium, since $\omega^* > m_e\nu_e/m_i$. Hence, the instability, and, consequently, thermal convection appear even when $g_m < a_0$. However, the effects are much smaller than those associated with the classical thermal conductivity of the ions, and can be neglected.

where

$$\eta = \left(\frac{\chi_\perp^7\alpha}{A^4q_T}\right)^{1/5} \approx \beta\xi^{-1/5}\sqrt{\frac{m_i}{m_e}}. \qquad (7.40)$$

According to Eq. (7.39) the instability will occur only for perturbations characterized by

$$m < m_c = (\eta a_0)^{-5/2}. \qquad (7.41)$$

When $\eta a_0 \geq 1$, i.e., $\beta > a_0^{-1}\xi^{1/5}\sqrt{\frac{m_e}{m_i}}$, all modes are stable including the one characterized by $m = 1$. Furthermore, if $m_c^2x_{mc} < r$, i.e., $\eta a_0 > \xi^{1/5}$, the individual convection cells do not overlap and can be treated separately.

We shall assume that g_m is not much greater than the critical value $g_m = a_0$. Then l can be assumed to be equal to $1/2$, so that $T_+ = -T_-$. In this approximation, using Eq. (7.35), we have

$$\frac{d^2V}{dt^2} + g_m\left(\frac{1}{t} - t^2\right)V = \frac{1}{2t^2}V^2(1+V), \qquad (7.42)$$

where $t = x/x_m$; $V = A\nu T_+$.

If g_m is slightly greater than a_0, Eq. (7.42) can be solved by perturbation theory, by taking $V = BV_0$, where V_0 is the solution of Eq. (7.38), while B is an unknown amplitude which can be found from the orthogonality condition, i.e., by multiplying Eq. (7.42) by V_0 and then integrating with respect to x. Neglecting the quantity V on the right side of Eq. (7.42) as compared with unity, we find

$$B = 2D_0(g_m - a_0), \qquad (7.43)$$

where

$$D_0 = \int_0^\infty\left(\frac{1}{t} - t^2\right)V_0^2dt \Big/ \int_0^\infty V_0^3t^{-2}dt.$$

The quantity V_0 is normalized in such a way that $D_0 = 1$.

When $g_m \gg a_0$, the first term on the left side of Eq. (7.42) is small compared with the second and can be replaced approximately by $a_0(1/t - t^2)V$, i.e., by the value that it assumes at small values of $g_m - a_0$. In this approximation we find

$$V^2 + V = 2(g_m - a_0)t(1 - t^3). \qquad (7.44)$$

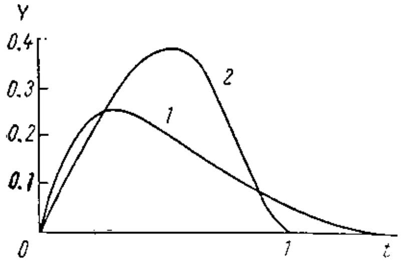

Fig. 15. The function V^2/t near the stability threshold. 1) $y = (D_0 V)^2/t$; 2) $y = t(1-t^3)^2$.

In what follows, we will use the approximate solution (7.44) for both large and small values of $(g_m - a_0)$. As shown in Fig. 15, for small values of $(g_m - a_0)$ the solution in (7.44) and $V = BV_0$ lead to values of V^2/t that are not very different and which determine the contribution to the thermal flux

$$q_T = -\chi_\perp \frac{dT_0}{dx} + \frac{A}{2x} T_+^2 = -\chi_\perp \frac{dT_0}{dx} + \frac{q_T}{2g_m} \frac{V^2}{t} . \quad (7.45)$$

It follows from Eq. (7.45) that

$$-\frac{dT_0}{dx} = \frac{q_T}{\chi_\perp}\left(1 - \frac{1}{2g_m}\frac{V^2}{t}\right). \quad (7.46)$$

If we neglect V compared with V^2 in Eqs. (7.44), then in Eq. (7.46)

$$-\frac{dT_0}{dx} = \frac{q_T}{\chi_\perp}\left[t^3 + \frac{a_0}{g_m}(1 - t^3)\right]. \quad (7.47)$$

When $g_m \gg a_0$ we can use Eq. (7.33) which, by virtue of the relation $x_{1m}/x_m = (1/8g_m)^{1/7}$, can be written approximately in the form

$$-\frac{dT_0}{dx} \simeq \frac{q_T}{\chi_\perp}\left[t^3 + g_m^{-3/7}(1 - t^3)\right]. \quad (7.48)$$

It is evident that Eq. (7.47) becomes (7.48), roughly speaking, only when $g_m > a_0^2 \approx 8$; hence, in practice it is sufficient to make use of the solution in (7.47)

3. Heat Flux in the Presence of Convection

The convection which develops as a result of the instability leads to an additional heat flux.

We shall first consider the case of nonoverlapping cells, in which case $m_c^2 x_{mc} < r$, i.e., in accordance with the condition in (7.41)

$$\beta > \sqrt{\xi \frac{m_e}{m_i}} . \quad (7.49)$$

We must now find the relation between the mean value of the temperature gradient $S_0 = -\left\langle \frac{dT_0}{dx} \right\rangle$ and the heat flux q_T. The quantity $S_0 = q_T \nu + \sum_m \rho_m x_m \left\{\int_0^1 \left(\frac{dT_0}{dx}\right)_m dt - q_T \nu\right\}$ where ρ_m is the density of convection cells denoted by subscript m, i.e., the number of such cells per unit length, while $\left(\frac{dT_0}{dx}\right)_m$ is the value found above from (7.47) and (7.48) for the gradient inside the convection cell. The quantity ρ_m can be found as follows. Let $q = m/n$. When n changes by Δn, the quantity q varies by $\Delta q = -\frac{m}{n^2}\cdot \Delta n = -q^2 \frac{\Delta n}{m}$. But $\Delta q = q'\Delta x$ and, consequently, the number of fractions of the form m/n per unit length is equal to $\frac{\Delta n}{\Delta x} = \frac{m}{q^2} q'$. However, of the fractions of the form m/n, some are reducible, i.e., they represent the higher harmonics of the basic modes in each convective cell. Let P_1 be the probability for the appearance of a nonreducible fraction, P_2 the probability that the fraction is divisible by a factor of two, etc. It is evident that the probability $P_s = (1/s^2) P_1$, i.e., this probability is equal to the product of the probability $1/s^2$ (the numerator and the denominator divided by s) multiplied by P_1, the probability that after reduction by s there will remain a nonreducible fraction. From the normalization condition $\sum_1^\infty P_s = P_1 \sum_{s=1}^\infty \frac{1}{s^2} = \frac{\pi^2}{6} P_1 = 1$ we find $P_1 = 6/\pi^2$. Thus,

$$\rho_m = \frac{\pi^2}{6} \cdot \frac{q'}{q^2} m \quad (7.50)$$

and, consequently,

$$S_0 = q_T \nu + \frac{\pi^2}{6} \frac{q'}{q^2} r \left(\frac{AS_0}{\alpha}\right)^{1/3} \sum_{m=1}^{m_c} m^{1/3} \left\{\int_0^1 \left(\frac{dT_0}{dx}\right)_m dt - q_T \nu\right\} . \quad (7.51)$$

Now replacing $\left(\frac{dT_0}{dx}\right)_m$ everywhere by the values found above in (7.47) and (7.48), replacing the summation over m by integration, and taking account of $g_m = a_0(m_c/m)^{2/3}$, we find

$$\sum_{m=1}^{m_c} m^{1/3}\left\{\int_0^1 \left(\frac{dT_0}{dx}\right)_m dt - q_T\nu\right\} \simeq -q_T\nu\frac{3}{8}m_c^{4/3} = -\frac{3}{8}\cdot\frac{A^2 q_T^3 \nu^5 r}{a_0^2}\left(\frac{AS_0}{\alpha}\right)^{2/3}.$$

Whence

$$S_0 = q_T\nu - \frac{\pi^2}{16}\cdot\frac{q'}{q^2 a_0^2}A^2 q_T^2 \nu^5 \frac{AS_0}{\alpha} r^3,$$

i.e.,

$$S_0 = -\frac{dT_0}{dr} = \frac{\nu}{1 + \frac{\pi^2}{16}\frac{q' A^3 r^3}{q^2 a_0^2 \chi_\perp^5 \alpha} q_T^3} q_T. \tag{7.52}$$

This formula can only be used when the second term in the denominator is smaller than unity. If this requirement is not satisfied we must take account of overlapping of the convection cells and the role of the transverse thermal conductivity for the larger cells will be played by the convection in the smaller cells. Equation (7.52) indicates that for increasing $\nu = 1/\chi_\perp$ (a reduction in the transverse thermal conductivity) the temperature gradient S_0 for a given flux q_T first increases with ν. Then, when the second term in the denominator becomes of order unity, the temperature gradient starts to fall off rapidly. This reduction is clearly not physical and is due to the fact that we have not taken account of the interaction between cells.

We now consider in greater detail the interaction between overlapping cells. Let us first consider a group of cells with a specified azimuthal number m. The cells in this group interact with both large-scale (m' < m) and small-scale (m' > m) perturbations. The interaction with perturbations appreciably smaller than those being considered, i.e., m' ≫ m, obviously leads to an increase in the thermal conductivity; for these perturbations the perturbation being considered plays the role of a background, i.e., it does not differ from the average distribution T_0. In turn, the effect of perturbations characterized by m on m', and also m' ≪ m on m, leads to a macroscopic flow with a velocity v_0 which, as an approximation, can be regarded as a constant within the limits of the cell being considered. Thus, if we neglect the change in the nature of the interaction of the perturbations for m' ~ m, taking account of the interaction, Eq. (7.17) for a cell specified by the number m can be written in the form

$$\frac{AT_m}{x}\frac{d}{dx}(T_{0m} + T_m) = -\alpha m^2 x^2 T_m + \frac{d}{dx}\left(\chi_m\frac{dT_{0m}}{dx} + \chi_m\frac{dT_m}{dx}\right) - v_{0m}\frac{dT_m}{dx}, \tag{7.53}$$

where $T_m \equiv T_+$; χ_m is the effective thermal conductivity, which is determined by all perturbations with m' > m; v_{0m} is the effective convection flow rate, due to perturbations characterized by m' < m; T_{0m} is the mean temperature close to the cell being considered [in Eq. (7.53) as an approximation we write $l = 1/2$, so that $T_+ = -T_-$]. To simplify the analysis we neglect thermal conductivity along y although this conductivity actually makes a contribution of the same order as the thermal conductivity along x.

Let us first consider the linear approximation

$$\frac{AT_m}{x}\cdot\frac{dT_{0m}}{dx} = -\alpha m^2 x^2 T_m + \frac{d}{dx}\left(\chi_m\frac{dT_m}{dx} + \chi_m\frac{dT_{0m}}{dx}\right) - v_{0m}\frac{dT_m}{dx}. \tag{7.54}$$

The thermal flux q_T can be written in the form

$$q_T = -\chi_\perp\frac{dT_0}{dx} + \sum_m \frac{1}{2}v_m T_m = -\chi_m\frac{dT_{0m}}{dx} + \sum_{m'<m}\frac{1}{2}v_{m'}T_{m'} + \frac{1}{2}v_m T_m, \tag{7.55}$$

where χ_m is the effective thermal conductivity which is determined by the higher (m' > m) convective cells. In general, the quantity χ_m is a rapidly varying function of x. However, Eqs. (7.54) and (7.55) can be averaged over x (i.e., over the fast fluctuations). In this case, we find the mean values $\left\langle \chi_m\frac{dT_m}{dx}\right\rangle$ and $\left\langle \chi_m\frac{dT_{0m}}{dx}\right\rangle$, but it is obvious from Eq. (7.55) that $\chi_m\frac{dT_{0m}}{dx}$ does not contain a part that oscillates rapidly in x. A similar statement can be made for the quantity $\chi_m\frac{dT_m}{dx}$, as can be established by integration of Eq. (7.54) over x. Consequently,

$$\frac{dT_m}{dx} = \chi_m^{-1}\left\langle \chi_m\frac{dT_m}{dx}\right\rangle;\ \frac{dT_{0m}}{dx} = \chi_m^{-1}\left\langle \chi_m\frac{dT_{0m}}{dx}\right\rangle.$$

Averaging these relations, we obtain $\left\langle \chi_m \frac{dT_m}{dx} \right\rangle = \langle \chi_m^{-1} \rangle^{-1} \left\langle \frac{dT_m}{dx} \right\rangle$ and a similar expression for $\frac{dT_{0m}}{dx}$. Thus, in Eq. (7.54) and (7.55) all quantities can be taken to be the values as averaged over the small-scale fluctuations ($m' \gg m$) and, under these conditions, χ_m can be understood to be the mean value in the sense $\langle \chi_m^{-1} \rangle^{-1}$.

In Eq. (7.55) the sum over the large-scale fluctuations ($m' < m$) can be regarded as approximately constant and in Eq. (7.54) we replace $\chi_m \frac{dT_{0m}}{dx}$ by $\frac{AT_m^2}{2x}$. In a linear approximation this term can be neglected. Carrying out the further substitution of variables

$$t = x/x_m; \quad V_m(t) = A\chi_m^{-1} T_m = Z_m(t) \exp\left(\frac{v_{0m} x_m}{2\chi_m} t\right), \tag{7.56}$$

we reduce Eq. (7.54) to the form

$$\frac{d^2 Z_m}{dt^2} + g_m \left(\frac{1}{t} - t^2\right) Z_m - b_m Z_m = 0, \tag{7.57}$$

where

$$g_m = A x_m \chi_m^{-1} \frac{dT_{0m}}{dx}; \quad b_m = v_{0m}^2 x_m^2 / 4\chi_m^2.$$

Neglecting the fluctuations, by dT_{0m}/dx and v_{0m}^2 we shall understand the values as averaged over the large-scale cells. Now let us consider the quantities g_m and b_m. The effective thermal conductivity χ_m is obviously a monotonically diminishing function of m which approaches some limiting value $\chi_\perp$ for $m \to \infty$. The quantity x_m also diminishes with m; on the other hand, v_{0m}^2 increases with m.

We now assume that g_m is large. Then the ground state of Eq. (7.57) corresponds to a function which is localized in the region $t < g_m^{-1}$, so that we can neglect t^2 compared with t in the second term; Eq. (7.57) then assumes the form of a Schrödinger equation for the hydrogen atom. The ground state obtains when $b_m = g_m^2/4$, and it is only when $\sqrt{b_m} < g_m/2$ that the instability appears. Substituting the values for g_m and b_m that have been obtained above, we can write the condition for the onset of convection in the form

$$v_{0m} < A \frac{dT_{0m}}{dx}. \tag{7.58}$$

This condition is independent of both χ_m and x_m.

If Eq. (7.58) is satisfied the quantity g_m is still not determined. In order to determine this quantity we must find a relation, independent of Eq. (7.57), that establishes a link between g_m and b_m. We note that the contribution to the convective flux of a cell denoted by subscript m is proportional to $\rho_m T_m^2 / x_m \sim m V_m^2 / \chi_m^2 x_m$, so that it must diminish with m; on the other hand, the quantity $m\chi_m^{-2} x_m^{-1}$ increases with m, so that V_m must diminish with increasing m. It then follows that perturbations characterized by a high value of m are only slightly beyond threshold.

We note further that for large values of g_m the characteristic solution Z_m of Eq. (7.57) goes as $\exp\left(-\frac{|v_{0m}| x_m}{2\chi_m} t\right)$ for $1/g_m < t < 1$; consequently, when $v_{0m} < 0$, in accordance with Eq. (7.56), the function V_m is localized in the region $t \sim g_m^{-1}$; when $v_{0m} > 0$ it is localized in the region $t \sim 1$. Consequently, when $v_{0m} > 0$ the amplitude of the perturbation characterized by subscript m must be smaller than for $v_{0m} < 0$. Without dwelling in detail on this difference, to be definite we shall take $v_{0m} > 0$. In this case, the localization region $t \sim 1$ and, to simplify the calculation, in what follows we shall assume some definite profile for V_m (the main problem lies in finding the amplitude of the fluctuations and not the amplitude profile). If it is assumed that g_m is close to the critical value g_m^c for which only that instability arises, then, as in the conversion from Eq. (7.42) to Eq. (7.44), as an approximation we replace the linear part by $(g_m - g_m)V_m(1/t - t^2)$, in which case Eq. (7.53) assumes exactly the same form as Eq. (7.44); specifically,

$$V_m^2 + V_m = 2(g_m - g_m^c)\, t\,(t - t^3), \tag{7.59}$$

where $g_{cm} \neq a_0$, and is related to a_0 by a relation which can be approximated as follows: $b_m = \frac{1}{4} g_m^c (g_m^c - a_0)$. When $b_m \to 0$, we find $g_m^c = a_0$, and when $g_m^c \gg a_0$ we have $b_m = (g_m^c/2)^2$.

We now express χ_m and v_{0m} in terms of V_m:

$$v_{0m}^2 = \sum_{m'<m} \rho_{m'} \frac{\chi_{m'}^2}{2x_{m'}} \int_0^1 V_{m'}^2 \frac{dt}{t^2}; \tag{7.60}$$

$$\chi_m S_0 = \sum_{m'>m} \rho_{m'} \frac{\chi_{m'}^2}{2A} \int_0^1 V_{m'}^2 \frac{dt}{t}. \tag{7.61}$$

If we take $V_{m'}$ to be described by some profile which is independent of m', we can write $\int_0^1 V_{m'}^2 \frac{dt}{t} = d_0 \int_0^1 V_{m'}^2 \frac{dt}{t^2}$, where d_0 is a constant of order unity. In this case,

$$S_0 \frac{d\chi_m}{dm} = -\rho_m \frac{\chi_m^2}{2A} \int_0^1 V_m^2 \frac{dt}{t} = -\frac{\chi_m d_0}{A} \cdot \frac{dv_{0m}^2}{dm}. \tag{7.62}$$

For values of m that are reasonably small the quantity v_{0m}^2 is small and we can use the earlier solution (7.44) according to which the approximation $V_m^2 = 2g_m t(1-t^3)$ holds when $g \gg a_0$. Thus, assuming that $g_m = AS_0 x_m/\chi_m$, we have

$$\frac{d\chi_m}{dm} = -\frac{\pi^2}{8} \cdot \frac{q' r \xi^{1/3}}{q^2} m^{1/3} \chi_m. \tag{7.63}$$

If we introduce the transverse thermal conductivity in approximate fashion by replacing g_m by $g_m - a_0$, Eq. (7.63) is replaced by

$$\frac{d\chi_m}{dm} = -\frac{\pi^2}{8} \cdot \frac{q' r \xi^{1/3}}{q^2} m^{1/3} \chi_m \left(1 - \frac{a_0 m^{2/3}}{AS_0 r \xi^{1/3}} \chi_m\right). \tag{7.64}$$

In terms of the variable $u = 8a_0 q \chi_m / \pi \sqrt{6q'r\xi}\, AS_0 r$, $\tau = (m/m_0)^{4/3}$, where $m_0^{4/3} = 32q^2/3\pi^2 q' r \xi^{1/3}$, we obtain the equation

$$\frac{du}{d\tau} = -u(1 - u\sqrt{\tau}).$$

The substitution $u = we^{-\tau}$ reduces this equation to the form $dw/d\tau = \sqrt{\tau} e^{-\tau} w^2$ and we have

$$u = \frac{w_0 e^{-\tau}}{1 - w_0 \int_0^\tau \sqrt{\tau}\, e^{-\tau} d\tau}. \tag{7.65}$$

The only solution that goes as $\tau^{-1/2}$ as $\tau \to \infty$ is obtained when

$$w_0 = \int_0^\infty \sqrt{\tau}\, e^{-\tau} d\tau = \sqrt{\pi}/2$$

$$u = e^{-\tau} \left\{ \int_\tau^\infty \sqrt{\tau}\, e^{-\tau} d\tau \right\}^{-1}. \tag{7.66}$$

Thus, $\chi_0 = \frac{\sqrt{6\pi\xi q' r}\, AS_0 r}{4a_0 q}$. Now, making the substitutions $a_0 = 2.9$; $\xi = AS_0/\alpha r^3$; $\alpha = \chi_\parallel \left(\frac{q' H_\theta}{q r H_0}\right)^2$; $A = \frac{3}{2} \frac{q c j_0}{q' H_\theta T_0 \sigma_0}$ and recalling that $q = 2\pi r H_z / L H_\theta$, we obtain the approximate expression

$$\chi_0 = \frac{L}{3\pi} \cdot \frac{q^{1/2}}{q'^2 \sqrt{\chi_\parallel}} \left(\frac{cj_0}{T_0 H_\theta \sigma_0} \cdot \frac{dT_0}{dr}\right)^{3/2}. \tag{7.67}$$

This is the expression that we have been seeking for the coefficient of turbulent thermal conductivity. We note that Eq. (7.67) can be written in the form

$$\chi_0 \cong \sqrt{\frac{2q}{3\xi}} \left(\frac{q}{q' r}\right)^{7/2} \left(\frac{L}{2\pi}\right)^4 \frac{1}{\chi_\parallel} \left(\frac{cE_0}{H_z T_0} \frac{dT_0}{dr}\right)^2, \tag{7.68}$$

which is very similar to the rough estimate that has been given earlier [20]

$$\chi_0 \sim 10 \left(\frac{L}{2\pi}\right)^4 \frac{1}{\chi_\parallel} \left(\frac{cE_0}{H_z T_0} \cdot \frac{dT_0}{dr}\right)^2.$$

It is evident that the estimate is valid when $q \sim 1$, $q'r \sim 1$, $\xi \sim 10^{-2}$.

We now wish to estimate the quantity v_{0m}, using the relation in (7.62) for this purpose. We substitute in this expression the value obtained above for χ_m and integrate with respect to m, writing $v_{0m} = 0$ when $m = 0$; thus we have

$$v_{0m} = \frac{A^2 S_0^2}{d_0 a_0} \int_0^\tau \frac{\sqrt{\pi}}{2} u(\tau) \sqrt{\tau}\, d\tau, \tag{7.69}$$

where $\tau = (m/m_0)^{4/3}$.

It is then evident that when $m \sim m_0$ the quantity v_{0m} is of order AS_0, i.e., this quantity is of the same order as the convection flow rate inside a small-scale cell. This means that when $m \sim m_0$, the small-scale cells have an effect on the flow in the large cells. For this reason the nature of the convection becomes highly complicated when $m > m_0$.

However, this complication has essentially no effect on cells of the basic scale size $m \sim m_0$; as soon as these cells overlap, the depression of the small cells by the large cells can be neglected because the overlap region is so small. On the other hand, the effect of the small-scale fluctuation can be taken into account by introducing a macroscopic coefficient of turbulent thermal conductivity χ_m. The coefficient χ_{0m} that corresponds to the value $m = m_0$ does not differ from χ_0 by more than a factor of 2.

An analysis of the pattern we have been studying leads to the following simple method for approximate evaluation of the coefficients of turbulent transport. At the outset we introduce into the equation of motion effective transport coefficients which take account of the small-scale fluctuations; the values of these coefficients are chosen to make the growth rates vanish for the instabilities with largest localization.

The values of the transport coefficients obtained in this way are approximately equal to the true values since they take account of contributions from all cells, except the very largest. This method is the basis of the analysis in the sequel.

For simplicity we have taken n = const above. In the presence of a density gradient the thermal convection and the heat transport also produce diffusion. Further, since a current-convective instability can develop on the density gradient, the density gradient leads to an additional convection with large-cell localization (we recall that the localization of the density perturbation can be of order ρ_i/θ). However, for values of θ that are reasonably large, and for sufficiently high temperature T, the diffusion effect is not large and we shall defer the analysis to §10.

In concluding this subsection we note further that the mean value is used for the density of the cell ρ_m. Close to the singular points corresponding to small values of m, cells with higher values of m' are crowded together. As a result the effective density ρ_m near these points can be appreciably different from the mean value, and this leads to a regular variation of χ_0 close to the singular points with small m. This has been called the "magic-number" effect [41]. We shall not dwell on this question here.

§8. TRANSPORT OF PARTICLES AND HEAT BY THE TEMPERATURE DRIFT INSTABILITY

1. Turbulent Thermal Conductivity

We now wish to investigate the anomalous thermal conductivity that results from the temperature drift instability. This instability has already been considered in the linear approximation in §3.

For reasons of simplicity we start with the case in which the density is constant in the equilibrium state (n_0 = const), in which there is only a temperature gradient. We assume, furthermore, that $\rho_i \ll \theta a$, so that the individual cells of the drift oscillations are highly localized, in which case the gradient of the mean temperature can be assumed to be constant within a cell. The density is written in the form $n = n_0 + n'$. When $\rho_i \ll \theta a$, the density perturbation is small ($n' \ll n_0$) and the longitudinal resistance and resonance electrons can be neglected, and we have

$$\varphi = \frac{T_e}{e} \cdot \frac{n'}{n_0}. \tag{8.1}$$

When $\theta \ll 1$, the width of the convective cells is appreciably greater than ρ_i. In this case, to which we limit ourselves here, the effect of the finite Larmor radius can be neglected and we can use the drift approximation of the kinetic equation for the ions:

$$\frac{\partial f}{\partial t} + v_\parallel \mathbf{h}\nabla f + \frac{cT_e[\mathbf{h}\nabla n']}{eHn_0}\nabla_\perp f - \frac{e}{m}\mathbf{h}\nabla\varphi\frac{\partial}{\partial v_\parallel}f = 0. \tag{8.2}$$

Here we have taken account of Eq. (8.1).

We now write the distribution function f in the form

$$f = f_0 + f' \equiv f_0 + n'\psi, \tag{8.3}$$

where f_0 is the Maxwellian equilibrium function, while the function $\psi(\mathbf{v})$ is normalized to unity $\int \psi d\mathbf{v} = 1$.

The problem lies in the analysis of the nonlinear oscillations that develop as a consequence of the instability. In order to carry out the analysis we must first direct attention to the essential difference between the process being studied here and the convection process studied in §7. This difference appears in the linear approximation: in the current-convective instability the growth rate for small perturbations is a weak function of wave number k (more precisely, m and n); in the temperature instability, however, the growth rate γ is proportional to k (and is of the order of the frequency ω).

In other words, in the temperature instability the small-scale oscillations are the first ones to develop. Furthermore, since $t_k \sim \gamma_k^{-1}$, since the characteristic time for a significant change in the amplitude of a perturbation of scale size k is smaller than the time for distortion of a large-scale perturbation, the trapping of small perturbations by the large perturbations, which leads to an ordered flow pattern in each cell in the current-convective instability, does not operate in the present case. Hence, a chaotic turbulent motion must result. The maximum scale for the turbulent fluctuations is determined by the localization width of the perturbation, and is of order $\lambda_{max} = \lambda_0 \sim \rho_i/\theta$.

The perturbations in temperature T' and density n' corresponding to this scale are obviously determined by the variation of T and n in a length λ_0, i.e., $T'_{\lambda_0} \sim T\lambda_0/a$ and $n'_{\lambda_0} \sim n\lambda_0/a$. Perturbations on the smaller scale λ can grow by virtue of temperature gradients associated with large-scale perturbations as well as the mean temperature gradient.

Thus, for any given scale size λ perturbations of larger wavelength play the role of a source, while perturbations of smaller wavelength, which are fed from the λ perturbations, provide damping. In other words, in the turbulence being considered here, as in the usual turbulence of an incompressible fluid, there is a flow along the spectrum into the region of small λ. If the motion of the plasma were incompressible, and if there were no damping, the integral $\int T^2 d\mathbf{r}$ would remain constant, i.e., it would be analogous to the total energy of the turbulent fluid, which is conserved in the absence of viscosity. In this case, the magnitude of the fluctuation in T'_λ on scale λ can be found from the following considerations.

Since the quantity T^2_λ remains constant in moving along the spectrum in the direction of smaller λ, there must be a constant flux $(1/\tau_\lambda)T^2_\lambda$, where τ_λ is the characteristic time for deformation of T_λ. But the quantity τ_λ is proportional to the product of the wave number and the gradient for fluctuations of somewhat larger scale, i.e., it is proportional to T_λ/λ^2. Thus,

$$T_\lambda \sim \lambda^{2/3} \quad \text{or} \quad T^2_k dk \sim \frac{dk}{k^{7/3}}, \tag{8.4}$$

i.e., the spectrum falls off rapidly with k.

Actually, the oscillations are characteristic of a compressible medium. However, since there is no systematic compression or rarefaction in the motion along **k**, on the average the compressibility does not affect the spectrum. The damping is a more important factor. The nonlinear interaction between waves leads to a flux into the region of waves with short wavelength in the direction of the magnetic field, in which case ion Landau damping can become important; for this reason, the spectrum falls off more rapidly than is indicated by the relation in (8.4). The damping due to the ions is then evidently the basic mechanism for the dissipation of the shortest wavelength perturbations of scale size $\lambda \lessdot \rho_i$.

Our problem lies not so much with determining the fluctuation spectrum as in the determination of the effective thermal conductivity χ. In finding χ we shall make use of an analogy with the current-convective instability by proceeding in the following manner. To Eq. (8.2) we add a term which takes account of the damping due to smaller-scale fluctuations:

$$\frac{\partial f'}{\partial t} + v_\parallel \mathbf{h}\nabla f' + \frac{cT_e[\mathbf{h}\nabla n']}{eHn_0}\nabla_\perp f - \frac{e}{m}\mathbf{h}\nabla\varphi\frac{\partial}{\partial v_\parallel} f = \operatorname{div}(\chi\nabla_\perp f'). \tag{8.5}$$

The quantity χ is different for different scale sizes and increases with increasing wavelength. We now choose χ in such a way that all perturbations aside from the perturbations characterized by maximum scale length will provide damping. The value of χ found in this way can be regarded as an approximate macroscopic coefficient of thermal conductivity. The point here is that we have taken account of the effect of all perturbations except those of maximum size. It is also possible to consider the contribution from these large-scale perturbations, for example, by using a quasilinear theory. But if χ is chosen in such a way that the growth rate for these perturbations vanishes, then the amplitudes of these perturbations will be negligibly small and, in effect, we have treated all the perturbations. Since the amplitudes of the large-scale perturbations are small, we can replace f by f_0 in the third term in Eq. (8.5) and, as a result, we obtain a linear equation which can be solved approximately.

It should be noted that the form of the term on the right side of Eq. (8.5) which we have chosen is not valid in a strict sense. The point here is that the oscillations in question have a longitudinal phase velocity $\omega/k_\parallel \sim v_i \ll v_e$. Hence, the electrons must set up a Boltzmann distribution along the lines of force. But in the presence of magnetic surfaces, the Boltzmann distribution must obtain over the entire magnetic surface and, from the equilibrium equation for the electrons,

$$\nabla p_e = -en\nabla\varphi - \frac{en}{c}[\mathbf{vH}] \tag{8.6}$$

it follows that the component of the electron velocity normal to the magnetic surface must vanish. In other words if electron inertia, the resonance-electron interaction with the waves, and ion friction are neglected, there can be no diffusion across the magnetic field. Hence, the nonlinear terms, the effect of which appear in the term on the left side of Eq. (8.5), can only give rise to a heat flux in the x direction (and other higher moments of the distribution function), but not a density flux. Thus, a more precise x component for the flux would be written in the form $-\chi n'\nabla\psi$, rather than $-\chi\nabla f'$.

In these oscillations the relative change in the velocity distribution is of the order of the relative change in the density; hence, we can neglect this anisotropy effect. We are then to understand that in the approximation being used here the oscillations do not cause diffusion of plasma across magnetic surfaces.

In Eq. (8.5) we substitute the perturbation f' in the form $f' \exp(-i\omega t + ik_y y + ik_z z)$, replace f by f_0, and integrate the entire equation with respect to $v_\perp$, which appears only as a parameter; thus,

$$+ i(\omega - k_\parallel v_\parallel) f' + ik_y v_0 f_0 \left(\frac{mv_\parallel^2}{2T} - \frac{1}{2}\right)\frac{1}{n_0}\int f' dv_z - \chi_\perp k_y^2 f' +$$
$$+ ik_\parallel \frac{T_e}{m}\cdot\frac{n'}{n_0}\cdot\frac{\partial f_0}{\partial v_\parallel} = -\chi_\perp \frac{d^2 f'}{dx^2}, \quad (8.7)$$

where $v_0 = \frac{c}{cH}\cdot\frac{dT}{dx}$ is the drift velocity.

We wish to find that value of $\chi_\perp$ for which only one solution corresponds to the condition $\mathrm{Im}\,\omega = 0$, while all other solutions are damped. In order to simplify the calculations we shall make use of an approximate method which is reminiscent of the method of separation of variables. To the left- and right-hand sides of Eq. (8.7) we add a term $\chi_\perp u(x) f'$, choosing it in such a way that the right side becomes small. Then, in the zeroth approximation, we have

$$i(\omega - k_\parallel v_\parallel) f' - ik_y v_0 f_0 \left(\frac{mv_\parallel^2}{2T} - \frac{1}{2}\right)\frac{n'}{n^0} - \chi_\perp k_y^2 f' +$$
$$+ \chi_\perp u f' + ik_\parallel \frac{T_e}{m}\cdot\frac{n'}{n_0}\cdot\frac{\partial f_0}{\partial v_\parallel} = 0. \quad (8.8)$$

If we express f' everywhere in terms of n' by means of the relation $f' = (A/B)n'$, where $A = k_y v_0 f_0\left(\frac{1}{2} - \frac{mv_\parallel^2}{2T}\right)$, and $B = \omega - k_\parallel v_\parallel + i\chi_\perp k_y^2 - i\chi_\perp u$, and make use of the relation $\int f' dv_\parallel = n'$, we obtain a dispersion relation for determining ω in the linear approximation. We shall use the symbols γ_0 and ω_0 to denote the "local" growth rate and frequency of the drift waves in the semiclassical approximation (i.e., $k_x \to 0$). We have

$$\omega + i\chi_\perp k_y^2 - i\chi_\perp u = \omega_0 + i\gamma_0. \quad (8.9)$$

The quantities $\omega_0(x)$ and $\gamma_0(x)$ have been given earlier (cf. Fig. 7). Thus, Eq. (8.9) determines the function u(x):

$$u(x) = k_y^2 - \frac{\gamma_0 + i\omega_0}{\chi_\perp} + \frac{i\omega}{\chi_\perp}. \quad (8.10)$$

In order to find the next approximation $f'_{(1)}$ we must solve an equation such as (8.8) with a right side:

$$f'_1 - \frac{A}{B} n'_{(1)} = \frac{\chi_\perp}{B}\left\{uf' - \frac{d^2 f'}{dx^2}\right\}, \quad (8.11)$$

where A and B are the functions introduced above. If Eq. (8.11) is integrated with respect to $v_\parallel$, taking account of the dispersion relation in (8.9) we obtain a zero on the left-hand side. This integral relation is the condition that must be satisfied in order to obtain a solution, and is the conventional one that appears when perturbation theory is used. As an approximation we shall assume that B = const and take it out from under the integral (in accordance with the accuracy we have used above in which we have neglected the anisotropy in transport along the magnetic surfaces and across the magnetic surfaces). In this approximation,

$$\frac{d^2 n'}{dx^2} - u(x) n' = 0 \quad (8.12)$$

or, in another form,

$$\chi_\perp \frac{d^2 n'}{dx^2} - k_y^2 \chi_\perp n' + (\gamma_0 + i\omega_0) n' = i\omega n'. \quad (8.13)$$

It is evident that we have effectively added the quantity $\chi_\perp \Delta_\perp$ to the local growth rate, and that we must now choose $\chi_\perp$ in such a way that the resulting frequency ω is real.

We now convert to dimensionless variables in Eq. (8.13):

$$\xi = \frac{k_\parallel v_i}{\omega_T^*} = \frac{\theta v_i}{a v_0} x; \quad \nu = \frac{i\omega - k_y^2 \chi_\perp}{k_y v_0}; \quad \nu_0 = (\gamma_0 + i\omega_0)/k_y v_0$$
$$\lambda = \chi_\perp \left(\frac{\theta v_i}{a v_0}\right)^2 / k_y v_0$$

and write this equation in the form

$$\lambda \frac{d^2 n'}{d\xi^2} + \nu_0 n' = \nu n'. \quad (8.14)$$

However, from the analysis of instability carried out in the linear approximation it follows that ν_0 can be approximated roughly by the function

$$\nu_0 = \frac{1}{4} - \frac{1}{2}|\xi| + i|\xi| \quad (8.15)$$

(cf. Fig. 7). Substituting this expression in Eq. (8.14), and introducing the new variable $t = \xi + \xi_0$, where $\xi_0 = \frac{-\nu + 1/4}{-\frac{1}{2} + i}$, we reduce this equation to the form (assuming $\xi > 0$)

$$\frac{d^2 n'}{d\xi^2} + b\xi n' = 0. \tag{8.16}$$

Here,

$$b = \left(-\frac{1}{2} + i\right)\lambda^{-1}.$$

The solution of Eq. (8.16) which decays as $\xi \to \infty$ can be expressed in terms of the Airy function $\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(ux + \frac{u^2}{3}\right) du$:

$$n' = \Phi(\sqrt[3]{b}). \tag{8.17}$$

Since the function ν_0 is an even function of ξ, the solution over the entire ξ axis must be either even or odd with respect to ξ. This means that n' must have either a maximum or must vanish when $\xi = 0$. We shall be interested in the solution which has a minimum number of nodes along the ξ axis. For this reason we choose the solution that exhibits the first maximum of the function (8.17) at the point $\xi = 0$. The first maximum of the Airy function $\Phi(x)$ occurs at the point $x = s = -1.02$.

Thus, $\sqrt[3]{b}\,\xi_0 = s$ and, consequently,

$$\lambda = \frac{(1/4 - \nu)^3}{(i - 1/2)^2 s^3}. \tag{8.18}$$

Converting to the usual variables we find

$$\chi_\perp = \left(\frac{av_0}{\theta v_i}\right)^2 k_y v_0 \frac{1}{s^3\left(i - \frac{1}{2}\right)^2}\left(\frac{1}{4} - k_y\frac{\chi_\perp}{v_0} - \frac{i\omega}{k_y v_0}\right)^3. \tag{8.19}$$

The quantity ω is found from the condition that χ_L must be real and is found to be $\omega = 0.86 k_y v_0 \left(\frac{1}{4} - \frac{k_y \chi_\perp}{v_0}\right)$. Substituting this value for ω in Eq. (8.19), and then finding the maximum $\chi_\perp$ with respect to k_y, we have

$$\chi \cong \frac{1}{40}\cdot\frac{av_0^2}{\theta v_i}. \tag{8.20}$$

This is the quantity we have been seeking, the turbulent thermal conductivity for n_0 = const. We recall that $v_0 = \frac{e}{cH}\cdot\frac{dT}{dx}$; $v_i = \sqrt{\frac{2T}{m_i}}$;

$\theta = \frac{a}{L} r \frac{d\iota}{dr} = \frac{a}{Rq^2}\cdot\frac{dq}{dr}$. The corresponding optimum value $k_y = v_0/16\chi_\perp \sim \pi/\Delta x$, where $\Delta x \sim \rho_i/\theta$ is the localization width of the cell. In this case, $\xi_0 = -\frac{3}{8}\cdot\frac{1 - 0.86i}{1 - 2i}$, and the solution in (8.17) assumes the form

$$n' = \Phi\left(s\frac{\xi}{\xi_0} + s\right) \simeq \Phi[s + 4.25(1 - 0.4i)\xi]. \tag{8.21}$$

In the presence of a density gradient the growth rate and the effective thermal conductivity χ are both reduced; if $0 < \eta \equiv \frac{d\ln T}{d\ln n} < 0.95$, the growth rate will generally be equal to zero, and when $-1 < \eta < 0$, the growth rate is exponentially small. This feature can be introduced in an approximate way by the use of a factor F in Eq. (8.20), this factor being defined by

$$F(\eta) = \begin{cases} 1 & \text{for } |\eta| > 2, \\ |\eta| - 1 & \text{for } 1 < |\eta| < 2, \\ 0 & \text{for } |\eta| < 1. \end{cases}$$

2. Interaction Between Cells (Quasi-Modes)

Everywhere above, in considering the thermal conductivity in the largest cells we have investigated convection in an individual large cell. This approach is valid only if the large cells are statistically independent, i.e., if the convection in any one cell is not correlated with the convection in neighboring cells. Actually, however, if the cells are small a resonance coupling between cells can appear, and in this case the interaction between cells cannot be neglected. The possibility of a coupling of this kind was first considered by Roberts and Taylor [42]. The authors showed that even in the linear approximation there is a possibility of obtaining a perturbation with a large region of localization in x if a solution is taken in the form of a quasi-mode, i.e., a superposition of highly localized solutions with the same azimuthal number m but different k_z. This solution has different localization points, and thus can be constructed in a series along r by joining one solution to another. Let $\varphi_1 = \varphi_1(r - r_0)e^{ik_z z + im\vartheta}$ be one of the localized solutions. Then a quasi-mode, a solution with a large region of localization, can be formed by writing

$$\varphi(z) = \int A(r_0)\,\varphi_1(r - r_0)\,e^{-ik_z z + im\vartheta}\,dr_0, \qquad (8.22)$$

where $A(r_0)$ is some smoothly varying function of r_0, for example, $A(r_0) = A_0 e^{-(r_0 - r_{00})^2/l^2}$, where l is the localization width of the quasi-mode while r_{00} is the center of localization.

Since the point r_0 is determined from the condition $k_{\parallel} = 0$, i.e., $(m/r_0)H_\vartheta - k_z H_0 = 0$, then the quantity r_0 is related to k_z by $k_z = (m/R_0) \cdot [1/q(r_0)]$, and Eq. (8.17) can be written in the form of a superposition of solutions with different k_z. Assuming that $l \ll a$, we get $k_z = k_z^0 - k_y(\theta/a) \cdot (r_0 - r_{00})$, where k_z^0 corresponds to the localization point r_{00}. Then Eq. (8.22) assumes the form

$$\varphi(r) = \int \varphi_1(r - r_0)\,B_0 e^{-(k_z - k_z^0)^2 L^2}\, e^{-ik_z z + im\vartheta}\,dk_z, \qquad (8.23)$$

where B_0 = const, $L^2 = a^2/l^2 k_y^2$; $r_0 \quad r_{00} - (R_0 q^2/mq')(k_z - k_z^0)$.

It is then obvious that when l increases the localization width of the quasi-mode at the point $z = 0$ also increases but that there is also a reduction in the localization length L along the magnetic field over which the phases of the localized perturbations are approximately the same.

In a toroidal geometry, $k_z = n/R_0$ can only assume discrete values and hence the quasi-modes can only be constructed by a superposition of cells with different values of n. Since $q = m/n$, then δx, the distance between neighboring ($\delta n = 1$) cells characterized by the same azimuthal number m, is determined by the relation

$$q'\delta x \approx \frac{m}{n^2}\,\delta n \approx \frac{1}{m}\,q^2, \quad \text{i.e.,} \quad \delta x \approx \frac{q^2}{q'}\,m^{-1}.$$

Evidently, when $q'/q \approx 1/a$, $\delta x \approx qa/m \approx qk_y^{-1}$, i.e., $\delta x > k_y^{-1} \sim \rho\ /\theta$. In other words, cells with the same azimuthal number m are separated by a distance δx which exceeds the localization distance $\sim \rho_i/\theta$.

However, this result does not mean that a resonance interaction cannot operate between cells in a toroidal geometry. The point is that the interaction between cells is essentially nonlinear and cannot be described within a linear theory.

A nonlinear flow represents a superposition of many modes, and if two or more cells with somewhat different m and n are close together, it might be assumed that these cells would add constructively in the regions of ϑ and ζ where their phases $[m\vartheta - (n/R_0)\zeta]$ are approximately the same.

In other words, a given cell can contain a "mixture" of a neighboring cell, and thus reinforce the neighboring cell. It is obvious that this resonance capture of one cell by another can occur only when the values of ϑ and ζ are such that the phases $m\vartheta - (n/R_0)\zeta$ are approximately the same in the cells, i.e., the difference in the azimuthal phases $m - m'$ and the longitudinal numbers $n - n'$ must be much smaller than m and n. Since the number of cells with azimuthal numbers smaller than or of the order of m (and, consequently, $n \sim m/q$) is proportional to m^2, the mean distance between cells for any m and n not exceeding the given values is of order $\delta x \sim q'/m^2$. Consequently, the distance between resonance cells is always much larger $\delta x - q'/m^2 \sim 1/a k_y^2$. It then follows that the resonance interaction can occur only when $\rho_i/\theta \gg a/k_y^2 a^2$ or $a k_y = m \gg \sqrt{\rho_i/a\theta}$.

The existence of a resonance coupling means that the most effective heat transfer will be realized by coupled cells with effective transverse dimension $x \sim 1/k_y \ll a\sqrt{\rho_i/a\theta}$. This dimension can be appreciably greater than the localization of an individual isolated cell ρ_i/θ but, as we have seen, not by more than a factor $\sqrt{\theta a/\rho}$. In other words, even when $\theta a/\rho \sim 10^2$ the effect of resonance coupling increases the coefficient in (8.20) by less than a factor of ten.

3. Turbulent Diffusion

In analyzing the thermal conductivity we have assumed that the electrons maintain a Boltzmann distribution and we have neglected electron diffusion. In addition to modifying the thermal conductivity, however, the oscillations also cause electron diffusion. In order to find the appropriate diffusion coefficient we assume that the oscillations in the potential that develop φ are of order $\frac{T}{e}\,x/a \sim \frac{T}{e}\rho_i/\theta a$, where x is the localization width of the cell. Hence, electrons with longitudinal velocities $v_\parallel < \sqrt{\frac{2\varphi}{m_e}} \sim v_e\sqrt{\frac{\rho_i}{\theta a}}$ fall into an essentially nonlinear region, since they can be trapped by potential wells in the drift waves. On the other hand, in the region $v_\parallel > v_e\sqrt{\rho_i/\theta a}$ the electrons can pass freely through the potential barriers in φ and the diffusion arises only by virtue of resonance interaction of electrons with waves whose phase velocity coincides with the electron velocity. Since the oscillation amplitude $\varphi \sim \frac{T}{e}\cdot\frac{\rho_i}{\theta a}$ is taken to be small compared with T/e, in describing diffusion of resonance electrons we can use a quasi-linear approximation. In the quasi-linear approximation, in the expression for the particle flux

$$q_n = -D\frac{dn_0}{dx} = -\langle n'v_x'\rangle = -\left\langle \int f'dv\,\frac{cE_y}{H}\right\rangle \tag{8.24}$$

the perturbation f' is expressed in terms of the linear approximation:

$$f_{\mathbf{k}}' = \frac{i}{\omega - k_\parallel v_\parallel + i0}\left\{\frac{\partial f_0}{\partial x} + \frac{eH}{cT}\cdot\frac{v_\parallel k_\parallel}{k_y}f_0\right\}\frac{cE_{y\mathbf{k}}'}{H}, \tag{8.25}$$

where $f_{\mathbf{k}}'$ is the Fourier component of f', and $E_{\mathbf{k}}$ is the Fourier component of the electric field. From Eqs. (8.24) and (8.25) we can obtain the very approximate relation

$$D \sim \int v_{\mathbf{k}}^2\pi\delta(\omega - k_\parallel v_\parallel) f_0/n_0 dv d\mathbf{k}, \tag{8.26}$$

where $v_{\mathbf{k}}^2 = \left(\frac{cE_y}{H_0}\right)_{\mathbf{k}}^2$ is the spectral function of the velocity v_x.

We now take account of the fact that $\chi \sim v_x^2/\omega \sim \int \frac{v_{\mathbf{k}}^2}{\omega_{\mathbf{k}}}d\mathbf{k}$ and that the waves are concentrated in the region of phase velocities $\omega/\mathbf{k}_\parallel \sim v_i$. Using Eq. (8.26), we find

$$D \sim \int \frac{v_{\mathbf{k}}^2 d\mathbf{k}}{v_e k_\parallel} \sim \chi\frac{v_i}{v_e}\ln\frac{k_{\parallel\max}}{k_{\parallel\min}}, \tag{8.27}$$

where v_e is the mean thermal velocity of the electrons, v_i is the mean thermal velocity of the ions, $k_{\parallel\min} \sim \omega/v_e$ corresponds to the longitudinal phase velocity of order v_e, while $k_{\parallel\max}$ corresponds to the phase velocity $v_e\sqrt{\rho_i/\theta a}$ below which the nonlinear region is found. Thus, the resonance-electron contribution to the diffusion is smaller than

$$D \sim \chi\sqrt{\frac{m_e}{m_i}}\ln\sqrt{\frac{\theta a}{\rho_i}}. \tag{8.28}$$

As far as the electrons trapped by the potential wells are concerned, we note that since the fraction of such electrons (compared with the total number) is less than $\sqrt{\rho_i/\theta a}$, the corresponding contribution to the diffusion coefficient cannot exceed $\chi\sqrt{\rho_i/\theta a}$; when collisions are taken into account, these collisions tending to restore the Maxwellian velocity distribution of the electrons, this contribution becomes smaller still [by the factor $S(S+\sqrt{m_e/m_i})^{-1}$ where $S = \lambda_e\rho_i/a^2$]. The total coefficient D is then given in order-of-magnitude terms by

$$D \sim \left\{\sqrt{\frac{m_e}{m_i}}\ln\sqrt{\frac{a\theta}{\rho_i}} + \frac{S}{S+\sqrt{\frac{m_i}{m_e}}}\sqrt{\frac{\rho_i}{\theta a}}\right\}\chi. \tag{8.29}$$

Thus, the diffusion coefficient associated with the temperature drift instability is appreciably smaller (approximately by one order of magnitude) than the thermal conductivity, as is the case for ordinary Coulomb collisions. The quantity D is small because the electrons try to maintain a Boltzmann distribution and that it is difficult for electrons to be displaced from the magnetic surfaces. However, if the magnetic surfaces are disturbed, then the diffusion of particles will proceed simultaneously with the transport of heat and the coefficient D will be of order χ.

§9. ANOMALOUS DIFFUSION DUE TO TRAPPED PARTICLES

As the plasma temperature is increased or as the density is reduced, the frequency of ion–ion collisions is reduced and the instability due to trapped particles can become important. The first instability to appear is the dissipative instability associated with electron collisions. In considering the nonlinear oscillations that develop by virtue of the trapped-particle instability it is convenient to simplify the problem by averaging along the lines of force; we make use of the following simplified two-dimensional equations:

$$\frac{\partial n_+}{\partial t} + \frac{c[\mathbf{h}\nabla\varphi]}{H}\nabla n_+ = 0; \tag{9.1}$$

$$\frac{\partial n_-}{\partial t} + \frac{c[\mathbf{h}\nabla\varphi]}{H}\nabla n_- = -\nu_{ef}(n_- - \delta n_0). \tag{9.2}$$

Here, n_+ is the density of trapped ions; n_- is the density of trapped electrons; δ is the fraction of trapped particles in the equilibrium state (in an axially symmetric torus this fraction is given by $\sqrt{\varepsilon} = \sqrt{r/R_0}$). The right side of Eq. (9.2) takes account of the conversion of trapped particles into transiting particles by virtue of collisions, where $\nu_{\text{eff}} = \nu_e/\delta^2$ is the effective collision frequency. We assume that n_+, n_-, and φ are functions of the two variables r and ϑ only, so that Eqs. (9.1) and (9.2) describe the nonlinear oscillations of the trapped particles. The potential that appears in Eqs. (9.1) and (9.2) is determined from the neutrality condition

$$n_+ - \frac{e\varphi}{T}n_0 = n_- + \frac{e\varphi}{T}n_0, \tag{9.3}$$

in which we have taken account of both the trapped particles and the transiting particles, which exhibit a Boltzmann distribution. We note that terms of the form $\mathbf{v}_m \nabla n$, (associated with magnetic drift) have been neglected in Eqs. (9.1) and (9.2).

In the case being considered here, a dissipative instability, the frequency ν_e has been assumed to be large, so that we can neglect the time derivative of n_- in Eq. (9.2); furthermore, in the left side of Eq. (9.2) we replace n_- by its equilibrium value δn_0. Thus,

$$n_- = \delta n_0 - \frac{\delta^2}{\nu_e} \frac{c\,[\mathbf{h}\nabla\varphi]}{H} \nabla\delta n_0. \tag{9.4}$$

If we substitute this expression in Eq. (9.3) and express φ in terms of n_+ and n_-, using an expansion in $1/\nu_e$, then, by means of the expression that has been obtained for φ [and Eq. (9.1)], we obtain the following nonlinear equation for $n'_+ = n_+ - \delta n_0$:

$$\frac{\partial n'_+}{\partial t} - v_* \frac{\partial n'_+}{\partial y} = -\frac{v_*^2\delta^2}{\nu_e} \cdot \frac{\partial^2 n'_+}{\partial y^2} + N\left(\frac{\partial n'_+}{\partial x} \cdot \frac{\partial^2 n'_+}{\partial y^2} - \frac{\partial n'_+}{\partial y} \cdot \frac{\partial^2 n'_+}{\partial x \partial y}\right), \tag{9.5}$$

where

$$v_* = \frac{cT}{2eHn_0} \cdot \frac{d}{dr}\delta n_0; \quad N = \frac{cTv_*\delta^2}{2eHn_0^2\nu_e}.$$

Equation (9.5) is written in the plane-layer approximation, which is valid for small-scale instabilities; specifically, we have introduced the coordinates x and y in place of r and ϑ, and have neglected terms that contain $\partial^2 n_0/\partial x^2$ and $(\partial n_0/\partial x)^2$.

As is evident from Eq. (9.5), in the linear approximation we are dealing with a diffusion equation, but with a negative diffusion coefficient. Since the growth rate $\gamma \sim k_y^2$, at first glance we see that perturbations with short wavelengths will grow, and this leads to an effective diffusion with respect to the longwave perturbations.

In turn, the longwave perturbations appear as a supplementary mechanism for driving the shortwave instabilities. This feature can be seen directly from the structure of the nonlinear term. If we consider a longwave perturbation n'_1, in which the second derivative can be neglected and the first derivative can be taken as a constant, for a shortwave perturbation n'_{+2} (plane wave) the nonlinear term assumes the form $k_y\left(k_y \frac{\partial n'_{+1}}{\partial x} - k_x \frac{\partial n'_{+1}}{\partial y}\right) n'_{+2}$. When $\frac{\partial n'_{+1}}{\partial y} \neq 0$ the sign of this expression is not determined, so that we can always find a perturbation n'_2 that grows by virtue of $\nabla n'_{+1}$.

The feeding of shortwave perturbations by longwave perturbations leads to chaotic turbulent motion. In this case, the small-scale perturbations play the role of a diffusion mechanism with respect to the large-scale perturbations and their effect can be taken into account by an additional term of the form $D_t \Delta n'_+$ on the right side of Eq. (9.5). The quantity D_t, the coefficient of thermal diffusion of the trapped particles, is determined from the condition that the large-scale perturbations be stationary. We assume that because of the strong randomness the motion will be isotropic, i.e., $k_y^2 \approx k_x^2$; thus, by comparison with Eq. (9.5), we have

$$D_t = \frac{v_*^2\delta^2}{2\nu_e}. \tag{9.6}$$

Evidently, D_t does not depend on k^2. On the other hand, it is clear that D_t must increase as k^2 is reduced, because waves of larger and larger scale participate in the effective transport of matter. A self-consistent pattern can be obtained only under the assumption that D_t is determined by perturbations of the minimum scale size. At this scale size the transition from the dissipative trapped particle instability (to the collisionless trapped-particle instability) occurs, i.e., at the value of k_y determined by the relation $\nu_e/\delta^2 = \omega = \omega_* \delta^{3/2}$.

All of the perturbations of larger scale size are not excited. The effective coefficient for these D_t is found to be somewhat larger than the critical value (9.6), so that these decay in time. We can take the value given by Eq. (9.6) for the effective diffusion coefficient D_t. Assuming that D_t derives only from the trapped particles, we can now write the total effective diffusion coefficient:

$$D = \delta D_t = \frac{c^2T^2\delta^3}{8e^2H^2n_0^2\nu_e}\left(\frac{d}{dr}\delta n_0\right)^2. \tag{9.7}$$

For a circular torus with an axis of symmetry we take $\delta = \sqrt{r/R_0}$; as an approximation we remove δ from the differentiation sign, thereby obtaining

$$D = \frac{c^2T^2r^{5/2}}{8e^2H^2\nu_e R_0^{5/2}}\left(\frac{1}{n_0} \cdot \frac{dn_0}{dr}\right)^2. \tag{9.8}$$

In addition to causing the transport of density, the trapped-particle oscillations also cause a heat transfer. Since these oscillations represent an interchange of "tubes" of trapped particles, i.e., a convection of what are essentially incompressible plasma tubes, the coefficient of thermal conductivity both for the electrons χ_e and the ions χ_i must be approximately equal to D:

$$\chi_e = \chi_i = \frac{c^2 T^2 r^{5/2}}{8e^2 H^2 \nu_e R_0^{5/2}} \left(\frac{1}{n_0} \cdot \frac{dn_0}{dr} \right)^2. \tag{9.9}$$

Since $\nu_e \sim T^{3/2}$, the coefficients in (9.8) and (9.9) increase very rapidly with temperature. For a sufficiently high temperature, where ν_e/δ^2 becomes smaller than the oscillation frequency $\omega = \omega^* \delta^{3/2}$ (corresponding to perturbations with the maximum possible localization m ≈ g)

$$S \equiv \frac{\lambda_e \rho_i}{a^2} > \sqrt{\frac{m_i}{m_e}}\, q^{-1} \delta^{-5/2} \approx 10^2, \tag{9.10}$$

the dissipative trapped-particle instability becomes a collisionless instability and the magnetic drift must be introduced into Eqs. (9.1) and (9.2). Under these conditions, the perturbations are no longer localized, and it is not meaningful to introduce the notion of a local diffusion coefficient. We can only make an estimate of an effective diffusion coefficient D, which is averaged over the entire volume, to describe the convection of charged particles. Since the growth rate $\gamma \approx \varepsilon q \rho_i v_i / a^2$, while the localization width for the convective cells is of order a, then

$$D \approx \delta \gamma a^2 \approx \left(\frac{a}{R_0} \right)^{3/2} q \rho_i v_i, \tag{9.11}$$

i.e., this coefficient is of the same order as the Bohm diffusion coefficient $D_B = \frac{1}{16} \cdot \frac{cT}{eH}$. Convection due to trapped particles leads to the loss of trapped particles. However, the Coulomb collisions can fill the trapped-particle cone rather rapidly with transiting particles; moreover, if the distribution function has marked "bumps," it is reasonable to expect a rapid filling as a result of high-frequency electrostatic oscillations.*

Equations (9.8) and (9.9) have been obtained under the assumption that $T_i = T_e = T$. Returning to Eqs. (9.3)-(9.5) we see that if $T_i \neq T_e$ we must replace $1/T$ by $1/T_i + 1/T_e$ in Eq. (9.5) and in Eqs. (9.8) and (9.9). In other words,

$$D = \chi_e = \chi_i = \frac{c^2 T_i^2 T_e^2 r^{5/2}}{2e^2 H^2 \nu_e (T_i + T_e)^2 R_0^{5/2}} \left(\frac{1}{n_0} \cdot \frac{dn_0}{dr} \right)^2. \tag{9.12}$$

*In addition, for fluctuations on the scale $\lambda \sim a$, the fluctuations in the potential φ can convert trapped particles into transiting particles, and vice versa.

Here, the factor $(r/R_0)^{5/2}$ represents δ^5, where δ is the fraction of trapped particles. In traps with an inhomogeneous magnetic field (for example, if there are helical windings) the quantity δ can be of order unity.

We note that in a highly nonisothermal rarefied plasma (in which collisions are unimportant) the trapped-particle instability is easily stabilized, as can be seen from Eq. (4.30).

§10. DIFFUSION IN SMALL-SHEAR SYSTEMS

1. Drift-Dissipative Instability

It has been assumed in the foregoing that the shear $\theta \sim 10^{-1} \gg \sqrt{m_e/m_i}$. If the shear is reduced, the macroscopic effects due to all instabilities become stronger. In particular, the thermal conductivity due to the temperature drift instability (for $\eta \gg 1$) increases. Thus, in a plasma that is not too dense, we can assume $\eta \leq 1$, in which case the temperature drift instability does not develop. Furthermore, for very small values of θ those instabilities become important which are either stabilized when $\theta \sim 10^{-1}$, or which have very narrow localization ranges. This statement pertains to the collisionless drift instability, the dissipative drift instability, and the finite-orbit instability, which is closely related to these two; a peculiar kind of "smearing viscosity" is important in this latter instability. This viscosity derives from the deviation of the ions from the line of force in the drift motion. We shall start our analysis with the drift-dissipative instability.

When the localization of an instability is smaller than, or of the order of, the ion Larmor radius the effect of the instability is always weak; thus, we need only consider the case $\rho_i \ll x$, for which the magnetohydrodynamic equations can be used. Furthermore, for reasons of simplicity we shall assume that the ion temperature is zero (according to the linear theory, increasing the temperature from $T_i = 0$ to $T_i = T_e$ does not increase the growth rate by more than a factor of 2). The starting equations will be the equation of continuity

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} = 0, \tag{10.1}$$

the sum of the equations of motion for the electrons and ions

$$m_i n \frac{d\mathbf{v}}{dt} + \nabla p = \frac{1}{c} [\mathbf{jH}] \tag{10.2}$$

and the longitudinal component of the equation of motion for the electrons, in which we neglect the inertia term:

$$\mathbf{h}\nabla p_e = en\mathbf{h}\nabla\varphi + \frac{en}{\sigma} j_{\parallel}. \tag{10.3}$$

In the instability at hand the effect of any inhomogeneity in the magnetic field is unimportant, so that we can write $H = H_0 = \text{const}$. It then follows from the transverse component of the ion equation of motion that if inertia is neglected, the transverse velocity $\mathbf{v}_{\perp}$ is

$$\mathbf{v}_{\perp} = \frac{c\,[\mathbf{h}\nabla\varphi]}{H}, \tag{10.4}$$

so that the transverse motion is incompressible.

Now, applying the operation curl_z to Eq. (10.2), and taking $n = \text{const}$ in the inertia term, we find

$$\frac{d\Gamma}{dt} = \frac{\partial\Gamma}{\partial t} + \mathbf{v}\nabla\Gamma = -\frac{H_0}{cm_i n_0}\,\text{div}\,\mathbf{j}_{\perp} = \frac{H_0}{cm_i n_0}\,\text{div}\,\mathbf{j}_{\parallel} = \frac{H_0}{cm_i n_0}\,\mathbf{h}\nabla j_{\parallel}, \tag{10.5}$$

where $\Gamma = (\text{curl})_z\,\mathbf{v} = \frac{c}{H_0}\Delta_{\perp}\varphi$.

Recalling that $\mathbf{h}\nabla = -\theta\frac{x}{r^2}\cdot\frac{\partial}{\partial\vartheta}$, and taking account of (10.3), we write (10.5) in the form

$$\frac{d\Gamma}{dt} = \frac{H_0\theta^2x^2\sigma}{cr^4m_in_0}\cdot\frac{\partial}{\partial\vartheta}\left[\frac{1}{en}\cdot\frac{\partial p_e}{\partial\vartheta} - \frac{\partial\varphi}{\partial\vartheta}\right]. \tag{10.6}$$

We first consider the oscillations under the assumption that the electron temperature is constant along the lines of force. Then T_e and σ can be regarded as independent of ϑ. Furthermore, since $\partial p_0/\partial\vartheta = 0$, and by virtue of the smallness of the region of localization, the perturbation $p' = n'T_0$ is small, and the factor $1/en$ in front of $\partial p_e/\partial\vartheta$ in the square brackets in Eq. (10.6) can also be taken as constant. Thus, for perturbations of the form $\exp(im\vartheta)$, Eq. (10.6) can be written

$$\frac{d\Gamma}{dt} = \frac{\partial\Gamma}{\partial t} + \mathbf{v}\nabla\Gamma = -Bx^2\left(\frac{T_0n'}{en_0} - \varphi\right), \tag{10.7}$$

where the quantity $B = \frac{m^2H_0\theta^2\sigma}{cr^4}$ is assumed to be constant within a given cell.

In the linear approximation, Eqs. (10.1)-(10.3) describe the drift dissipative instability, in which the growth rate is proportional to the wave number $k_y = m/r$. This means that shortwave perturbations develop first and that the development of the instability must lead to chaotic turbulent motion. In order to determine the coefficient of turbulent diffusion D we again use an approximation method, introducing into the equation of continuity (10.1) for the large-scale cells a diffusion term which takes account of the effects due to small-scale perturbations:

$$\frac{\partial n}{\partial t} + \mathbf{v}\nabla n = D\Delta_{\perp}n. \tag{10.8}$$

Here, the term $D\Delta_{\perp}n$ represents an average, over the small perturbations, of the nonlinear term $\mathbf{v}\nabla n$. The same form of the nonlinear term $\mathbf{v}\nabla\Gamma$ appears in Eq. (10.7) for the vorticity Γ since the same small-scale fluctuations that cause the diffusion also lead to a mixing of regions with different values of Γ. These fluctuations must then cause the diffusion of Γ, which is equivalent to the appearance of a term with a viscosity $\nu\Delta_{\perp}\Gamma$ on the right side of Eq. (10.7):

$$\frac{\partial\Gamma}{\partial t} + \mathbf{v}\nabla\Gamma = Bx^2\left(\frac{T_0n'}{en_0} - \varphi\right) + \nu\Delta_{\perp}\Gamma. \tag{10.9}$$

Since the nonlinear terms associated with viscosity and diffusion are identical we can write $\nu = D$. The value of the effective diffusion D can again be found from the requirement that perturbations of maximum scale size must be neutrally stable. Now, taking $\Gamma = (c/H_0)\Delta_{\perp}\varphi$, in the linear approximation we have

$$-i\omega n' + i\omega^* n_0\psi = D\Delta_{\perp}n'; \tag{10.10}$$

$$-i\omega\Delta_{\perp}\psi - B_0x^2\left(\frac{n'}{n_0} - \psi\right) = D\Delta_{\perp}\Delta_{\perp}\psi, \tag{10.11}$$

where ω^* is the drift frequency; $\psi = \frac{e\varphi}{T_0}$; $B_0 = \frac{H_0^2\theta^2m^2\sigma}{c^2r^2n_0m_i} = \frac{\theta^2m^2}{r^4}\Omega_i\Omega_e\tau_e$.

We note that $B_0x^2 = -\omega_s\Delta_{\perp}$, where $\omega_s = \frac{k_{\parallel}^2}{k_{\perp}^2}\Omega_i\Omega_e\tau_e$. In Eq. (10.11) we have replaced ν by the equivalent quantity D.

We note that (10.10) and (10.11) do not have localized solutions when $D = 0$. Localization is produced by terms on the right side, i.e., nonlinear effects which are taken into account by the introduction of viscosity and diffusion.

The effective diffusion coefficient is found from the condition that there be no imaginary part in the characteristic frequency ω associated with (10.10) and (10.11). We now convert to a Fourier representation, making the substitutions $\Delta_\perp \to -k_y^2 - k^2$, $x^2 \to -d^2/dk^2$. Then, the new variable, $\chi = (\omega - iD\Delta - \omega^*)$ is governed by the second-order equation

$$\frac{d^2\chi}{dk^2} = -\frac{i\left(k_y^2+k^2\right)\left[\omega+iD\left(k_y^2+k^2\right)\right]^2}{B_0\left[\omega+iD\left(k_y^2+k^2\right)-\omega^*\right]}\chi. \qquad (10.12)$$

It will be convenient to convert to dimensionless variables in this equation:

$$\xi = k/k_y;\ \lambda = \omega/Dk_y^2;\ \lambda_* = \omega^*/Dk_y^2.$$

Then,

$$\frac{d^2\chi}{d\xi^2} = U\chi = \mu\frac{(1+\xi^2)\left[-i\lambda+(1+\xi^2)\right]}{(1+\xi^2)-i(\lambda-\lambda_*)}\chi, \qquad (10.13)$$

where $\mu = \dfrac{Dk_y^6}{B_0} = \dfrac{Dm^4}{r^2\theta^2\Omega_i\Omega_e\tau_e}$.

This second-order equation with a "complex well" (10.13) contains two real parameters, λ and μ. These parameters must be chosen in such a way that χ represents a characteristic function that decays as $x \to \pm\infty$.

We note that the quantity $\lambda^4\mu$ does not depend on $k_y = m/r$, being proportional to D^{-3}. Since we are interested in the solution with the largest possible D, the free parameter λ in Eq. (10.13) must be chosen in such a way that $\mu\lambda^4$ reaches a maximum value.

We now divide the potential U on the right side of Eq. (10.13) into real and imaginary parts $U = U_1 + iU_2$, where

$$U_1 = \mu\frac{(1+\xi^2)^2}{(1+\xi^2)^2+\alpha^2}\left[(1+\xi^2)^2+\alpha^2-\lambda_*^2\right]; \qquad (10.14)$$

$$U_2 = -\mu\frac{(1+\xi^2)^2}{(1+\xi^2)^2+\alpha^2}\left[(2\lambda_*-\alpha)(1+\xi^2)^2-\alpha(\lambda_*-\alpha)^2\right], \qquad (10.15)$$

while $\alpha = \lambda_* - \lambda$.

Multiplying Eq. (10.13) by the complex conjugate χ^*, and integrating over ξ, we have

$$\int_{-\infty}^{\infty} U_1|\chi|^2 d\xi = -\int_{-\infty}^{\infty}\left|\frac{d\chi}{d\xi}\right|^2 d\xi;$$

$$\int_{-\infty}^{\infty} U_2|\chi|^2 d\xi = 0. \qquad (10.17)$$

It follows from Eqs. (10.14) and (10.16) that λ cannot be a small quantity, i.e., $\lambda_*^2 \geq 1 + \alpha^2$. Since we are interested in the minimum possible value of λ_* we must take $\lambda_* \sim 1$. It is evident from Eq. (10.15) and (10.17) that $\alpha \sim 1$, so that $\mu \sim 1$. Thus, the minimum value of the parameter $\lambda_*^4\mu$, which we will denote by s, is of order unity: $s = \min\lambda_*^4\mu \sim 1$. Whence, recalling the expressions for λ_* and μ, in terms of D and ω^* we have

$$D = \frac{cT}{eH}\cdot\frac{1}{(\Omega_e\tau_e)^{1/3}}\left[\frac{\rho_{ie}^2 r^2}{\theta^2 s^2}\left(\frac{1}{n_0}\cdot\frac{dn_0}{dr}\right)^4\right]^{1/3}, \qquad (10.18)$$

where $\rho_{ie}^2 = T_e/m_i\Omega_i$, $s \sim 1$. In order-of-magnitude terms,

$$D \sim \frac{cT}{eH}\left(\frac{\rho_{ie}}{a\theta}\right)^{2/3}(\Omega_e\tau_e)^{-1/3}. \qquad (10.19)$$

Even with the existing parameters in toroidal devices, the quantity $\Omega_e\tau_e$ is extremely large, being of order 10^6; thus, even when $\theta \sim 10^{-3}$-10^{-2} the diffusion coefficient in (10.19) is relatively small and need not be considered.

We have assumed above that there is no perturbation of electron temperature. But, as we have shown in subsection 9 of §3, when $k_\parallel\lambda_e < \sqrt{m_e/m_i}$ an instability associated with perturbations in the electrons temperature can arise. Since the growth rate of this instability cannot exceed the quantity $(m_e/m_i)\nu_e$, while the localization width $x < \rho_i/\theta\sqrt{(m_e/m_i)}(1/S)$, the effective diffusion coefficient $D \sim \gamma x^2$ cannot exceed the value $D_c\dfrac{m_e}{m_i\theta^2S^2}$, where $D_c \approx \dfrac{m_e\nu_e}{m_i}\rho_i^2$ is the classical diffusion coefficient. If S is reasonably large the coefficient D due to this instability can also be neglected.

2. Finite-Orbit Instability

The diffusion coefficient (10.19) due to the drift dissipative instability can be found from the following simple estimates. It follows from the dispersion equation in the semiclassical approximation (3.38a) that the maximum growth rate $\gamma \simeq \omega^*$ is reached when $\omega^* \sim \omega_s = (k_\parallel^2/k_\perp^2)\Omega_i\Omega_e\tau_e$. Taking $k_y \sim k_x \sim k_\perp \sim x^{-1}$, we find the width of the localization region $x \sim (a\rho_i/\theta^2\Omega_e\tau_e)^{1/3}$; then, $D \sim \gamma x^2 \sim \omega^* x^2 \sim \rho_i v\,(\rho_i^2/a^2\theta^2\Omega_e\tau_e)^{1/3}$.

In the presence of drift motion in an inhomogeneous magnetic field, as we have established in subsection 3 of §4, there is an enhanced viscosity because of "mixing," i.e., the departure of ions from the lines of force becuase of the magnetic drift. This effect can be taken into account roughly if we reduce ω_s by a factor ρ^2/Λ^2, where Λ is the mean-square value of the ion deviation from the magnetic surface due to the drift motion. There is a corresponding increase in the diffusion coefficient and the latter is found to be given approximately by

$$D \simeq \rho_i v_i (\Lambda^2/a^2\theta^2\Omega_e\tau_e)^{1/2}. \qquad (10.20)$$

In the Model-C Stellarator, for example, the quantity Λ can reach a value $\sim a$, and the diffusion coefficient (10.20) can approach the Bohm value $D_B \sim 10^{-1}\rho_i v_i$. The estimate in (10.20) obviously no longer applies if it gives a value larger than D_B; this means that the localization width of the perturbation region becomes of order a.

3. Drift Instability

The drift instability is intimately related to the drift-dissipative instability; in essence these are two aspects of the same instability, the one appearing in the collision-free regime and the other in the collision-dominated regime. Hence, limiting ourselves to the simplified case $T_i = 0$, which does not differ qualitatively from the general case, we again use Eq. (10.5) and the equation of continuity for the ions. Now, however, in place of Ohm's law we make use of a relation which can be obtained from the collisionless kinetic equation for the electrons in the drift approximation:

$$\frac{\partial f_e}{\partial t} + v_\parallel \mathbf{h}\nabla f + \frac{c\,[\mathbf{EH}]}{H^2}\nabla f - \frac{e}{m_e} E_\parallel \frac{\partial f}{\partial v_\parallel} = 0. \qquad (10.21)$$

Considering the perturbation corresponding to the cell of the largest scale size, we again take account of the small-scale perturbations by replacing ω by $\omega - iD\Delta_\perp$. In this case, for simplicity, we shall assume in Eq. (10.21) that the average function f_0 is essentially a Maxwellian shifted by an amount corresponding to the velocity of the current u. Actually, the drift oscillations will distort the longitudinal velocity distribution function to some extent, but if the plasma is not too rarefied, i.e., if $S = \lambda_e\rho_i/a \sim 1$, this distortion is removed to some degree by collisions. Neglecting the quasi-linear distortion of f_0 increases the effective coefficient D to some extent.

We now use the linearized equation (10.21) to determine the longitudinal component of the electron current $j_{\parallel e}$, and also take account of the ion longitudinal current; then the system in (10.10) and (10.11) is replaced by the following two equations:

$$(\omega - iD\Delta_\perp)\, n' = \omega^* n_0 \psi; \qquad (10.22)$$

$$(\omega - iD\Delta_\perp)\,\rho_{ie}^2\Delta_\perp\psi =$$

$$\left(\omega - iD\Delta_\perp - \omega^* + i\sqrt{\pi}\,\omega\,\frac{\omega - \omega^* + k_\parallel u}{v_e\,|k_\parallel|} - \frac{k_\parallel^2 T_e}{m_i\omega}\right)\psi. \qquad (10.23)$$

It is evident that in this formulation Eq. (10.23) for ψ is independent of the density equation (10.22) and, consequently, the quantity D can be found from the condition that the characteristic value ω of Eq. (10.23) be real. When $\theta \ll \sqrt{m_e/m_i}$, $\theta \ll \rho_i/a$, we can assume the magnetic field to be uniform and we can assume that $k_\parallel$ is an arbitrary quantity that does not depend on x (but $k_\parallel^{-1} \leqslant \min\left(\frac{v_e}{\omega}, \frac{c_A}{\omega}\right)$. In the case u = 0 and $\beta > m_e/m_i$, we then obtain from Eq. (10.23) the same result that is obtained in the weak-turbulence approximation [16]:

$$D \sim \frac{m_e}{m_i\beta} \cdot \frac{\rho_i^2 v_i}{a}. \qquad (10.24)$$

In the presence of a longitudinal current $u > v_i$ it is possible to excite drift waves with a transverse wavelength of order a and the diffusion coefficient can reach values of the order $D \sim (u/v_e)D_B \sim (u/v_e)\rho_i v_i$ [43].

If the magnitude of the current velocity u is large enough, drift waves can be excited even when θ is not very small. The ion inertia does not play a role in these oscillations, so that the left side of Eq. (10.23) can be neglected. If we assume, furthermore, that the oscillation frequency is close to ω^*, and if we write $\omega = \omega^* + \lambda$, where $\lambda \ll \omega^*$, the following equation is obtained:

$$D\Delta_\perp\psi + \left(i\lambda - \sqrt{\pi}\,\omega^* \frac{k_\parallel u}{|k_\parallel|\,v_e} - i\,\frac{k_\parallel^2 T_e}{m_i\omega^*}\right)\psi = 0. \qquad (10.25)$$

Thus, in order-of-magnitude terms we find $D/x^2 \sim \lambda \sim \omega^* \frac{u}{v_e} \sim \frac{k_y^2 T_e\theta^2}{m_i\omega^* a^2}x^2$, which yields the following estimate for the diffusion coefficient:

$$D \sim \left(\frac{u}{v_e}\right)^{3/2} \frac{\rho_i^2 v_i}{\theta a}. \qquad (10.26)$$

When $u/v_e \sim 10^{-1}$, this diffusion coefficient can reach a value of the order of the thermal-diffusion coefficient due to the temperature drift instability (8.20). It should be kept in mind, however, that if $u/v_e < \theta$, this instability is stabilized.

§11. SUMMARY OF FORMULAS

The above analysis of macroscopic effects such as enhanced diffusion and enhanced thermal conductivity allows us to set up a system of equations that describes the behavior of turbulent plasma in toroidal systems. We have treated the individual instabilities separately. Under actual conditions, however, several different interacting oscillations can develop; for example, oscillations of one kind can feed oscillations of another kind. But since the macroscopic effects are determined by perturbations with the largest localization, it is not reasonable to anticipate stabilization of the most dangerous perturbations by the less dangerous oscillations. In other words, for any combination of instabilities the most dangerous must be considered before the others. Hence, in the equations that describe the macroscopic large-scale motion of the plasma, we can simply sum all of the coefficients of thermal conductivity and diffusion that have been found above, thus assuming that the equations themselves will pick out the largest of these.

The coefficient for anomalous thermal conductivity appears in the heat-transport equation in precisely the same way as the usual coefficient and the generalization of this equation does not represent any great difficulty. The diffusion coefficient can be introduced into the equations of motion for the ions and electrons in the form of a supplementary transverse frictional force, i.e., as an effective reduction in the transverse conductivity. However, in incompressible convection flow the diffusion flux is proportional to the density gradient and not to the pressure gradient, as is the case in ordinary diffusion. It would appear therefore that a corresponding change should be made in the transverse thermal force. But there is no particular advantage in introducing the effective diffusion in this complicated way. The point is that the equations with the averaged effect of the instability only apply for slow flows, in which case it is possible to establish a stationary spectrum of turbulent fluctuations. Since the time required to establish this condition is determined by the growth rate γ (for the drift instabilities this time does not exceed the transit time v_i/a), the equations for the averaged flow apply only for inertialess motion. Under these conditions the equation of continuity and the thermal-conductivity equation suffice.

Since the expressions found above for D and χ contain certain numerical factors and are rather complicated functions of temperature, density, etc., in practical applications it is convenient to convert to a useful system of units. Below we express the temperature in electron volts, and take the unit of density to be 10^{13} part/cm^3. The time will be measured in milliseconds and the length in centimeters. Furthermore, in place of the radius r we introduce the dimensionless parameter $x = r/a$, where a is the radius of the plasma. The

longitudinal magnetic field H_0 will be measured in kilogauss and in place of the azimuthal field H_ϑ we introduce the quantity $\mu = Rq_0H_\vartheta/rH_0 = q_0/q$, where $q_0 =$ const is a maximum stability margin to which it is convenient to refer q. In the units used here, in accordance with [3] we also assume that the longitudinal conductivity is given by $\sigma_\parallel = 1.18 \cdot 10^{13} T_e^{3/2}$, the longitudinal thermal conductivity by $\chi_\parallel = \frac{2}{3}\frac{\varkappa_\parallel^e}{n} = 8.5 \cdot 10^6 \frac{T_e^{5/2}}{n}$, and the transverse ion thermal conductivity by $\chi_\perp = \frac{2}{3}\frac{\varkappa_\perp^i}{n} = 9.4 \cdot 10^5 \frac{n}{H_0^2} T_e^{-1/2}$.

Taking account of the possibility of a time variation in the longitudinal field H_0 for the purpose of compressing the plasma, we now write the system of equations that describes the behavior of the plasma with turbulence effects included:

$$\frac{\partial \mu}{\partial t} = \frac{1}{2H_0} \cdot \frac{dH_0}{dt} x \frac{\partial \mu}{\partial x} + \frac{A_0}{x} \cdot \frac{\partial}{\partial x}\left(\frac{\alpha}{T_e^{3/2}}\right), \quad \alpha = \frac{1}{x}\frac{\partial}{\partial x}(x^2\mu); \tag{11.1}$$

$$n\frac{dT_e}{\partial t} - D\frac{\partial n}{\partial x} \cdot \frac{\partial T_e}{\partial x} - \frac{n}{2H_0} \cdot \frac{dH_0}{dt} x \frac{\partial T_e}{\partial x} - \frac{2}{3} \cdot \frac{1}{H_0} \times$$
$$\times \frac{dH_0}{dt} nT_e = \frac{1}{x} \cdot \frac{\partial}{\partial x}\left(xn\chi_e \frac{\partial T_e}{\partial x}\right) + B_0 \frac{1}{T_e^{3/2}}\left[\frac{1}{x} \cdot \frac{\partial}{\partial x}(x^2\mu)\right]^2 +$$
$$+ Q_e + C_0 \frac{n^2}{T_e^{3/2}}(T_i - T_e) - gnT_e; \tag{11.2}$$

$$n\frac{\partial T_i}{\partial t} - D\frac{\partial n}{\partial x} \cdot \frac{\partial T_i}{\partial x} - \frac{n}{2H_0} \cdot \frac{dH_0}{dt} x \frac{\partial T_i}{\partial x} - \frac{2}{3}\frac{1}{H_0} \times$$
$$\times \frac{dH_0}{dt} nT_i = \frac{1}{x} \cdot \frac{\partial}{\partial x}\left(xn\chi_e \frac{\partial T_i}{\partial x}\right) + \delta_0 B_0 \frac{1}{T_e^{3/2}}\left[\frac{1}{x} \cdot \frac{\partial}{\partial x}(x^2\mu)\right]^2 +$$
$$+ Q_i - C_0 \frac{n^2}{T_e^{3/2}}(T_i - T_e) - gnT_i; \tag{11.3}$$

$$\frac{\partial n}{\partial t} - \frac{1}{2H_0} \cdot \frac{dH_0}{dt} x \frac{\partial n}{\partial x} - \frac{1}{H_0} \cdot \frac{dH_0}{dt} n =$$
$$= \frac{1}{x} \cdot \frac{\partial}{\partial x}\left(xD\frac{\partial n}{\partial x}\right) + gn, \tag{11.4}$$

where $A_0 = 6 \cdot 10^3/a^2$, $B_0 = 2 \cdot 10^7 H_0^2/R^2q_0^2$, $C_0 = 2.3 \cdot 10^2$.

Equation (11.1) describes the diffusion of the magnetic field. The first term on the right corresponds to radial transport of the frozen-in lines of force, while the second describes the diffusion due to finite conductivity. We shall assume that the longitudinal current increases slowly, and that the current density $j \sim \alpha$ does not experience strong changes which would lead to a screw instability and the associated anomalous diffusion of the magnetic field by virtue of the development of helical cells. When Eqs. (11.1)-(11.4) are solved on an electronic computer the absence of a screw instability can be checked by means of Eq. (2.25); in the notation of the present section, this equation becomes

$$\frac{d^2\psi}{dx^2} - \frac{m^2}{x^2}\psi = \frac{\psi}{x(\mu - q_0 n/m)} \cdot \frac{d\alpha}{dx}. \qquad (11.5)$$

Equations (11.2) and (11.3) are the heat-balance equations for the electrons and ions, respectively. On the left side of these equations the second term describes the transport of heat with the diffusion velocity; the third term describes the transport of heat by virtue of magnetic compression, and the fourth describes the adiabatic heating (due to compression). On the right side, the first term corresponds to the turbulent thermal conductivity; the second term in Eq. (11.2) takes account of Joule heating, while the second term in Eq. (11.3) takes account of the generation of Joule heat by the ions in the presence of an ion acoustic instability; Q_e is the loss due to radiation; Q_i is the heat generated by external sources or by thermonuclear reactions; the fourth term describes heat exchange as a result of electron-ion collisions; finally, the last term arises by virtue of the flow of cold gas, i.e., g is the number of neutrals that appear in the plasma per unit time. The same effect appears in Eq. (11.4) in the last term, while the next-to-last term in Eq. (11.4) describes the anomalous diffusion. Thus, all of the turbulence effects are included in the coefficients χ_i, χ_e, and D.

Each of the coefficients (χ_i, χ_e, and D) consists of a sum of terms, each of which, in turn, corresponds to a given instability:

$$\chi_i = \chi_c + \chi_T + \chi_s + \chi_t, \quad \chi_e = \chi_s + \chi_t, \quad D = D_T + D_s + D_t + D_d + D_f.$$

Here, χ_c is the classical thermal conductivity, which is given the following by (when $q^2 \gg 1$):

$$\chi_c = \frac{10^2 q_0^2}{a^2} \cdot \frac{n}{H^2\mu^2\sqrt{T_i}}; \qquad (11.6)$$

χ_T is the coefficient for turbulent thermal conductivity due to the temperature drift instability. In accordance with Eq. (8.20), in the notation of the present section we can write:

$$\chi_T = \frac{Rq_0}{2a^4} \cdot \frac{F(\eta)}{H^2 x \left|\frac{d\mu}{dx}\right| \sqrt{T_i}} \left(\frac{dT_i}{dx}\right)^2, \qquad (11.7)$$

where

$$\eta = \left|\frac{d\ln T}{d\ln n}\right|; \quad F(\eta) = \begin{cases} 0 & \text{for } 0 < \eta < 1, \\ 1 - \eta & \text{for } 1 < \eta < 2, \\ 1 & \text{for } \eta > 2. \end{cases}$$

In general, the coefficient χ_T is a rather complicated function of T_e. However, since its basic dependence is determined by T_i (any change in T_e does not change the order of magnitude of χ_T) we take the value of χ_T corresponding to the condition $T_i = T_e$ and extrapolate to the case $T_i \neq T_e$.

The quantity χ_s corresponds to the thermal conductivity due to the current-convective instability. Under conditions for which this quantity is important the convection cells have a region of localization which is appreciably greater than ρ_i; in this case the contributions to the ion and electron thermal conductivities and the diffusion coefficients are approximately the same. Using Eq. (7.67), we find

$$\chi_s = D_s = 3.5 \cdot 10^3 \frac{R\sqrt{q_0}}{a^3} \cdot \frac{\sqrt{n}\,|\alpha|^{3/2}}{T_e^5 x^{3/2}\mu^2} \left(\frac{dT_e}{dx}\right)^2 \left(\frac{d\mu}{dx}\right)^{-2}. \qquad (11.8)$$

The coefficient $\chi_t = D_t$ is associated with the trapped-particle instability. In addition to the effects we have considered above, there are a number of others that influence the thermal conductivity; these include a small term which corresponds to the electron temperature instability. In the presence of magnetic cells there is also an additional heat transfer that results from the destruction of the magnetic surfaces.

In the diffusion coefficient D, in addition to the term in (11.8), there is a term D_T that corresponds to the temperature drift instability; as an approximation for this term we can write

$$D_T = 0.1\chi_T. \qquad (11.9)$$

There is also a term D_t that corresponds to the trapped-particle instability which, in accordance with Eq. (9.12), is given by

$$D_t = \chi_t = 4 \cdot 10^2 H^2 \frac{T_e^{7/2} T_i^2 x^{5/2}}{(T_e + T_i)^2 a^{3/2} R^{5/2}} \left(\frac{1}{n} \cdot \frac{dn}{dx}\right)^2. \qquad (11.10)$$

In addition there are the following terms: D_d which derives from the drift instability, and D_f, which derives from an instability due to the finite-orbit nature of the drift motion. These terms are important in systems with small shear. In accordance with §10:

$$D_d \sim \begin{cases} 0 \quad \text{for} \quad u/v_e < 0, \\ \sim 10\left(\frac{u}{v_e}\right)^{3/2} \frac{Rq_0}{a^4} \cdot \frac{T_e^{3/2}}{H^2 x \left|\frac{d\mu}{dx}\right|}, \quad \frac{u}{\vartheta_e} > 0 \end{cases} \qquad (11.11)$$

where

$$u/v_e = 12\alpha H_0 Rq_0 n \sqrt{T_e},$$

$$\theta = \frac{a}{Rq_0} x \left|\frac{d\mu}{dx}\right|,$$

$$D_f \sim \frac{10^2 T_e}{a^2 H}\left(\frac{\Lambda^2}{a^2\theta^2\Omega_e\tau_e}\right)^{1/3}, \qquad (11.12)$$

where Λ is the mean-square ion displacement in the drift motion.

The quantity Q_e takes account of losses due to radiation, which comprise the bremsstrahlung Q_B, the magnetic synchrotron radiation Q_m, and impurity radiation Q_T. In the units adopted here,

$$Q_B = -7.2 \cdot 10^{-4} n^2 \sqrt{T_e}. \qquad (11.13)$$

As far as Q_m is concerned, we find from [44] that this quantity can be approximated by the expression

$$Q_m = 6 \cdot 10^{-14} n T_e^{5/3} H_0^2 \sqrt{\frac{H_0}{na}}. \qquad (11.14)$$

The quantity Q_r can be approximated by an expression given in [45]:

$$Q_r = \xi Z^4 1.4 \cdot 10^{-2} n^2 T_e^{-1/2} + \xi Z^6 1.9 \cdot 10^{-1} n^2 T_e^{-3/2}, \qquad (11.15)$$

where ξ is the impurity concentration and Z is the effective charge number of the impurity.

The quantity Q_i, which is the energy generated in nuclear reactions in an equal mixture of deuterium and tritium, is

$$Q_i = 1.8 \cdot 10^7 \frac{n^2}{T_i^{3/2}} \exp(-200/T_i^{1/3}). \qquad (11.16)$$

The quantity δ_0 takes account of the possibility of energy transfer to the ions by virtue of ion-acoustic instabilities. As we have argued in §5, the development of the instability can lead to an energy transfer to the ions of the order of the Joule heat developed in the electron component; however, the effect is nonvanishing only when $T_i < T_e/3$. Since a departure from equal temperatures ($T_i \ll T_e$) occurs only when $u \gg c_s = \sqrt{T_e/m_i}$, i.e., when the condition for the excitation of the ion-acoustic instability is satisfied, δ_0 can be approximated by

$$\delta_0 = \frac{T_e^2}{T_e^2 + 10T_i^2}. \qquad (11.17)$$

for example, $\delta \sim 10^{-1} \ll 1$ when $T_e = T_i$ and $\delta_0 = 1$ when $T_i \ll T_e$. The quantity g is the flux of neutral gas into the plasma as determined by the wall conditions.

We note further that under conditions of plasma equilibrium the quantity β,

$$\beta = \frac{8\pi p}{H_0^2} = \frac{10^{-4}}{H_0^2}\int_0^1 n(T_e + T_i)\, 2x\,dx \qquad (11.18)$$

must be small, of order a/Rq^2.

Equations (11.1)-(11.17) represent the equations of motion for a turbulent plasma in a toroidal system. Strictly speaking, these equations apply to a circular torus such as the Tokomak. To extend them to stellarator systems, in Eqs. (11.7), (11.8), (11.11), and (11.12), the quantity $d\mu/dx$, which is proportional to the shear θ, must be replaced by $(q_0/2\pi)(d_\iota/dx)$, where ι is the total rotational transform.

Estimates show that the plasma flow described by the turbulence coefficients χ and D is of the order of the values that are observed experimentally.

CONCLUSION

In the present review we have tried to take account of all of the basic plasma instabilities that pertain to a toroidal geometry, and have investigated the associated macroscopic effects, a summary of which is given in §11. The possibility is not excluded that some of the results given here will undergo significant changes in the future. In any case, a great deal of work still remains to be done in order to establish the relations between the theoretical predictions given here and the experimental data. Rough qualitative agreement has been obtained, but a more complete comparison of theory and experiment will evidently require numerical calculations, since the macroscopic equations are very sensitive to the radial distributions of current, density, and temperature in the plasma. Even without the results of these calculations, and without a reliable comparison of theory and experiment, it is still of interest, at this point, to consider the feasibility of obtaining a self-sustaining thermonuclear reaction within the framework of the models proposed here for a turbulent plasma.

Since an increase in the dimensions of the apparatus leads to a reduction in loss, the question before us can be formulated as follows. For what apparatus dimensions will a thermonuclear reaction become self-sustaining even in the presence of anomalous diffusion?

In an equal mixture of deuterium and tritium a self-sustaining reaction requires that the following condition be satisfied:

$$H^2\tau > 6 \cdot 10^7/\beta, \qquad (1)$$

where τ is the confinement time; $\beta = 8\pi p/H^2$; and H is the magnetic field (a longitudinal field in the present case).

The confinement time τ can be conveniently referred to the Bohm time by writing

$$\tau = \pi a^2 \frac{eH}{cT} \cdot \frac{1}{\alpha}, \qquad (2)$$

where the factor $\alpha < 1$ takes account of the possibility of a reduction in the effective diffusion coefficient (or in the thermal conductivity) as compared with the Bohm value [36].

The relations in (1) and (2) can then be combined to yield the following condition for a self-sustaining reaction:

$$a^2 H^3 > \frac{\alpha}{\beta} 2 \cdot 10^7 \frac{cT}{eH}. \qquad (3)$$

It is then evident that a self-sustaining reactor requires the use of the maximum possible magnetic field.

At the present time, in principle it is possible to produce a magnetic field $H = 10^5$ g by means of superconducting windings. If we take $H = 10^5$ g and $T = 10^4$ eV in (3), the requirement becomes

$$a > 140\sqrt{\frac{\alpha}{\beta}}. \qquad (4)$$

Thus, with $\beta \approx 10^{-2}$ and the Bohm diffusion coefficient ($\alpha = 1$) the minor radius of the torus must be 14 meters, a dimension that is too large to be acceptable. However, if $\alpha = 10^{-2}$, the dimension $a = 140$ cm becomes completely reasonable, and the question of achieving controlled fusion thus reduces to the possibility of reducing the turbulent diffusion coefficient to a value which is two orders of magnitude smaller than the Bohm value. The results of the present review indicate that this reduction is feasible.

The point here, as we have established above, is that in a rarefied plasma one of the most dangerous instabilities is the trapped-particle instability. However, with $\lambda_e/a < \frac{1}{\theta}\sqrt{\frac{m_i}{m_e}} \sim 10^3$ this instability leads to a loss that does not exceed the losses associated with the drift instabilities. With $T = 10^4$ eV, $n = 10^5$ part/cm^{-3}, and H $(1\text{-}2)\cdot 10^5$ g, which corresponds to $\beta \approx 10^{-2} \approx \theta^2$, the mean-free path is $\lambda_e \approx 10^5$ cm, i.e., it is of order 10^3 a, so that the trapped-particle instability is not an overwhelming obstacle. On the other hand, the drift instabilities lead to the development of highly localized cells with localization widths that do not exceed ρ_i/θ. Hence, the corresponding coefficients of thermal conductivity and diffusion do not exceed values of the order of $\rho_i/\theta a$ of the Bohm value (here we are taking account of the presence of a small factor of order γ^2/ω^2 in the expressions for D and χ). With $T = 10^4$ eV and $H = 10^5$ g, the mean ion Larmor radius is $\rho_i \sim 10^{-1}$ cm, and thus, with $a \sim 10^2$ cm and $\theta \sim 10^{-1}$, the factor α is $\rho^1/\theta a \approx 10^{-2}$. Consequently, with these parameters, the possibility of achieving the value $\alpha \sim 10^{-2}$ is completely realizable. Furthermore, the results of the present review indicate that at lower temperatures, where the trapped-particle instabilities are stabilized, the estimates given above refer only to the thermal conductivity; the diffusion coefficient can be appreciably smaller.

Now let us consider the heating problem. In a high-temperature plasma in which the shear θ is reasonably large the longitudinal current does not lead to an additional instability; furthermore, it provides a convenient means for producing shear. Thus, it would appear that the most convenient method of heating a plasma is Joule heating. The energy generated in 1 cm^3 per unit time is

$$j^2/\sigma = \frac{c^2 H_\vartheta^2}{4\pi^2 a^2 \sigma}, \qquad (5)$$

where H_ϑ is the value of the azimuthal magnetic field at the edge of the plasma (the current is assumed to be distributed uniformly). Equating this expression to the loss $(1/\tau)3nT$, and assuming that $2nT = \beta H^2/8\pi$, we find

$$H \gtrsim \alpha\beta \frac{H^2}{H_0^2} \cdot \frac{T\sigma}{cc} . \tag{6}$$

Substituting T = 5 keV ($\sigma = 4 \cdot 10^{18}$), which corresponds to the ignition temperature, (for which energy generated by nuclear reactions balances the bremsstrahlung), we find

$$H \gtrsim 10^9 \alpha\beta \frac{H^2}{H_\vartheta^2} g . \tag{7}$$

It appears that in a highly curved torus the quantity H^2/H_ϑ^2 can approach values of the order of 10. Then, with $\beta \sim 10^{-2}$ and $H \sim 10^5$ g it would be possible to satisfy (7) if it were possible to reduce α to 10^{-3} for a period of the order of the heating time. Theoretical considerations indicate that α can be reduced on a transient basis. For example, it is possible to exploit the fact that the diffusion coefficient is small compared with the thermal conductivity. Thus, during the heating period the plasma can be isolated from the walls either by changing the magnetic field or by displacing the limiters. Furthermore, it would be possible to add impurities at the periphery of the plasma in order to reduce the conductivity, and to increase the Joule heating in this region. Thus, the possibility is not excluded that Joule heating could be used exclusively; however, this would require a highly curved torus in order to obtain the largest possible value of H_ϑ/H with $q > 1$.

The results of the present review would seem to be weighted more toward optimistic rather than pessimistic conclusions. It appears possible to obtain a controlled and thermonuclear reaction even in the presence of turbulent diffusion and turbulent thermal conductivity, although the technological penalties would be rather severe.

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