PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #3

(1) Show that for time scales sufficiently greater than γ^{-1} that the solution $x(t)$ to the Langevin equation $\ddot{x}+\gamma\dot{x}=\eta(t)$ describes a Markov process. You will have to construct the matrix M defined in eqns. 2.213 and 2.214 of the lecture notes. You should assume that the random force $\eta(t)$ is distributed as a Gaussian, with $\langle\eta(s)\rangle=0$ and $\langle\eta(s)\,\eta(s')\rangle=\Gamma\,\delta(s-s')$.

(2) Provide the missing steps in the solution of the Ornstein-Uhlenbeck process described in §2.4.3 of the lecture notes. Show that applying the method of characteristics (see appendix IV) to Eqn. 2.61 leads to the solution in Eqn. 2.62.

(3) Consider a discrete one-dimensional random walk where the probability to take a step of length 1 in either direction is $\frac{1}{2}p$ and the probability to take a step of length 2 in either direction is $\frac{1}{2}(1-p)$. Define the generating function

$$\hat{P}(k,t)=\sum_{n=-\infty}^{\infty} P_n(t)\,e^{-ikn}\ ,$$

where $P_n(t)$ is the probability to be at position n at time t. Solve for $\hat{P}(k,t)$ and provide an expression for $P_n(t)$. Evaluate $\sum_n n^2\,P_n(t)$.

(4) Numerically simulate the one-dimensional Wiener and Cauchy processes discussed in §2.6.1 of the lecture notes, and produce a figure similar to Fig. 2.3.

Hint: To generate normal (Gaussian) deviates with a distribution $p(y)=(4\pi D\varepsilon)^{-1/2}\exp(-y^2/4D\varepsilon)$, we must invert the relation

$$x(y)=\frac{1}{\sqrt{4\pi D\varepsilon}}\int\limits_{-\infty}^{y}\!ds\,e^{-s^2/4D\varepsilon}=\tfrac{1}{2}+\tfrac{1}{2}\,\mathsf{erf}\bigg(\frac{y}{\sqrt{4D\varepsilon}}\bigg)\quad .$$

This is somewhat unpleasant. A slicker approach is to use the *Box-Muller method*, which you can read about on Wikipedia.

(5) Due to quantum coherence effects in the backscattering from impurities, one-dimensional wires don't obey Ohm's law in the limit where the 'inelastic mean free path' is greater than the sample dimensions, which you may assume here. Rather, let $\mathcal{R}(L)=e^2R(L)/h$ be the dimensionless resistance of a quantum wire of length L, in units of $h/e^2=25.813\,\mathrm{k}\Omega$. The dimensionless resistance of a quantum wire of length $L+\delta L$ is then given by

$$\begin{aligned}\mathcal{R}(L+\delta L)&=\mathcal{R}(L)+\mathcal{R}(\delta L)+2\,\mathcal{R}(L)\,\mathcal{R}(\delta L)\\&\qquad\qquad+2\cos\alpha\,\sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]}\ ,\end{aligned}$$

where α is a *random phase* uniformly distributed over the interval $[0,2\pi)$. Here,

$$\mathcal{R}(\delta L)=\frac{\delta L}{2\ell}\ ,$$

is the dimensionless resistance of a small segment of wire, of length $\delta L \lesssim \ell$, where ℓ is the 'elastic mean free path'.

(a) Show that the distribution function $P(\mathcal{R}, L)$ for resistances of a quantum wire obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell} \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R}\,(1+\mathcal{R})\, \frac{\partial P}{\partial \mathcal{R}} \right\} .$$

(b) Show that this equation may be solved in the limits $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$, with

$$P(\mathcal{R}, z) = \frac{1}{z}\, e^{-\mathcal{R}/z}$$

for $\mathcal{R} \ll 1$, and

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\, \frac{1}{\mathcal{R}}\, e^{-(\ln \mathcal{R} - z)^2/4z}$$

for $\mathcal{R} \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle \mathcal{R} \rangle$ in the former case, and $\langle \ln \mathcal{R} \rangle$ in the latter case.