PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #1 SOLUTIONS

Reading: Please read the chapter 3 material from PHYS 210A: Approach to Equilibrium. You may also find it useful to review some of the worked examples. Then review 200A material on Hamiltonian Mechanics.

(1) Study the solutions to problems 3.3 and 3.4 from the worked examples for PHYS 210A, chapter 3. This will provide an introduction to the mathematics of Markov chains, which we will discuss later in the quarter.

Next, consider the case of the U.S. Supreme Court, which consists of nine justices. Suppose each justice may be characterized as either liberal (L) or conservative (C). There are thus ten possible "configurations" for the court: $\mathrm{L}^n\,\mathrm{C}^{9-n}$ with $n \in \{0, \ldots, 9\}$. While the degeneracy of each configuration is $g_n = \binom{9}{n}$, it should become clear that you won't have to account for degeneracies to solve this problem.

Assume for each vacancy that an L justice is appointed with probability p and a C justice with probability $q \equiv 1 - p$. Thus, for each n, the transition matrix $Q_{m,n} = P(m, t+1 \,|\, n, t)$ has nonzero elements only for $|m - n| \le 1$.

(a) Assuming that retirements occur randomly, find expressions for $Q_{n+1,n}$, $Q_{n,n}$, and $Q_{n-1,n}$ in terms of n and p (you may use $q \equiv 1 - p$ as convenient). Your transition matrix should satisfy $\sum_m Q_{m,n} = 1$. Assuming you've derived the correct expressions for $Q_{n+1,n}$ and $Q_{n-1,n}$, you can guarantee this by taking

$$Q_{n,n} = 1 - Q_{n+1,n} - Q_{n-1,n} \quad .$$

The transition matrix $Q(p)$ is thus a band diagonal 10×10 matrix whose elements are (linear) functions of p.

(b) Write a computer program to find the eigenvalues and right eigenvectors of $Q(p)$ (you will surely want to use available linear algebra commands or subroutines in whatever software you are using). The equilibrium distribution $P_n^{(\mathrm{eq})}(p)$ is the right eigenvector of $Q(p)$ with eigenvalue $\lambda_1(p) = 1$, normalized so that $\sum_n P_n^{(\mathrm{eq})}(p) = 1$. Make histograms of this distribution for the cases $p = 0.1$, $p = 0.3$, $p = 0.5$, $p = 0.7$, and $p = 0.9$.

(c) Suppose there are two vacancies per presidency, and that the presidential party alternates with each consecutive term. Then after four years of a Democrat and four years of a Republican in the White House, the transition matrix will be $\mathcal{Q}(p) \equiv Q^2(p)\, Q^2(1-p)$. What do you expect the equilibrium distribution will resemble? Plot the equilibrium distribution histograms for $\mathcal{Q}(p)$ for the p values in (b).

(d) Starting with a random configuration of the court, the distribution after successive two term D/R presidential cycles will converge to the equilibria $\mathcal{P}_n^{(\mathrm{eq})}(p)$ you found in part (c). The convergence is exponential, and the time scale (measured in eight year (*i.e.* two term) cycles is given by $\tau(p) = -1/\ln|\lambda_2(p)|$, where $\lambda_2(p)$ is the next leading eigenvalue. It is necessarily the case that $|\lambda_2(p)| < 1$. Plot $\tau(p)$ as a function of p.

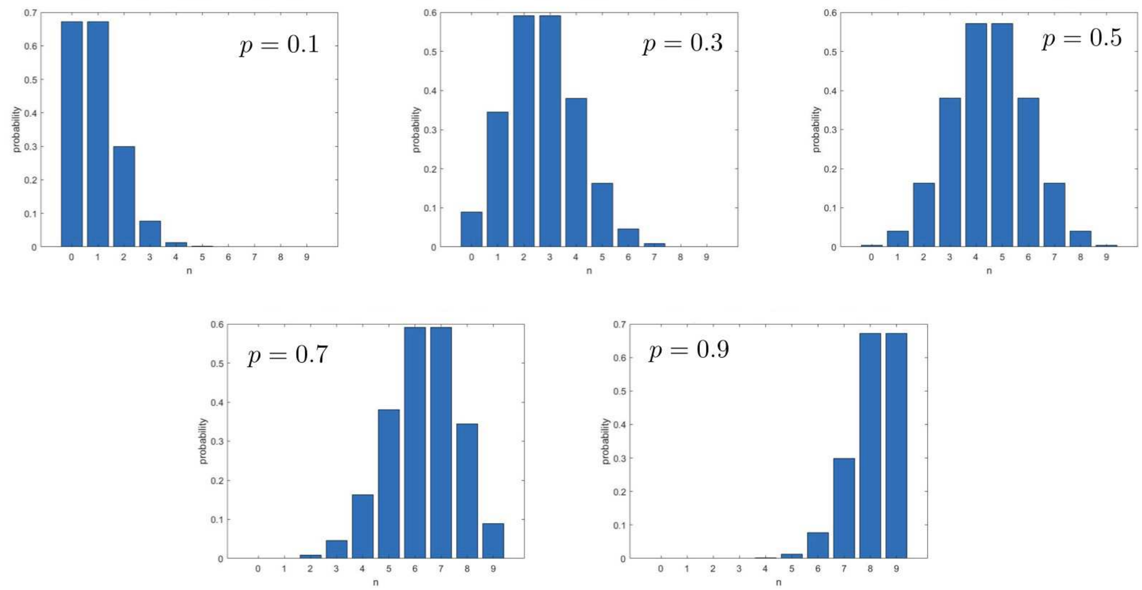

Figure 1: Equilibrium distributions $P_n^{(\mathrm{eq})}(p)$ for $Q(p)$. Credit: Haodong Qin.

Solution :

(a) The probability that there is a liberal vacancy on a $\mathrm{L}^n\,\mathrm{C}^{9-n}$ court is $\frac{n}{9}$, while the probability of a conservative vacancy is $1-\frac{n}{9}$. Thus,

$$Q_{n+1,n} = \left(1-\tfrac{n}{9}\right)p \quad , \quad Q_{n-1,n} = \tfrac{n}{9}\,(1-p) \quad ,$$

where $n \in \{0, \ldots, 9\}$, and therefore

$$\begin{aligned} Q_{n,n} &= 1 - Q_{n+1,n} - Q_{n-1,n} \\ &= \left(1-\tfrac{n}{9}\right)(1-p) + \tfrac{n}{9} \quad . \end{aligned}$$

We also have $Q_{m,n} = 0$ if $|m-n| > 1$.

(b) Histograms of $P^{(\mathrm{eq})}(p)$ for $Q(p)$ are shown in Fig. 1.

(c) Histograms of $P^{(\mathrm{eq})}(p)$ for $\mathcal{Q}(p) = Q^2(p)\,Q^2(1-p)$ are shown in Fig. 2.

(d) One finds $\lambda_2(p) = 0.6243$ and therefore $\tau(p) = -1/\log\lambda_2(p) = 2.123$, independent of p.

(2) Find a (non-pornographic) pixelated image and find a way to evolve it using the generalized Arnold cat map,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ k & k+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \ \mathsf{mod}\ \mathbb{Z}^2 \quad ,$$

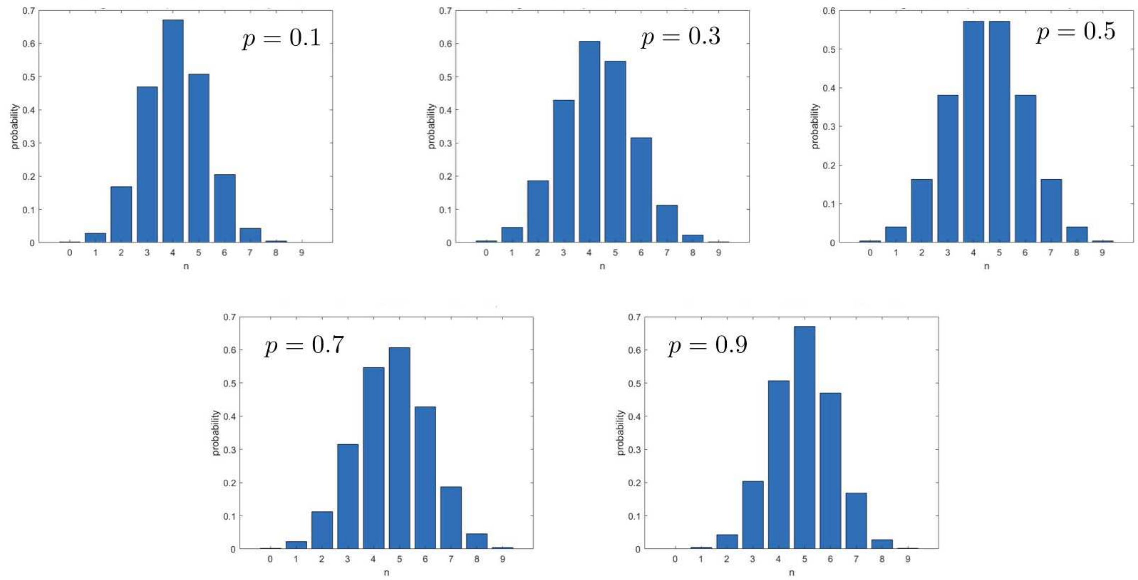

Figure 2: Equilibrium distributions $P_n^{(\mathrm{eq})}(p)$ for $\mathcal{Q}(p) = Q^2(p)Q^2(1-p)$. Credit: Haodong Qin.

where you get to choose $k \in \mathbb{Z}$. `Matlab` is a good tool for this project. If you don't know how to manipulate images in `Matlab`, find a fellow student with whom you can collaborate. Confirm that your pixelated image is recurrent under the action of the map. See problem 3.6 from the worked examples of PHYS 210A, chapter 3.

Solution : Any map

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{M} \begin{pmatrix} x \\ p \end{pmatrix} ,$$

will do, provided $a, b, c, d \in \mathbb{Z}$ and det $M = ad - bc = 1$, which is to say $M \in \mathsf{SL}(2, \mathbb{Z})$, called the *modular group*. Arnold's cat map $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and its generalizations $M = \begin{pmatrix} 1 & 1 \\ p & p+1 \end{pmatrix}$ are elements of $M \in \mathsf{SL}(2, \mathbb{Z})$. In general, they give rise to a mixing flow, and with a relatively small number of iterations, almost the entire torus is covered. A pixelated image exhibits Poincaré recurrence, as we see in Figs. 4 and 6 below.

(3) Is the following four-dimensional map canonical?

$$\begin{aligned} x_{n+1} &= 2\alpha x_n - \gamma x_n^2 - p_n + X_n^2 \\ p_{n+1} &= x_n \\ X_{n+1} &= 2\beta X_n - P_n + 2x_n X_n \\ P_{n+1} &= X_n \quad . \end{aligned}$$

Solution : The strategy here is to check whether this map preserves the symplectic structure of the Hamiltonian equation, namely whether the Jacobian of the transformation M

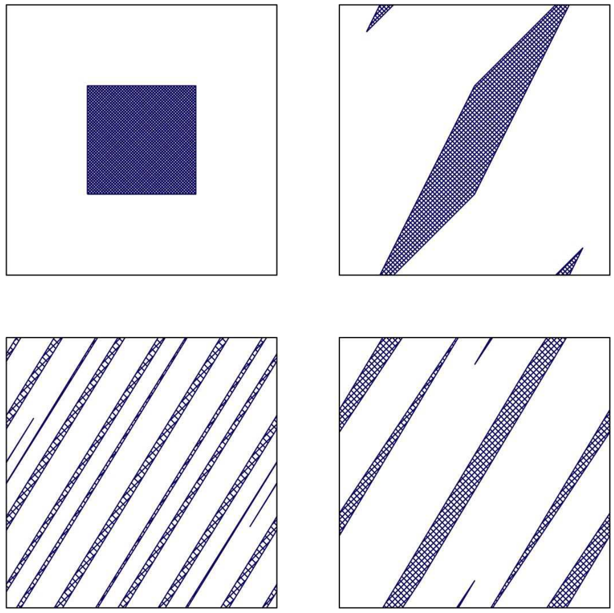

Figure 3: Zeroth, first, second, and third iterates of the generalized cat map with $p = 1$ (*i.e.* Arnold's cat map), acting on an initial square distribution (clockwise from upper left).

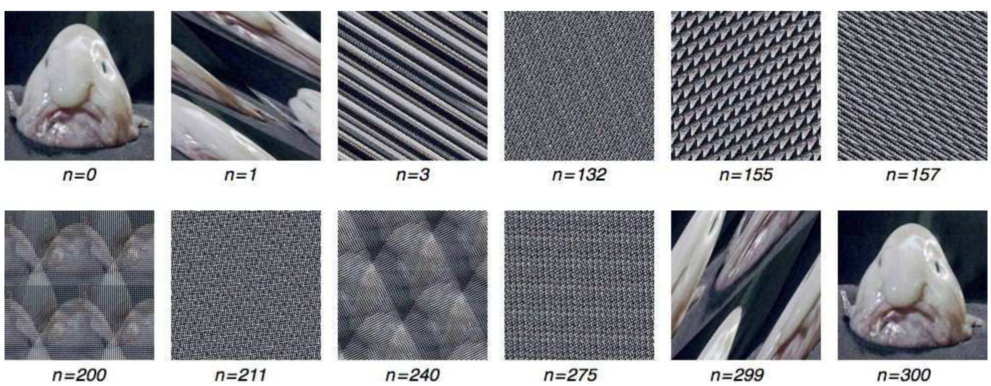

Figure 4: Evolution of a pixelated blobfish under the Arnold cat map.

satisfies $M\mathbb{J}M^T = \mathbb{J}$. Define the original vector $\xi = (x_n, X_n, p_n, P_n)$ and the transformed vector $\Xi = (x_{n+1}, X_{n+1}, p_{n+1}, P_{n+1})$. Explicitly, the Jacobian is:

$$M = \frac{\partial \Xi}{\partial \xi} = \begin{pmatrix} 2\alpha - 2\gamma X & 2X & -1 & 0 \\ 2X & 2x + 2\beta & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then it is straightforward to show that, indeed,

$$M\mathbb{J}M^T = \mathbb{J}$$

Therefore M is symplectic and the transformation is canonical.

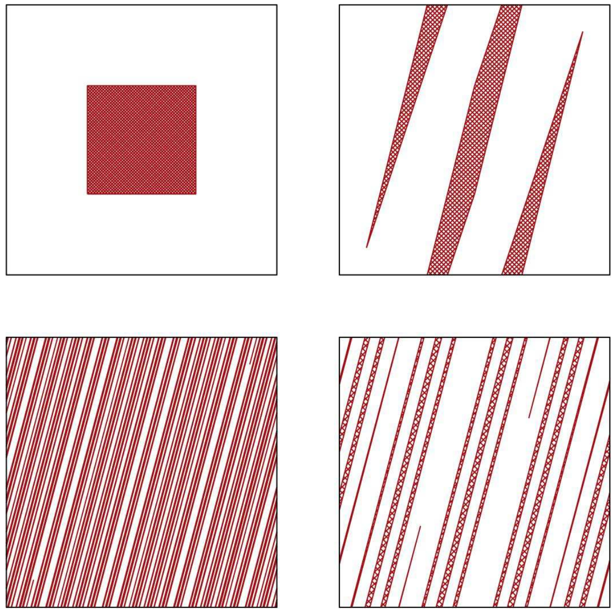

Figure 5: Zeroth, first, second, and third iterates of the generalized cat map with $p = 2$, acting on an initial square distribution (clockwise from upper left).

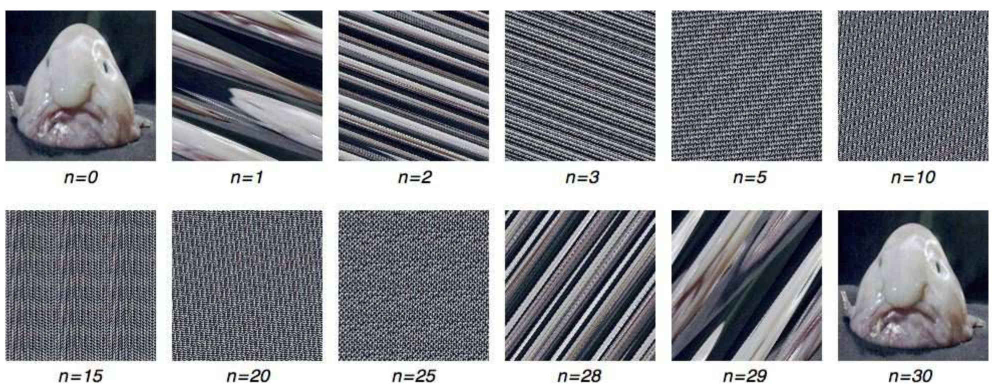

Figure 6: Evolution of a pixelated blobfish under the $p = 2$ generalized cat map.

(4) Consider the Hamiltonian $H(\boldsymbol{J}, \boldsymbol{\phi}) = H_0(\boldsymbol{J}) + \epsilon\, H_1(\boldsymbol{\phi})$, where

$$H_0(\boldsymbol{J}) = \Lambda J_1^{3/2} + \Omega J_2$$

$$H_1(\boldsymbol{\phi}) = \cos\phi_1 \sum_{-\infty}^{\infty} V_n\, e^{in\phi_2} \quad .$$

(a) Obtain an expression for $J_1(t)$ valid to first order in ϵ.

(b) Which tori are destroyed by the perturbation?

Hint : Follow the steps outlined in §15.8.4 of the 200A lecture notes, and eqs. 15.229 and

15.230 in particular. When the denominator in 15.230 vanishes, there is a resonance and the torus is destroyed. You should obtain a condition on the value of the action J_1 which involves the constants Λ, Ω, and a set of integers which index the resonances.

Solution : From the unperturbed part, we obtain the zeroth order of the two frequencies:

$$\nu_{1,0} = \frac{\partial H_0}{\partial J_1} = \tfrac{3}{2}\Lambda J_1^{1/2}$$
$$\nu_{2,0} = \frac{\partial H_0}{\partial J_2} = \Omega$$

We proceed formally, and reach the differential equation that determines S:

$$\nu_{1,0}\frac{\partial S_1}{\partial \phi_{1,0}} + \nu_{2,0}\frac{\partial S_1}{\partial \phi_{2,0}} = \langle H_1 \rangle - H_1 = -\cos\phi_1 \sum_{n=-\infty}^{\infty} V_n\, e^{in\phi_2}$$

The solution is given by:

$$S_1 = \frac{i}{2}\sum_{n=-\infty}^{\infty}\left(\frac{V_n\, e^{in\phi_{2,0}+i\phi_{1,0}}}{n\nu_{2,0}+\nu_{1,0}} + \frac{V_n\, e^{in\phi_{2,0}-i\phi_{1,0}}}{n\nu_{2,0}-\nu_{1,0}}\right)$$

Therefore,

$$J_{1,0} = J_1 + \epsilon\,\frac{\partial S}{\partial \phi_{1,0}} = J_1 + \epsilon\sum_{n=-\infty}^{\infty}\left(\frac{V_n\, e^{in\phi_{2,0}+i\phi_{1,0}}}{n\nu_{2,0}+\nu_{1,0}} - \frac{V_n\, e^{in\phi_{2,0}-i\phi_{1,0}}}{n\nu_{2,0}-\nu_{1,0}}\right)$$

where

$$\phi_{1,0}(t) = \phi_{1,0}(0) + \nu_{1,0}t$$
$$\phi_{2,0}(t) = \phi_{2,0}(0) + \nu_{2,0}t$$

When the ratio between $\nu_{1,0}$ and $\nu_{2,0}$ is a integer, one of the terms in the series diverges, implying the breaking down of the perturbation theory. As a consequence, the tori specified by the following condition:

$$\frac{\nu_{1,0}}{\nu_{2,0}} = \frac{3\Lambda J_1^{1/2}}{2\Omega} = n \in \mathbb{Z}$$

are destroyed by arbitrarily small perturbation.