Chapter 1.4

MHD Description of Plasma

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1.4.1. Modes of description of a plasma

A plasma is a collection of charged particles. These charged particles generate electromagnetic fields through their elementary charges and currents. In order to evaluate these fields it would be necessary to know the position and velocity of every particle at all times. The motions of the charges themselves must be followed in the

Handbook of Plasma Physics, Eds. M.N. Rosenbluth and R.Z. Sagdeev
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fields they generate and those externally imposed. This program is beyond what is possible except in the simplest possible situations.

Fortunately there is a cruder description of the plasma that is often sufficiently accurate to give gross behavior to the extent desired.

Instead of specifying the plasma in terms of each of its particles a more macroscopic description of the plasma can be pursued in which the emphasis is on its fluid nature. Depending on circumstances that will be discussed below this fluid description may be a one-fluid, a two-fluid, or a many-fluid approach.

The one-fluid approach will be considered first. Every cm^3 of plasma must contain a definite number ρ g of plasma. The rate of change of this density is controlled by mass flow $\boldsymbol{U}$ out of the walls of this cm^3. The momentum ρU in any cm^3 is itself controlled by the forces acting on it. These are normally electrical, magnetic, and gravitational forces acting on its volume, and pressure forces acting on its walls. Because the plasma is a conducting fluid its current can be found from Ohm's law in some form, while the direct electrical forces are usually small. The current can be used to find the magnetic field by the Biot–Savart law and the changing magnetic field gives the induced part of the electric field, while the remainder, the electrostatic part, follows from the condition that the current driven by the electric field be divergence-free. The determination of the pressure forces is often the weakest part of this one-fluid description since the pressure is not usually a scalar, particularly if the plasma is collisionless. In addition the heat flow is often quite large. (Microscopically, particles together in a small cube remain together for only a short time.) However, many plasma phenomena of interest do not depend on the pressure in any essential way so that even an inappropriate treatment by an assumed equation of state for a scalar pressure can give a reasonable description of the phenomena in their grosser aspects. (The more basic properties of the plasma are governed by its electrical nature.)

For a more detailed description of plasmas in which interest is centered on plasma temperatures and energy densities, the two-fluid description is more appropriate. In this description the electron and ion fluids are treated separately. Although the mean velocities are nearly equal, the electron and ion temperatures are often quite different due to the weak energy exchange rates between ions and electrons. The two-fluid approach is also appropriate for a weakly ionized plasma. Here the ion cyclotron frequency may be less than the ion neutral frequency, while the electron cyclotron frequency is greater than the electron neutral collision frequency. The resulting electron and ion flows can be quite different under these circumstances.

Finally, when the plasma is nearly collisionless but the pressure terms play a central role, an even more detailed, but still approximate, description becomes appropriate, the guiding center description. In this description the magnetic field is strong enough that the plasma is still hydromagnetic in a direction perpendicular to the magnetic field, since the gyration frequency is large for both species. However, the particle flows along the lines need not be fluid-like, so it is necessary to keep track of the distribution of velocities parallel to the line by a one-dimensional kinetic equation. Even in this case the description may be simplified to a fluid description that preserves the independent plasma behavior along and across the lines. Two

equations of state for the two independent components of the pressure tensor are needed, and this is supplied by the Chew–Goldberger–Low or double adiabatic equations.

In summary, although any real plasma is extremely complicated, some of its main properties may often be captured by simple macroscopic sets of equations. These can only describe the slower more macroscopic properties of a plasma that occur on long enough time and space scales that microscopic processes such as collisions and gyrations can establish sufficient consistency in the plasma to enable it to be considered as a coherent fluid.

1.4.2. Collisional plasma

As described in the introduction, the fluid picture of a plasma is most appropriate when the plasma is at least somewhat collisional. Then the electrons and ions separately relax to a local thermodynamic equilibria on a time short compared with that in which substantial changes in plasma conditions occur, and in regions small compared with the size of the plasma. Thus, we may assign a density ρ, mean velocity $\boldsymbol{U}$, and scalar pressure p to each of the plasma components.

In the simplest description of the one-fluid plasma we may ignore the differences in the electron and ion properties and simply lump them together. We consider this description first.

The one-fluid description

On this level the plasma is in many ways like a highly conducting molten metal. The fluid equations describing its density, velocity and pressure are

$$\partial\rho/\partial t + \nabla\cdot(\rho \boldsymbol{U}) = 0, \tag{1}$$

$$\rho(\partial \boldsymbol{U}/\partial t) + \rho \boldsymbol{U}\cdot\nabla \boldsymbol{U} = \boldsymbol{j}\times\boldsymbol{B} - \nabla p + \rho \boldsymbol{g}, \tag{2}$$

$$(\mathrm{d}/\mathrm{d}t)(p/\rho^{\gamma}) = 0. \tag{3}$$

Equation (1) is the equation of continuity. Equation (2) is Euler's equation for fluid motion. The left-hand side represents the mass of a cm^3 of material times its acceleration *at any instant*. The acceleration is produced by the magnetic and gravitational forces acting on the same cm^3 and the surface force term represented by the pressure gradients. $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{j}$ the plasma current, and $\boldsymbol{g}$ a fixed gravitational field. The pressure is the sum of the separate partial pressures of the ions and electrons whose gradients are assumed to act together on the plasma rather than on each species separately.

In the third equation $\mathrm{d}/\mathrm{d}t \equiv (\partial/\partial t) + \boldsymbol{U}\cdot\nabla$ is the convective derivation and γ is the ratio of specific heats of the plasma. This last equation is the equation of state for each separate fluid element following the motion. It is only valid under conditions where the heat flow is small. Note that p/ρ^{γ} is related to the entropy per unit mass of a fluid element. If more general conditions prevail, e.g. ionization, radiation pressure, etc., are important, then (3) should be replaced by the condition of constant entropy following each fluid element. However, in most cases where the one-fluid theory is employed the simple power-law assumption is generally adequate. Note further that various limiting cases arise by taking $\gamma = 1$, isothermal, or $\gamma = \infty$ incompressible. It can be easily worded as "p/ρ^γ is a constant following the motion, but in general is different for different fluid elements".

It should be noted that the electrical force $\rho_E \boldsymbol{E}$, where ρ_E is the electrical charge and $\boldsymbol{E}$ the electric field, has been dropped in (2). This is because, as will soon appear, these forces are relativistically small compared with magnetic forces and must be neglected for consistency, since our theory is nonrelativistic.

We see that knowing $\boldsymbol{B}$ and $\boldsymbol{g}$, (1)–(3) form a complete set giving the forward time evolution of the fluid quantities ρ, $\boldsymbol{U}$ and p. The velocity $\boldsymbol{U}$ needed in (1) to advance ρ in time is determined by (2). The pressure needed in (2), to advance $\boldsymbol{U}$, is given by (3), etc.

The electromagnetic fields are controlled by Maxwell's equations:

$$\nabla \times \boldsymbol{B} = 4\pi \boldsymbol{j}, \tag{4}$$

$$\partial \boldsymbol{B}/\partial t = -c \nabla \times \boldsymbol{E}, \tag{5}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{6}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi \rho_E, \tag{7}$$

where c is the speed of light. The displacement current in (4) has been dropped since, as will appear, its effects are also relativistically small. Further, there is no need for (7) since the charge density ρ_E appears nowhere else in the equations.

The electromagnetic and fluid equations are coupled by Ohm's law, which in its simplest form can be written (Spitzer, 1962)

$$\boldsymbol{E} + (\boldsymbol{U} \times \boldsymbol{B})/c = \eta \boldsymbol{j}, \tag{8}$$

where η is the plasma resistivity. The combination $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}/c$ is the electric field seen by the plasma in its moving frame $\boldsymbol{U}$, and (8) states that in this frame $\boldsymbol{j}$ is parallel to and proportional to $\boldsymbol{E}'$

Equation (8) is not strictly accurate for a plasma. Because of the anisotropy of the field there will be Hall currents flowing perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$ that may actually be larger than that predicted by (8). However, the current in (8) is parallel to $\boldsymbol{E}'$ and represents dissipation of energy whereas the Hall currents do not. Thus the secular effects produced by this term are generally more significant than those due to the Hall terms. It is customary in the simplest form of the one fluid MHD equations to employ Ohm's law in the form (8).

Equations (4), (5), and (8) represent three vector equations for the three vectors $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{j}$. They may be combined into two equations by solving (8) for $\boldsymbol{E}$ and substituting from (4) to eliminate $\boldsymbol{j}$. We get

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) - \frac{c}{4\pi} \nabla \times (\eta \nabla \times \boldsymbol{B}). \tag{9}$$

If η is a constant, the last term becomes simply $(\eta c/4\pi)\nabla^2 \boldsymbol{B}$ so

$$\frac{\nabla \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \frac{\eta c}{4\pi} \nabla^2 \boldsymbol{B}. \tag{9a}$$

The first term on the right gives the change in magnetic field produced by convection of lines of force by the plasma. The second term gives the magnetic diffusion term, which tends to smooth out irregularities in the plasma perhaps induced by the first term. If there were no plasma motions, the diffuse term would smooth out any irregularities, in a characteristic time of order $4\pi L^2/\eta c$ where L is the irregularity size. (This is essentially the "L/R time" for a plasma considered as a lumped circuit.) This decay time is of order $10^{-7} T^{3/2} L^2$ s where T is the temperature of the plasma in eV. For high temperatures or large plasmas this time may be very long. The changes in $\boldsymbol{B}$ produced by the convective term often occur on a time so short compared with this diffusive term that the magnetic diffusion can be ignored altogether. That is, we may replace (9a) by the "infinite conductivity" equation

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}). \tag{10}$$

The subset of the above equations (1), (2), (3), (4), and (10) constitute the so-called ideal MHD equations. They are clearly an approximation to the true plasma equations, but they have so many nice properties that they are the preferred set for describing macroscopic plasma phenomena. Equation (10) gives the evolution of $\boldsymbol{B}$ as a result of plasma motions. Then making use of (4) $\boldsymbol{j}$ can be determined, and thus $\boldsymbol{j} \times \boldsymbol{B}$, to determine the evolution of the fluid quantities under the action of the electromagnetic forces.

The electric field $\boldsymbol{E}$ is no longer needed in this description but it may be obtained from the infinite-conductivity limit of Ohm's law:

$$\boldsymbol{E} + (\boldsymbol{U} \times \boldsymbol{B})/c = 0. \tag{11}$$

Then the electric force on the plasma $\rho_E \boldsymbol{E}$ can be estimated from (7) to be

$$\rho_E \boldsymbol{E} = \frac{\boldsymbol{E}\, \nabla \cdot \boldsymbol{E}}{4\pi} \approx \frac{U^2}{4\pi L c^2} B^2,$$

and it is seen, as mentioned earlier, that it is relativistically small compared with the magnetic force $\boldsymbol{j} \times \boldsymbol{B} \approx B^2/4\pi L$. In the same way we may show that inclusion of the displacement current $(1/c)(\partial \boldsymbol{E}/\partial t)$ has a relativistically small effect on the equations. Adding it to (4) will alter $\boldsymbol{j}$ by the small amount $\delta \boldsymbol{j}$ and this will produce an additional contribution to the electromagnetic force term in (2):

$$\delta \boldsymbol{j} \times \boldsymbol{B} = \frac{1}{4\pi c} \frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} = -\frac{\partial}{\partial t}\left(\frac{\boldsymbol{U} \times \boldsymbol{B}}{4\pi c^2}\right) \times \boldsymbol{B} \approx \frac{U B^2}{4\pi t c^2},$$

where t is a macroscopic time. Comparing this with the inertia term on the left we see that it is smaller by $B^2/4\pi\rho c^2$. In fact, the addition of this term can be thought of as adding the "mass" of the magnetic field to the mass of the plasma.

The ideal equations of MHD are best thought of as exactly describing an ideal infinitely conducting fluid with an adiabatic equation of state whose properties are sufficiently close to a plasma to be of interest, rather than an appropriate system of equations for a real plasma. For the moment imagine that there is such an ideal infinitely conducting fluid to study. It is immersed in some magnetic field. Then, by the condition of flux freezing, the evolution of the field may be expressed in terms of the distribution of magnetic lines of force bodily transmitted by the velocity $\boldsymbol{U}$. This means the field depends only on the net displacement of each element of the fluid and not on the history of the fluid displacements. The $\boldsymbol{j}\times\boldsymbol{B}$ force can readily be thought of as the magnetic tension and pressure contained in these lines of force. Similarly, ρ is given purely by the displacement of the fluid elements and, further, the pressure is also thus determined. This means that, at least in principle, the force on a fluid element is determined holonomically by its displacement and the displacement of its neighbors. It is this fact, plus the fact that the system is dynamical (given by a Lagrangian), that leads to the many very satisfying properties of this ideal system. In fact a considerable amount of macroscopic plasma physics is devoted to determining to what extent a real plasma can differ from its ideal counterpart. Some of these questions, magnetic reconnection for example, are among the most important of modern-day research problems (Petschek, 1964).

The two-fluid description

An alternative and more precise treatment of a fully ionized plasma is contained in the two-fluid description. The two fluids are the electrons and ions. If there is a single species of ions, we can assign a density, velocity and pressure to the electrons and to the ions. Then the three equations for a single fluid, (1)–(3), must be replaced by six equations, three for each fluid, describing the six independent quantities ρ_i, ρ_e, $\boldsymbol{U}_i$, $\boldsymbol{U}_e$, p_i, p_e. Now the one-fluid equations were written down on phenomonological grounds and were not extremely accurate except in the limit $\omega_{ce}\tau_e$ very small, where ω_{ce} is the electron cyclotron frequency and τ_e the electron collision frequency. On the other hand, considerable work has been devoted to deriving a set of equations accurate for any collision rate faster than the dynamic rates of change of ρ_i, ρ_e, etc. The generally accepted set of equations are those of Braginski (1965), that are now taken as standard. We give them here for reference.

The two continuity equations are

$$\partial n_i/\partial t + \nabla\cdot(n_i\boldsymbol{U}_i) = 0, \tag{12}$$

$$\partial n_e/\partial t + \nabla\cdot(n_e\boldsymbol{U}_e) = 0, \tag{13}$$

where n_i and n_e are the electron and ion particle densities. These equations are linked by the charge neutrality condition, $Zn_i = n_e$, where Z is the ion charge number.

The two vector equations of motion are

$$\rho_i\left(\frac{\partial\boldsymbol{U}_i}{\partial t} + \boldsymbol{U}_i\cdot\nabla\boldsymbol{U}_i\right) = -\nabla p_i - \nabla\cdot\boldsymbol{\pi}_i + Zen_i\left(\boldsymbol{E} + \frac{\boldsymbol{U}_i\times\boldsymbol{B}}{c}\right) - \boldsymbol{R}_{ei} + \rho_i\boldsymbol{g}, \tag{14}$$

$$\rho_e\left(\frac{\partial\boldsymbol{U}_e}{\partial t} + \boldsymbol{U}_e\cdot\nabla\boldsymbol{U}_e\right) = -\nabla p_e - \nabla\cdot\boldsymbol{\pi}_e - n_e e\left(\boldsymbol{E} + \frac{\boldsymbol{U}_e\times\boldsymbol{B}}{c}\right) + \boldsymbol{R}_{ei} + \rho_e\boldsymbol{g}. \tag{15}$$

In these equations p_i and p_e are the ion and electron scalar pressures, $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_e$ are the nonscalar parts of the stress tensors, $\boldsymbol{R}_{ei}$ is the rate of transfer of momentum from ions to electrons by collisions. They in turn are linked by the equation defining the current $\boldsymbol{j} = (Zn_i e/c)(\boldsymbol{U}_i - \boldsymbol{U}_e)$, where e is the electronic charge. We assume that Zn_i is much closer to n_e than $\boldsymbol{U}_i$ is to $\boldsymbol{U}_e$. Because, $\boldsymbol{j}$ cannot be too large without producing electromagnetic effects we can say that $\boldsymbol{U}_i$ and $\boldsymbol{U}_e$ are also close together.

The two energy equations are:

$$\tfrac{3}{2}n_i(\partial T_i/\partial t + \boldsymbol{U}_i\cdot\nabla T_i) + p_i\nabla\cdot\boldsymbol{U}_i = -\nabla\cdot\boldsymbol{q}_i - \boldsymbol{\pi}_i : \nabla\boldsymbol{U}_i + Q_i, \tag{16}$$

$$\tfrac{3}{2}n_e(\partial T_e/\partial t + \boldsymbol{U}_e\cdot\nabla T_e) + p_e\nabla\cdot\boldsymbol{U}_e = -\nabla\cdot\boldsymbol{q}_e - \boldsymbol{\pi}_e : \nabla\boldsymbol{U}_e + Q_e, \tag{17}$$

where the temperatures are defined by $p_i = n_i T_i$, $p_e = n_e T_e$ and the units of T are chosen to make Boltzmann's constant unity. The second term on the left of each equation is the $p\,dV$ work done by compression. $\boldsymbol{q}_i$ and $\boldsymbol{q}_e$ are the heat flows, $\boldsymbol{\pi}_i : \nabla\boldsymbol{U}_i$ and $\boldsymbol{\pi}_e : \nabla\boldsymbol{U}_e$ are the frictional heating terms due to nonuniform velocities while Q_i and Q_e represent energy exchange between the species and joule heating.

Equations (14)–(17) become more accurate as the collision time τ goes to zero. They consist of "fluid" terms and dissipative terms and the latter are smaller than the former roughly by τ/t. Thus, if τ were zero, collisions would be sufficient to maintain an isotropic velocity distribution in the frame moving with the fluid and the π terms would be small. However, because $\boldsymbol{U}$ is inhomogeneous, an isotropic distribution at one point does not match the isotropic distribution a mean free path away, and a certain mixing of these distributions leads to anisotropy of the distribution and to off-diagonal terms in the stress tensor. The other dissipative term $\boldsymbol{R}_{ei}$ is produced by unlike particle collisions and is the friction force between electrons and ions. Since the difference between the electron and ion velocities is the current, this friction includes the resistivity as well as thermoelectric effects. In most cases in practice $\boldsymbol{U}_i$ is close to $\boldsymbol{U}_e$ and can be identified with the mass flow of the plasma. If (14) is added to (15), the electron–ion friction force cancels out and the electron inertial term and gravitational terms are negligible. Thus, except for the viscosity terms $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_e$, we recover the one-fluid equation of motion, (2). On the other hand, if we express $\boldsymbol{U}_e$ in terms of $\boldsymbol{U}_i$ and $\boldsymbol{j}$ by solving

$$\boldsymbol{j} = n_i Z_e(\boldsymbol{U}_i - \boldsymbol{U}_e), \tag{18}$$

and neglect inertia in (18), we obtain a form of Ohm's law usually denoted as the generalized Ohm's law (Spitzer, 1962)

$$\boldsymbol{E} + \frac{\boldsymbol{U}\times\boldsymbol{B}}{c} = \frac{c}{n_e e}\boldsymbol{j}\times\boldsymbol{B} - \frac{\nabla p_e}{n_e e} - \frac{\nabla\cdot\boldsymbol{\pi}_e}{n_e e} + \frac{\boldsymbol{R}_{ei}}{n_e e}. \tag{19}$$

Equations (12)–(17) are the equations describing the electron and ion fluids separately. To complete them, we must add Maxwell's equations (4)–(6), where $\boldsymbol{j}$ is defined by (18). Again, we may consistently neglect the displacement current term in (4) and take $Zn_i = n_e$ so (13) is not needed. (This is the case for a low-frequency phenomenon. Although it is the case that the two-fluid equations may be used to derive some high-frequency wave phenomena provided thermal effects are small, these derivations are not really sound.) We also need the expressions for the various dissipation terms. These are given by Braginski (1965). Let the ion and electron collision times be defined as

$$\tau_i = \frac{3m_i^{1/2}T_i^{3/2}}{4\pi^{1/2}(\ln\Lambda)e^4Z^2n_i}, \tag{20a}$$

$$\tau_e = \frac{3m_e^{1/2}T_e^{3/2}}{4(2\pi)^{1/2}(\ln\Lambda)e^4Zn_e}, \tag{20b}$$

where $\ln\Lambda$ is the Coulomb logarithm and $m_{i,e}$ are the particle masses. The calculation is further limited to the case $Z=1$ and to the limit $\omega_{cs}\tau_s \gg 1$, where s indicates the particle species, i or e. Then from Braginski's article we have

$$\begin{aligned}\pi_s = {} & \eta_s^0(b\cdot\nabla U_s\cdot b - \tfrac{1}{2}\nabla\cdot U_s)(I_\perp - 2bb) - \eta_s^1(I_\perp\cdot\nabla U_s\cdot I_\perp + b\times\nabla U_s\times b)\\ & - \eta_s^2(bb\cdot W_s\cdot I_\perp + I_\perp\cdot W_s\cdot bb) + \tfrac{1}{2}\eta_s^3(b\times W_s\cdot I_\perp - I_\perp\cdot W_s\times b)\\ & + \eta_s^4(b\times W_s\cdot bb - bb\cdot W_s\times b),\end{aligned} \tag{21}$$

where

$$\begin{aligned} & b = B/B, && W_s = \nabla U + (\nabla U)^{tr}, && I_\perp \equiv I - bb,\\ & \eta_i^0 = 0.96 n_i T_i\tau_i, && \eta_e^0 = 0.73 n_e T_e\tau_e, && \\ & \eta_i^1 = 0.3 n_i T_i/\omega_{ci}^2\tau_i, && \eta_e^1 = 0.51 n_e T_e/\omega_{ce}^2\tau_e, && \eta_s^2 = 4\eta_s^1,\\ & \eta_i^3 = 0.5 n_i T_i/\omega_{ci}, && \eta_e^3 = -0.5 n_e T_e/\omega_{ce}, && \eta_s^4 = 2\eta_s^3. \end{aligned} \tag{22}$$

For R_{ei},

$$R_{ei} = en_e\frac{j\cdot b}{\sigma_\parallel}b + \frac{j_\perp}{\sigma_\perp} - 0.71 n_e b\cdot\nabla T_e b - \frac{3}{2}\frac{n_e}{\omega_{ce}\tau_e}(b\times\nabla T_e), \tag{23}$$

where $\sigma_\perp = e^2n_e\tau_e/m_e$, $\sigma_\parallel = 1.96\,\sigma_\perp$, and the last two terms of (23) represent thermal forces.

The heat flow terms q_s are given by

$$\begin{aligned} q_s = {} & -K_{s\parallel}b\cdot\nabla T_s b - K_{s\perp}I_\perp\cdot\nabla T_s + \frac{5}{2}\frac{n_sT_s}{\omega_{cs}m_s}b\times\nabla T\\ & + \left[0.71 n_eT_e(U_i - U_e) + \frac{3}{2}\frac{n_eT}{\omega_{ce}\tau_e}b\times(U_i - U_e)\right]\delta_{es}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} & K_{e\parallel} = 3.16 n_eT_e\tau_e/m_e, && K_{i\parallel} = 3.9 n_iT_i\tau_i/m_i,\\ & K_{e\perp} = 4.66 n_eT_e/m_e\omega_{ce}^2\tau_e, && K_{i\perp} = 2n_iT_i/m_i\omega_{ce}^2\tau_i, \end{aligned} \tag{25}$$

and the factor multiplying the bracket indicates that this term (the thermoelectric term) is present only for q_e.

The internal heating terms Q are given by

$$Q_e = -R_{ei}\cdot(U_i - U_e) - Q_\Delta, \tag{26}$$

where the first term is the joule heating term and the second

$$Q_i = Q_\Delta = 3\frac{m_e}{m_i}\frac{n_e}{\tau_e}(T_e - T_i), \tag{27}$$

the energy exchange term.

Equations (12)–(17) are a complete set of equations for the plasma quantities $n_i = n_e$, U_i, U_e, p_i, and p_e, all the quantities on the right being defined in terms of them. They allow a much richer set of plasma phenomena to be described than the one-fluid equations, particularly in the allowance for different electron and ion temperatures and the inclusion of nonideal effects such as thermal conductivity, viscosity, resistivity and thermoelectric effects. Thus, they are more useful for describing long-term phenomena in which nonideal effects play a significant role. It is possible to include such nonideal terms in the one-fluid equation. However, because ion and electron transport play different roles and because the temperature sensitivity of these is important, the modified one-fluid approach is usually highly inaccurate and misleading. Thus, one could possibly distinguish between the usefulness of the one-fluid and two-fluid approaches as follows. The one-fluid approach is preferable for short-time hydrodynamic effects in which nonideal effects play a minor role. Its great advantage is that its equations are considerably simpler to handle than the two-fluid approach. Finally, it can be used in longer-time problems to get an idea of at least some of the plasma behavior.

The two-fluid equations are more accurate and necessary for any precision in the discussion of phenomena where plasma transport or dissipation is involved. They are too complex to solve, however, for any problems except those with simple geometries. They can, of course, be used to form a good idea as to the accuracy of calculations based on the one-fluid approach.

1.4.3. Collisionless plasma

In Section 1.4.2 plasmas were discussed in which the collision time was the shortest time in the problem with the possible exception of the gyration period. Thus, a small element of mass of a plasma will relax quickly to a Maxwellian before it can change its properties, and a local description in terms of the parameters characterizing this Maxwellian is appropriate. This consistency justifies a fluid description. But in many important plasmas the collision time is so long that collisions should be ignored. It would appear that for such "collisionless" plasmas a fluid theory is not appropriate. However, even for weak magnetic fields, the cyclotron period is still shorter than any macroscopic period, and the plasma does have a two-dimensional consistency perpendicular to the magnetic field. This restores the possibility of a fluid theory to a limited extent and is the basis for the guiding center description of a plasma.

The guiding center limit of the Vlasov equation

A collisionless plasma is completely described by giving its velocity distribution functions f_s [$f_s(t, \boldsymbol{r}, \boldsymbol{v})\mathrm{d}^3r\mathrm{d}^3x$ is the number of particles in an element $\mathrm{d}^3r\mathrm{d}^3v$ at position $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ at time t]. Its time behavior is governed by the Vlasov equation with e_s the particle charge

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\nabla f_s + \frac{e_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\nabla_v f_s = 0, \tag{28}$$

where $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are the mean electric and magnetic fields produced by the smoothed-out plasma distributions f_s

$$\nabla\times\boldsymbol{B} = 4\pi\sum_s \frac{e_s}{c}\int f_s\boldsymbol{v}\mathrm{d}^3v + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{29a}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum e_s\int f_s\mathrm{d}^3v, \tag{29b}$$

$$\partial\boldsymbol{B}/\partial t = -c\nabla\times\boldsymbol{E}, \tag{29c}$$

$$\nabla\cdot\boldsymbol{B} = 0. \tag{29d}$$

These equations are more complicated than the fluid equations because they involve seven independent variables t, $\boldsymbol{r}$, $\boldsymbol{v}$ rather than four, t, $\boldsymbol{r}$. However, by an asymptotic expansion in the smallness of the gyration radiation $\rho = mcv/eB$ compared with the scale size of the plasma the effective number of variables in the kinetic equation can be reduced by two, because the gyration phase variable is irrelevant and the scalar perpendicular velocity is controlled by a constant of the motion, the adiabatic invariant (Chew et al., 1955; Kulsrud, 1962).

Further, to lowest order, the motion of the particles consists of an $\boldsymbol{E}\times\boldsymbol{B}$ velocity perpendicular to the magnetic field common to all particles, regardless of their peculiar velocities or species, and a parallel motion along the field. If the parallel electric field $E_\parallel = \boldsymbol{b}\cdot\boldsymbol{E}$, where $\boldsymbol{b} \equiv \boldsymbol{B}/B$, is small [cf., the discussion after (34)], it is well known that the magnetic lines of force can be assigned the same $\boldsymbol{E}\times\boldsymbol{B}$ velocity perpendicular to themselves (Newcomb, 1958). Thus, all particles will stay on the same line and it should be possible to concentrate our attention on a single line and derive a kinetic equation involving only two particle variables, position along the line and parallel velocity.

To derive the equations for this reduced system we may carry out a formal expansion in the quantity m/e (Kruskal, 1960). (If we regard macroscopic lengths and times to be fixed, then the small-gyration-radius limit is reached by taking a sequence of fictitious charged particles with different atomic properties m/e approaching zero. In this imagined series of experiments one expects results to be near their asymptotic value when the true values of m/e are reached, if the ratio of gyration radius to scale size is sufficiently small.) In point of fact, it turns out to be slightly more convenient to expand all quantities $\boldsymbol{E}$, $\boldsymbol{B}$, f in just the reciprocal charge, the quantity $1/e$ (Rosenbluth and Rostoker, 1958).

Consider first the Vlasov equation (28) and set $f = f_0 + f_1$ where $f_1 = \mathrm{O}(1/e)$ etc. From this point on the subscript s will be dropped when no confusion results. Then to lowest order

$$[\boldsymbol{E} + (\boldsymbol{v}\times\boldsymbol{B})/c]\cdot\nabla_v f_0 = 0. \tag{30}$$

Introduce the $\boldsymbol{E}\times\boldsymbol{B}$ velocity:

$$\boldsymbol{U}_\mathrm{E} = c(\boldsymbol{E}\times\boldsymbol{B})/B^2, \tag{31}$$

and set $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{U}_\mathrm{E}$. Equation (28) then becomes

$$[(\boldsymbol{v}'\times\boldsymbol{B})/c]\cdot\nabla_{v'} f_0 + E_\parallel\boldsymbol{b}\cdot\nabla f_0 = 0. \tag{32}$$

Next introduce cylindrical coordinates $v_\perp$, ϕ and $v_\parallel$ in $\boldsymbol{v}'$ space, by

$$\boldsymbol{v}' = \hat{\boldsymbol{x}}v_\perp\cos\phi + \hat{\boldsymbol{y}}v_\perp\sin\phi + \boldsymbol{b}v_\parallel. \tag{33}$$

Then (32) becomes

$$-\frac{B}{c}\frac{\partial f_0}{\partial\phi} + E_\parallel\frac{\partial f_0}{\partial v_\parallel} = 0. \tag{34}$$

If $E_\parallel \neq 0$, then (34) implies f_0 is constant along a helix in velocity space extending to infinite velocities, which is unphysical. Therefore, (30) has reasonable solutions only if $E_\parallel$ is expanded in $1/e$ also. That is $E_\parallel = \mathrm{O}(1/e)E$. (If this were not the case, the greatly more effective $E_\parallel$ would accelerate particles on a cyclotron period time scale until $E_\parallel$ is shorted out to the lowest order.) The resulting greatly reduced $E_\parallel$ can then produce a force comparable with the other forces. [See (19)]. It is simpler not to expand $\boldsymbol{E}$ and $\boldsymbol{B}$ further, but simply to regard $E_\parallel$ as smaller by one power of e.

If the $E_\parallel$ term in (34) is dropped, the lowest order Vlasov equation says that f_0 is independent of ϕ, but gives no further information on its dependence on t, $\boldsymbol{r}$, $v_\perp$ and $v_\parallel$. Proceeding to first order we have

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v}\cdot\nabla f_0 + \frac{e}{m}\left(\boldsymbol{E}_0 + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\nabla_v f_1 + \frac{e}{m}E_\parallel\frac{\partial f_0}{\partial v_\parallel} = 0. \tag{35}$$

Transforming to the cylindrical variables $v_\perp$, $v_\parallel$, yields

$$\frac{eB}{mc}\frac{\partial f_1}{\partial\phi} = \left(\frac{\partial f_0}{\partial t} + \boldsymbol{v}\cdot\nabla f_0\right) + \frac{e}{m}E_\parallel\frac{\partial f_0}{\partial v_\parallel}. \tag{36}$$

(The terms in parentheses are not yet so transformed but they must be.) This transformation is somewhat complex since at fixed $\boldsymbol{v}$, $v_\perp$, and $v_\parallel$ are dependent on $\boldsymbol{r}$ and t, because $\boldsymbol{b}$ and U_E are, through (31). It is easy to see that actually the transformation of the quantities in parentheses leads to a series of terms that are sines and cosines in ϕ. Once this transformation is accomplished it is easy to solve (36) for f_1. However, any constant term leads to an f_1 linear in ϕ and therefore not periodic with period 2π. Thus, in order to have a proper solution for f_1 a necessary and sufficient condition is that the average of the right-hand side of (36) vanish. Imagine the right-hand side transformed to $v_\perp$, $v_\parallel$ variables and averaged over ϕ. The details of this calculation are straightforward and the result is that (36) can be solved for f_1, if and only if

$$\frac{\partial f_0}{\partial t}+(\boldsymbol{U}_{\rm E}+v_\parallel \boldsymbol{b})\cdot\nabla f_0-\frac{v_\perp}{2}(\nabla\cdot\boldsymbol{U}_{\rm E}-\boldsymbol{b}\cdot\nabla\boldsymbol{U}_{\rm E}\cdot\boldsymbol{b}+v_\parallel\nabla\cdot\boldsymbol{b})\frac{\partial f_0}{\partial v_\perp}$$

$$+\left(-\boldsymbol{b}\cdot\frac{\mathrm{D}\boldsymbol{U}_{\rm E}}{\mathrm{D}t}+\frac{v_\perp^2}{2}(\nabla\cdot\boldsymbol{b})+\frac{e}{m}E_\parallel\right)\frac{\partial f_0}{\partial v_\parallel}=0, \tag{37}$$

where $\mathrm{D}\boldsymbol{U}_{\rm E}/\mathrm{D}t\equiv\partial\boldsymbol{U}_{\rm E}/\partial t+(\boldsymbol{U}_{\rm E}+\boldsymbol{b}v_\parallel)\cdot\nabla\boldsymbol{U}_{\rm E}$. This condition thus gives the time evolution of f_0. Strictly speaking we should go ahead and solve for f_1 once we are assured by (37) that this can be done. But it will appear shortly that we do not need f_1 for a lowest-order description of a guiding center plasma.

To complete the system we must add the equations for $\boldsymbol{E}$ and $\boldsymbol{B}$, Maxwell's equations (29a)–(29d). They involve f so that they also must be expanded in our small "parameter" $1/e$. To lowest order we have

$$0=4\pi\sum_s\frac{e_s}{c}\int f_{s0}\boldsymbol{v}\,\mathrm{d}^3v, \tag{38a}$$

$$0=4\pi\sum_s e_s\int f_{s0}\,\mathrm{d}^3v. \tag{38b}$$

Equation (38b) is the charge neutrality condition which states that to lowest order in $1/e$ the total charges of each species must be equal. For a $Z=1$ ion species this reduces to equality of the species densities. (Any finite charge density is produced by first-order differences in charge density because of the factor $1/e$). Similarly (38a) is the current neutrality condition. If we transform the velocity integration to cylindrical coordinates, we get for (38a)

$$0=4\pi\sum_s\frac{e_s n_{s0}}{c}\boldsymbol{U}_{\rm E}+4\pi\sum_s\frac{e_s}{c}\int f_0 v_\parallel 2\pi v_\perp\,\mathrm{d}v_\perp\,\mathrm{d}v_\parallel,$$

and the first term vanishes by virtue of (38b) so we have

$$0=\sum_s\boldsymbol{j}_{s-1}\cdot\boldsymbol{b}=\sum_s\frac{e_s}{c}\int f_0 v_\parallel\,\mathrm{d}^3v. \tag{39}$$

Equations (38b) and (39) are related by the continuity equation derivable from (37) or even from (28),

$$\sum_s e_s\left(\frac{\partial n_{0s}}{\partial t}+\boldsymbol{B}\cdot\nabla\frac{n_{0s}(\boldsymbol{U}_s\cdot\boldsymbol{b})}{B}\right) \tag{40}$$

so that if (39) is satisfied at some initial time t, and (38b) is satisfied (and the other guiding center equations are satisfied), then (39) will be satisfied for all t. Alternatively, if the charge neutrality condition is satisfied and (39) is satisfied at one point on each line at every time it will be satisfied everywhere.

Equations (38b) and (39) are extra conditions imposed on f_0 and do not serve to advance $\boldsymbol{E}$ and $\boldsymbol{B}$ in time. These conditions are essentially thought to be control on the magnitude of $E_\parallel$, which is usually chosen to ensure that they are satisfied. To complete our equations we must include (29c) and (29d) and proceed to one higher order in the expansion of (29a) and (29b). Thus, (29a) and (29b) become

$$\nabla\times\boldsymbol{B}=4\pi\sum_s\frac{e_s}{c}\int\boldsymbol{v}f_{1s}\,\mathrm{d}^3v+\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}, \tag{41a}$$

$$\nabla\cdot\boldsymbol{E}=4\pi\sum_s e_s\int f_{1s}\,\mathrm{d}^3v. \tag{41b}$$

It would appear that it is necessary to evaluate f_1 from (36) after all. However, full information on the dependence of f_1 is not needed. Transformation of (41a) to cylindrical coordinates shows we only need $\int f_1\,\mathrm{d}\phi$, $\int f_1\sin\phi\,\mathrm{d}\phi$, and $\int f_1\cos\phi\,\mathrm{d}\phi$. These may be obtained by multiplying (36) by 1, $\sin\phi$ and $\cos\phi$ and integrating over ϕ. An equivalent set of moments can be carried out on the exact Vlasov equation (28) and passing to the zeroth-order limit. But these are simply the MHD equations of Sections 1.4.1 and 1.4.2. Thus, $\boldsymbol{j}$ to zeroth order is determined by

$$\sum n_s m_s\left(\frac{\partial\boldsymbol{U}_s}{\partial t}+\boldsymbol{U}_s\cdot\nabla\boldsymbol{U}_s\right)=-\nabla\cdot\boldsymbol{P}+\boldsymbol{j}\times\boldsymbol{B}+\rho_E\nabla\cdot\boldsymbol{E}, \tag{42}$$

where the mass velocity $\boldsymbol{U}_s$ and the stress tensor $\boldsymbol{P}$ are defined by

$$n_s\boldsymbol{U}_s=\int f_s\boldsymbol{v}\,\mathrm{d}^3v,$$

$$\boldsymbol{P}=\sum_s m_s\int f_s(\boldsymbol{v}-\boldsymbol{U}_s)(\boldsymbol{v}-\boldsymbol{U}_s). \tag{43}$$

Note that the component of $\boldsymbol{U}_s$ perpendicular to $\boldsymbol{b}$ is $U_{\rm E}$, while by (39) the parallel mass velocities are the same for both species. Thus $\boldsymbol{U}=\boldsymbol{U}_s$. On transforming to cylindrical coordinates the stress tensor may be written

$$\boldsymbol{P}=p_\perp(\boldsymbol{I}-\boldsymbol{bb})+p_\parallel\boldsymbol{bb}, \tag{44a}$$

where $\boldsymbol{I}$ is the unit dyadic and

$$p_\perp=\sum_s m_s\int f_s\frac{v_\perp^2}{2}\,\mathrm{d}^3v, \tag{44b}$$

$$p_\parallel=\sum_s m_s\int f_s(v_\parallel-\boldsymbol{U}\cdot\boldsymbol{b})^2\,\mathrm{d}^3v. \tag{44c}$$

As advertised, (42) determines the part of $\boldsymbol{j}$ perpendicular to $\boldsymbol{b}$. The parallel part of $\boldsymbol{j}$ is a different moment of f_1 but can also be found from Maxwell's equations. We may continue this scheme but it is more efficacious at this point to change the emphasis from $\boldsymbol{E}$ to $\boldsymbol{U}$, regarding $\boldsymbol{U}$ as the primary variable and $\boldsymbol{E}$ as a secondary variable;

$$\boldsymbol{E}=-(\boldsymbol{U}\times\boldsymbol{B})/c, \tag{45}$$

from (31). This is particularly true since $\boldsymbol{E}$ is restricted to be perpendicular to $\boldsymbol{b}$, while $\boldsymbol{U}$ is not and determines $\boldsymbol{E}$ automatically to satisfy this condition.

Solving (29a) for j_0, substituting into (42) and making use of (45) gives

$$\rho\left(\frac{\partial \boldsymbol{U}}{\partial t}+\boldsymbol{U}\cdot\nabla\boldsymbol{U}\right)=-\nabla\cdot\boldsymbol{P}+\frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}+\frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{U}\times\boldsymbol{B})\times\boldsymbol{B}$$
$$+\frac{(\boldsymbol{U}\times\boldsymbol{B})\,\nabla\cdot(\boldsymbol{U}\times\boldsymbol{B})}{c^2}, \tag{46}$$

where $\rho=\Sigma n_s m_s$. Then substituting (45) into (29c) gives

$$\partial\boldsymbol{B}/\partial t=\nabla\times(\boldsymbol{U}\times\boldsymbol{B}). \tag{47}$$

Equations (46) and (47) are nearly self-contained except we need f_{0s} to compute ρ and $\boldsymbol{P}$. ρ is given by the continuity equation

$$\partial\rho/\partial t+\nabla\cdot(\rho\boldsymbol{U})=0, \tag{48}$$

but we cannot obtain $\boldsymbol{P}$ in any other way than from f_0. Thus, the equation determining f_0 and thus $\boldsymbol{P}$, (37), may be considered to determine the "equation of state" of the plasma. Finally, inspection of (37) shows it brings in $E_\parallel$, which must be determined by the charge neutrality condition (38b) or alternatively the parallel current condition of (39). It is possible by combining the separate moment equations to show that

$$E_\parallel=\sum_s(e_s/m_s)\boldsymbol{b}\cdot\nabla\cdot\boldsymbol{P}_s\Big/\sum_s\left(n_s e_s^2/m_s\right). \tag{49}$$

However, this is a little misleading since (49) arises from the second time derivative of the charge neutrality condition (38a) and in fact if one seeks equilibria, $E_\parallel$ actually drops out of (49).

Our complete system of guiding center equations are (45)–(48) with $\boldsymbol{P}$ defined by (44a)–(44c) and f_0 and $E_\parallel$ determined by (37) and (38a). Again as in the one-fluid theory we see that the last two terms of (46) may be dropped as relativistically small. The system then reduces to that of a one-fluid description with the main complication occurring through the equation of state. This complication can only be removed by solving an apparently five-dimensional equation for f_0. However, these five variables t, $\boldsymbol{r}$, $v_\perp$, $v_\parallel$ can be reduced to four by replacing $v_\perp$ by the new variable

$$\mu\equiv v_\perp^2/2B, \tag{50}$$

equal to the magnetic moment of the particle. Equation (37) then reduces to

$$\frac{\partial f_0}{\partial t}+(\boldsymbol{U}_{\mathrm{E}}+v_\parallel\boldsymbol{b})\cdot\nabla f_0+\left(-\boldsymbol{b}\cdot\frac{\mathrm{D}\boldsymbol{U}_{\mathrm{E}}}{\mathrm{D}t}+\mu B\,\nabla\cdot\boldsymbol{b}+\frac{e}{m}E_\parallel\right)\frac{\partial f_0}{\partial v_\parallel}=0 \tag{51}$$

where the coefficient of $\partial f/\partial\mu$ vanishes so that the effective number of variables is reduced by one. The variable μ occurs merely as a parameter in (52) and $v_\parallel$ is the only real variable in addition to $\boldsymbol{r}$ and t. Note that

$$\boldsymbol{U}_{\mathrm{E}}=\boldsymbol{U}_\perp\equiv\boldsymbol{U}-\boldsymbol{bb}\cdot\boldsymbol{U}. \tag{52}$$

The guiding center theory demonstrates how in the absence of collisions the magnetic field acts to give the plasma *almost* enough consistency for a hydrodynamic

description. It interferes strongly with motions across itself forcing all particles to move together so that all particles in one tube of force stay in that one tube of force.

Equation (51) may be reduced by two more dimensions in line with the remarks at the beginning of this section. To do this the Clebsch form is used for any divergence-free field as shown in Section 1.4.4; for any vector field $\boldsymbol{B}$ such that $\nabla\cdot\boldsymbol{B}=0$ one can find two scalars α and β such that

$$\boldsymbol{B}=\nabla\alpha\times\nabla\beta, \tag{53}$$

α and β are not uniquely determined, but if they once give $\boldsymbol{B}$ at some initial time t_0, they will continue to represent $\boldsymbol{B}$ by (53) for all time, provided they satisfy

$$\frac{\partial\alpha}{\partial t}+\boldsymbol{U}\cdot\nabla\alpha=0,\qquad\frac{\partial\beta}{\partial t}+\boldsymbol{U}\cdot\nabla\beta=0, \tag{54}$$

or, in other words, provided they are "frozen" in the fluid. Since α and β are flux labels, a line of force is always given by $\alpha=$ constant, $\beta=$ constant. This result is a precise mathematical expression of the fact that lines of force are frozen in a plasma. If we replace the general position variable $\boldsymbol{r}$ by new coordinates α, β and l, a parameter characterizing position along a line of force, then (52) can be reduced to a "one-dimensional" kinetic equation by transforming to the variables α, β, l, μ, $v_\parallel$, ϕ. It becomes, with s arc length along $\boldsymbol{B}$,

$$\frac{\partial f_0}{\partial t}+v_\parallel\left(\frac{\partial l}{\partial s}\right)\frac{\partial f_0}{\partial l}+\left(-\boldsymbol{b}\cdot\frac{\mathrm{D}\boldsymbol{U}_{\mathrm{E}}}{\mathrm{D}t}+\mu B\,\nabla\cdot\boldsymbol{b}+\frac{eE_\parallel}{m}\right)\frac{\partial f_0}{\partial v_\parallel}=0, \tag{55}$$

provided only that l satisfies $(\partial l/\partial t+\boldsymbol{U}_{\mathrm{E}}\cdot\nabla l)=0$.

For completeness we collect together the full systems of guiding center equations for the fundamental variables ρ, $\boldsymbol{U}$, $\boldsymbol{B}$, f_0, and $E_\parallel$.

$$\partial\rho/\partial t+\nabla\cdot(\rho\boldsymbol{U})=0, \tag{48}$$

$$\rho\left(\frac{\partial\boldsymbol{U}}{\partial t}+\boldsymbol{U}\cdot\nabla\boldsymbol{U}\right)=\frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}-\nabla\cdot\boldsymbol{P}, \tag{46}$$

$$\partial\boldsymbol{B}/\partial t=\nabla\times(\boldsymbol{U}\times\boldsymbol{B}), \tag{47}$$

$$\boldsymbol{P}=p_\perp\boldsymbol{I}+(p_\parallel-p_\perp)\boldsymbol{bb}, \tag{44a}$$

$$p_\perp=\sum_s m_s\int f_{0s}\frac{v_\perp^2}{2}\,\mathrm{d}^3v, \tag{44b}$$

$$p_\parallel=\sum_s m_s\int f_{0s}(v_\parallel-\boldsymbol{U}\cdot\boldsymbol{b})^2\,\mathrm{d}^3v, \tag{44c}$$

$$\frac{\partial f_{0s}}{\partial t}+(\boldsymbol{U}_{\mathrm{E}}+v_\parallel\boldsymbol{b})\cdot\nabla f_{0s}-v_\perp\left(\nabla\cdot\boldsymbol{U}_\perp-\boldsymbol{b}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{b}+v_\parallel\nabla\cdot\boldsymbol{b}\right)\frac{\partial f_{0s}}{\partial v_\perp}$$
$$+\left(-\boldsymbol{b}\cdot\frac{\mathrm{D}\boldsymbol{U}_{\mathrm{E}}}{\mathrm{D}t}+\frac{v_\perp^2}{2}\nabla\cdot\boldsymbol{b}+\frac{e}{m}E_\parallel\right)\frac{\partial f_{0s}}{\partial v_\parallel}=0, \tag{37}$$

$$\sum_s e_s\int f_{0s}\,\mathrm{d}^3v=0. \tag{38b}$$

The double adiabatic theory

As remarked in the previous subsection a collisionless plasma is subject to description by fluid equations with the single difficulty involving the determination of the evolution of the two pressure components $p_\perp$ and $p_\parallel$. Chew et al. (1956) showed that these quantities themselves can be expressed in terms of two equations of state:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\perp}{\rho B}\right) = 0, \tag{56a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\parallel B^2}{\rho^3}\right) = 0, \tag{56b}$$

which apply under the same restrictions as the adiabatic theory of the previous subsection but with an important additional restriction. The system must vary sufficiently slowly along the lines of force that little communication of particles from points of different behavior along the lines occurs. More explicitly (see Fig. 1.4.1), let points P_1 and P_2 be two points on a line of force at which the plasma properties, ρ, T, $\boldsymbol{B}$, etc., are significantly different. Then in a time $t \approx l/v$, particles from 1 and 2 will mix together and they can no longer be considered separate units. However, if significant changes occur at P_1 in a time short compared with t, the behavior at P_2 can exert no appreciable affect on P_1. Particles at P_1 can be considered to remain intact and the two-particle adiabatic invariants may be employed to determine the behavior at P_1. $p_\perp$ is proportional to $v_\perp^2$ averaged over all the particles and to the density ρ, while $\langle v_\perp^2 \rangle$, by the invariance of μ, is proportional to B, so we have

$$p_\perp \propto \langle v_\perp^2 \rangle \rho \propto \rho B.$$

This, of course, is true following the motion since it is the particles and not their location that is of importance.

The second invariant is not so familiar. It is $v_\parallel l$ where l is the "extension" of a fluid element along the line. The quantity l has an amount of uncertainty in its definition since the particles are dispersing at a considerable rate. However, it is known that even in free expansion of a one-dimensional gas the mean square dispersion of velocities decreases as the density does and moreover is inversely proportional to the length of the element of gas squared. (This can be seen for a gas initially of finite length, the particles of slowest velocity staying near the initial position.) For our case the length l is proportional to B/ρ since the volume of a tube of force is inversely proportional to ρ, while the cross sectional area is inversely

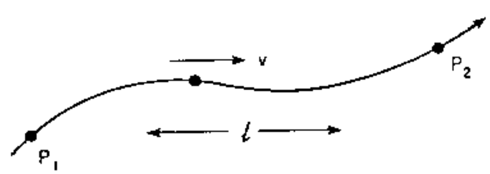

Fig. 1.4.1. A line of force, $\boldsymbol{B}$.

proportional to B. Thus, the parallel pressure goes as

$$p_\parallel \propto \rho \langle v_\parallel^2 \rangle \propto \rho / l^2 \propto \rho^3 / B^2.$$

A more formal derivation is as follows: The condition that points P_1 and P_2 remain intact clearly means that there is no significant heat exchange between points P_1 and P_2. Thus, in the second moment of the Vlasov equation we may neglect $\boldsymbol{Q}$ the heat flow tensor. Multiply (28) by $m_s(\boldsymbol{v}-\boldsymbol{U}_s)(\boldsymbol{v}-\boldsymbol{U}_s)$, integrate over all velocities at a fixed point $\boldsymbol{r}$. By charge and current neutrality $\boldsymbol{U}_s$ is the same for ions and electrons if a single ion species is assumed. Then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_s + \nabla\cdot\boldsymbol{Q}_s + \boldsymbol{P}_s\nabla\cdot\boldsymbol{U} + \boldsymbol{P}_s\cdot\nabla\boldsymbol{U} + (\boldsymbol{P}_s\cdot\nabla\boldsymbol{U})^{\mathrm{tr}} + \frac{e_s}{m_s c}(\boldsymbol{B}\times\boldsymbol{P}_s + \boldsymbol{P}_s\times\boldsymbol{B}) = 0, \tag{57}$$

where the superscript tr indicates transpose of the diadic, $\boldsymbol{P}_s$ is defined as in (43), and $\boldsymbol{Q}_s$ is the triad:

$$\boldsymbol{Q}_s \equiv m_s \int (\boldsymbol{v}-\boldsymbol{U}_s)(\boldsymbol{v}-\boldsymbol{U}_s)(\boldsymbol{v}-\boldsymbol{U}_s) f \,\mathrm{d}^3 v. \tag{58}$$

As before, we regard the last two terms as dominant because of the factor e/mc (the small gyration radius expansion). Thus, to lowest significant order, the pressure $\boldsymbol{P}_{s0}$ must satisfy

$$\boldsymbol{B}\times\boldsymbol{P}_{s0} = \boldsymbol{P}_{s0}\times\boldsymbol{B}. \tag{59}$$

The most general solution of this equation is

$$\boldsymbol{P}_{s0} = p_{\perp s}(\boldsymbol{I} - \boldsymbol{bb}) + p_{\parallel s}\boldsymbol{bb}, \tag{60}$$

where the two scalars (so far) are arbitrary functions of time and space.

Denote the left-hand side of (57) by $L\boldsymbol{P}_0$; then to next significant order in our expansion, (57) reads

$$L\boldsymbol{P}_{0s} = \frac{e_s}{m_s c}(\boldsymbol{P}_{s1}\times\boldsymbol{B} - \boldsymbol{B}\times\boldsymbol{P}_{s1}), \tag{61}$$

where $\boldsymbol{P}_{s1}$ is the first-order pressure. The necessary and sufficient condition that this can be solved for $\boldsymbol{P}_{s1}$ is that the trace of this equation vanish and also that it vanish when dotted with $\boldsymbol{b}$ on the right and left sides. Performing these operations, dropping $\boldsymbol{Q}$ and summing over s, gives

$$(\mathrm{d}/\mathrm{d}t)(2p_\perp + p_\parallel) + (2p_\perp + p_\parallel)\nabla\cdot\boldsymbol{U} + 2p_\perp(\nabla\cdot\boldsymbol{U} - \boldsymbol{b}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{b}) + 2p_\parallel\boldsymbol{b}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{b} = 0, \tag{62a}$$

$$(\mathrm{d}/\mathrm{d}t)p_\parallel + p_\parallel\nabla\cdot\boldsymbol{U} + 2p_\parallel\boldsymbol{b}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{b} = 0. \tag{62b}$$

$\boldsymbol{U}$ can be related to the rate of change of ρ and B by (48) and (47):

$$\mathrm{d}\rho/\mathrm{d}t = -\rho\nabla\cdot\boldsymbol{U}, \tag{63}$$

and

$$\mathrm{d}B/\mathrm{d}t = \boldsymbol{b}\cdot\mathrm{d}\boldsymbol{B}/\mathrm{d}t = \boldsymbol{b}\cdot[\nabla\times(\boldsymbol{U}\times\boldsymbol{B})+\boldsymbol{U}\cdot\nabla\boldsymbol{B}] = B(\boldsymbol{b}\cdot\nabla\boldsymbol{U}\cdot\boldsymbol{b} - \nabla\cdot\boldsymbol{U}), \quad (64)$$

so that (62b) becomes

$$\frac{\mathrm{d}p_\parallel}{\mathrm{d}t} = +\frac{3p_\parallel}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{2p_\parallel}{B}\frac{\mathrm{d}B}{\mathrm{d}t}. \quad (65)$$

This reduces immediately to (56b). Subtracting (62b) from (62a) and using (63) and (64) again yields

$$\frac{2\mathrm{d}p_\perp}{\mathrm{d}t} - \frac{2p_\perp}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{2p_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = 0,$$

which reduces to (56a).

Thus, the double adiabatic equations of state result from the guiding center equations and the dropping of the heat flow. We can reduce the expression for $\boldsymbol{Q}$ by making use of the special form of f_0, derived in the previous section from (34), that is its independence of gyration phase ϕ. $\boldsymbol{Q}$ can be written

$$\boldsymbol{Q} = 2q'_\perp(\boldsymbol{Ib} + \boldsymbol{bI} + \mathrm{tr}) + 2q'_\parallel\boldsymbol{bbb}, \quad (66)$$

where

$$q'_\perp = \sum_s m_s \int \frac{v_\perp^2}{2}(v_\parallel - \boldsymbol{U}\cdot\boldsymbol{b}) f \mathrm{d}^3v, \quad (66a)$$

$$q'_\parallel = \sum_s m_s \int (v_\parallel - \boldsymbol{U}\cdot\boldsymbol{b})^3 f \mathrm{d}^3v, \quad (66b)$$

and the symbol tr denotes the third possible transposition of the triad $\boldsymbol{Ib}$. $q'_\perp$ is the parallel heat flow of perpendicular energy while $q'_\parallel$ is the parallel flow of parallel energy. They are only small if f is nearly symmetric, the situation arising when macroscopic plasma parameters vary slowly along $\boldsymbol{B}$. Also

$$\mathrm{Tr}\,\nabla\cdot\boldsymbol{Q} = \boldsymbol{b}\cdot\nabla(10q'_\perp + 2q'_\parallel) - (10q'_\perp + 2q'_\parallel)(\boldsymbol{b}\cdot\nabla B)/B, \quad (67a)$$

and

$$\boldsymbol{b}\cdot(\nabla\cdot\boldsymbol{Q})\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\nabla(6q'_\perp + 2q'_\parallel) - 2(q'_\perp + q'_\parallel)(\boldsymbol{b}\cdot\nabla B)/B, \quad (67b)$$

so the derivative reduces the heat flow term by an additional factor proportional to the slowness of variation along $\boldsymbol{B}$.

To summarize the double adiabatic formalism, it is identical with the single-fluid theory, (1)–(4) and (10), with the single change that p is replaced by the divergence of the tensor pressure $\boldsymbol{P}$, with the two scalars $p_\perp$, $p_\parallel$ determined by the double equations of state, (56a) and (56b). Again it can be seen that the double adiabatic formalism is holonomic: all quantities can be expressed in terms of the displacement vector and can be reduced to a Lagrangian formalism.

These nice properties plus the apparent generalization allowed by a nonscalar pressure have made the double adiabatic theory quite popular. Unfortunately, the stringent conditions of very slow variation along magnetic lines of force imposed by the neglect of $\boldsymbol{Q}$ greatly limit its applicability, at least when accurate results are desired. On the other hand, the equations can be applied to solve problems beyond their limits of applicability, and the answers obtained are grossly inaccurate. This will be illustrated by an example in Section 1.4.5; namely, the computation of the criteria for stability against the mirror instability when a homogeneous magnetized plasma has unequal perpendicular and parallel pressures. This easy applicability of the formalism beyond the range of its validity makes it somewhat dangerous.

1.4.4. Consequences of the MHD description

The ideal MHD equations and, to a lesser extent, the double adiabatic equations and the guiding center equations possess some nice properties that often may be employed to draw some intuitive conclusions concerning plasma behavior without solving the equations in detail. They consist of some general global relations, conservation equations, and virial theorems, and also of the flux and line conservation equations which may be thought of as detailed conservation equations.

Conservation relations

The three quantities conserved by a plasma are linear momentum, energy, and angular momentum. To write them down for the ideal one-fluid system the force equation is first rewritten as:

$$\rho\frac{\partial\boldsymbol{U}}{\partial t} = -\rho\boldsymbol{U}\cdot\nabla\boldsymbol{U} + \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} - \nabla p - \rho\nabla\phi, \quad (68a)$$

where use has been made of (4) to eliminate $\boldsymbol{j}$ and the gravitational potential ϕ with $\boldsymbol{g} = -\nabla\phi$ has been introduced. Multiplying the continuity equation by $\boldsymbol{U}$ and adding gives:

$$(\partial/\partial t)(\rho\boldsymbol{U}) = -\nabla\cdot\boldsymbol{T} - \rho\nabla\phi, \quad (68b)$$

where

$$\boldsymbol{T} = +\rho\boldsymbol{U}\boldsymbol{U} - \left(\frac{B^2}{8\pi}\boldsymbol{I} - \frac{\boldsymbol{BB}}{4\pi}\right) - p\boldsymbol{I}. \quad (69)$$

$\boldsymbol{T}$ represents stresses exerted on any surface: the first terms are Reynold stresses; the second, magnetic stresses, magnetic pressure and tension; while the third term is the pressure stress. Integrating (69) over a fixed volume V, and employing Gauss's theorem gives:

$$\frac{\partial}{\partial t}\int_V \rho\boldsymbol{U}\,\mathrm{d}\tau = -\int_S \boldsymbol{T}\cdot\mathrm{d}\boldsymbol{s} + \int_V \rho\boldsymbol{g}\,\mathrm{d}\tau. \quad (70)$$

The term on the left is the rate of change of the plasma momentum in the volume, the first term on the right represents changes in this momentum due to forces exerted on the surfaces, and the last, changes in this momentum due to gravitational forces. If the system were isolated and g zero, then the total linear momentum would be conserved. [This is actually impossible (see the virial theorem below) but if the gravitational force is self-consistent, produced by the plasma, the gravitational force can be written as a divergence and the linear momentum is actually conserved, as for example in an isolated star.] In any event the linear momentum density of a plasma is simply ρU and includes no magnetic field contribution. Its change may be estimated by the forces on the surface. The electromagnetic contribution is relativistically small and not included in our equation.

A more significant conservation relation is that of energy. It is obtained by first multiplying (68a) by U and making use of the continuity equation to obtain

$$\frac{\partial}{\partial t}\left(\rho \frac{U^2}{2}\right) = + \frac{U\cdot(\nabla\times B)\times B}{4\pi} - U\cdot\nabla p - \rho U\cdot\nabla\phi - \nabla\cdot\left(\rho\frac{U^2}{2}U\right). \tag{71}$$

The left-hand side represents the rate of change of kinetic energy per unit volume. The kinetic energy is changed as a result of corresponding changes of the magnetic energy (the first term on the right), pressure energy (the second term) and gravitational energy (the third term). In fact, multiplying (10) by B gives:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right) = B\cdot\frac{\nabla\times(U\times B)}{4\pi}. \tag{72}$$

Equations (3) and (1) give:

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma-1}\right) = -\frac{U\cdot\nabla p}{\gamma-1} - \frac{\gamma p}{\gamma-1}\nabla\cdot U. \tag{73}$$

Equation (1) gives:

$$(\partial/\partial t)(\rho\phi) = -\nabla\cdot(\rho U)\phi, \tag{74}$$

(ϕ is assumed to be independent of time). The quantities on the left of (70)–(74) are the rates of change of the magnetic, pressure and gravitational energy densities respectively. Each of these expressions is equal to a term that corresponds to one of the terms on the right-hand side of (71). In other words, any change in these energies can produce changes in the kinetic energy density.

Adding (71)–(74), integrating over a fixed volume V, and making use of Gauss's theorem yields

$$\frac{d\mathcal{E}_V}{dt} = \frac{d}{dt}\int\left(\frac{\rho V^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma-1} + \rho\phi\right)d\tau$$
$$= -\int dS\cdot\left(\frac{\rho U^2}{2}U + \frac{B\times(U\times B)}{4\pi} + \frac{\gamma}{\gamma-1}pU + \rho U\phi\right). \tag{75}$$

Thus, we may safely identify the left-hand side with the time rate of change of $\mathcal{E}_V$, the total energy inside the volume V, and the integral on the right-hand side with the loss of energy through the surface S. The energy consists of four types: kinetic energy, magnetic energy, pressure energy, and gravitational energy. Almost any macroscopic plasma process consists of exchange of various forms of energy together with loss of energy through the surface. From (75) this loss can be seen to consist of direct loss of kinetic energy (first term), Poynting flux (second term, since $U\times B = -cE$), thermal energy and $p\,dV$ work [since $\gamma pU/(\gamma-1) = pU/(\gamma-1) + pU$], and finally of gravitational work represented by fluid entering at one potential and leaving at another. (The Poynting flux can also be thought of as loss of magnetic energy plus a magnetic $P\,dV$ work.)

If the system is effectively isolated, say by rigid infinitely conducting walls at which $B\cdot n = 0$ at some time, then $B\cdot n$ will continue to be zero at all times and $U\cdot n = 0$ so the right-hand side of (75) will vanish and the energy inside the volume will be conserved.

Finally, a conservation relation can be derived for angular momentum, in complete analogy to (70). Take any point O as the origin and let r be the radius vector from this point. Then

$$\frac{d}{dt}\int_V r\times\rho U\,d\tau = \int_S (r\times T)\cdot dS + \int_V \rho r\times g\,d\tau. \tag{76}$$

The angular momentum again resides solely in plasma motions. This relation is of considerable use in discussing outflow of angular momentum from the sun via the solar wind.

Another important integral relation for a plasma is the virial theorem. Define with respect to an origin 0 the tensor moment of inertia of a plasma inside a fixed volume V

$$I_V = \int_V \rho rr\,d\tau. \tag{77}$$

Differentiate twice with respect to time making use of the ideal MHD equations and neglect surface terms and gravity

$$\frac{dI_V}{dt} = \int_V \frac{\partial\rho}{\partial t} rr\,d\tau = -\int_V \nabla\cdot(\rho U)rr\,d\tau = \int \rho(Ur + rU)\,d\tau, \tag{78}$$

$$\frac{d^2 I_V}{dt^2} = -\int[(\nabla\cdot T)r + r\nabla\cdot T]\,d\tau = 2\int_V T\,d\tau. \tag{79}$$

Then if the plasma remains in a finite region of space over a long period of time, we may time-average (79) and drop the left-hand side. There results from (69)

$$\left\langle \int\left[\rho UU + \left(\frac{B^2}{8\pi}I - \frac{BB}{4\pi}\right) + pI\right]d\tau\right\rangle = 0. \tag{80}$$

This is the vector virial theorem. $\langle\ \rangle$ denotes a time average. Deviations from this equation can result from surface terms so this equation applies only to an isolated system. Taking the trace of (80) yields

$$\left\langle \int\left(\rho U^2 + \frac{B^2}{8\pi} + 3p\right)d\tau\right\rangle = 0. \tag{81}$$

Since the integral is clearly positive this then shows the impossibility of an isolated (without coils) force-free system. On the other hand, if a self-consistent gravitational term is included,

$$\left\langle \int \left(\rho U^2 + \frac{B^2}{8\pi} + 3p + \frac{\rho\phi}{2} \right) \mathrm{d}\tau \right\rangle = 0, \tag{82}$$

so gravitational energy, which is always negative, can balance the other three types of energy. [Note that the first term is twice the kinetic energy, the second term is just the magnetic energy, and the third term is $3(\gamma - 1)$ times the thermal energy, equal to two times for $\gamma = 5/3$, while the last term is the gravitational energy.]

A final important theorem concerning ideal MHD systems is that the system is derivable from a Lagrangian. In order to understand this theorem most easily it is necessary to regard each plasma fluid element as an entity. Any flow pattern between times t_0 and t_1 should be viewed as a set of time-dependent displacements $\boldsymbol{\xi}(\boldsymbol{r}_0, t)$ of each of the fluid elements from its initial position $\boldsymbol{r}_0$ at $t = t_0$ to its final position $\boldsymbol{r}_1 = \boldsymbol{r}_0 + \boldsymbol{\xi}$ at t_1. A possible motion consists of a dependence of the displacement $\boldsymbol{\xi}(\boldsymbol{r}_0, t)$ on t. Then Hamilton's principle for the ideal MHD equations states that the motion that makes

$$L = \int_{t_0}^{t_1} \mathfrak{L}\,\mathrm{d}t, \tag{83}$$

stationary, where

$$\mathfrak{L} = \int \left(\frac{\rho U^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{8\pi} \right) \mathrm{d}\tau, \tag{84}$$

is the true dynamical one that satisfies the ideal MHD equations, and conversely. It is to be understood that for any displacement function $\boldsymbol{\xi}(\boldsymbol{r}, t)$, dynamical or not, the quantities ρ, p, and $\boldsymbol{B}$ are to be determined by solving (1), (3), and (10) respectively. We know that these quantities are determined holonomically and do not depend on the detailed time dependence of $\xi(\boldsymbol{r}_0, t)$.

For the proof of this result let us consider a given motion $\xi(\boldsymbol{r}_0, t)$ and determine a neighboring motion by specifying the Eulerian function $\delta\boldsymbol{\xi}(\boldsymbol{r}, t)$ which is defined to be the difference between the position of the fluid element at time t that would have been at $\boldsymbol{r}$ under the unperturbed motion, and $\boldsymbol{r}$. Then it is easy to see that the perturbations in the quantities ρ, p, $\boldsymbol{B}$ at position $\boldsymbol{r}$ under the influence of the perturbation of motion are

$$\rho_1 = -\nabla\cdot(\rho\delta\boldsymbol{\xi}), \tag{85a}$$

$$p_1 = -\gamma p \nabla\cdot(\delta\boldsymbol{\xi}) - \delta\boldsymbol{\xi}\cdot\nabla p, \tag{85b}$$

$$\boldsymbol{B}_1 = \nabla\times(\delta\boldsymbol{\xi}\times\boldsymbol{B}). \tag{85c}$$

It remains to determine U_1. The perturbation in the fluid element velocity is

$$\partial\delta\boldsymbol{\xi}/\partial t + \boldsymbol{U}\cdot\nabla\delta\boldsymbol{\xi},$$

by definition of $\delta\boldsymbol{\xi}$. But this perturbation is at $\boldsymbol{r} + \delta\boldsymbol{\xi}$ and is therefore also equal to

$\boldsymbol{U}_1 + \delta\boldsymbol{\xi}\cdot\nabla\boldsymbol{U}$. Hence

$$\boldsymbol{U}_1 = \partial\delta\boldsymbol{\xi}/\partial t + \boldsymbol{U}\cdot\nabla\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi}\cdot\nabla\boldsymbol{U}. \tag{85d}$$

Substituting these perturbations into the corresponding perturbations of (83) and (84) gives:

$$\begin{aligned} \delta L &= \int \delta\mathfrak{L}\,\mathrm{d}t \\ &= \int \mathrm{d}t \int \mathrm{d}\tau \left[\rho\boldsymbol{U}\cdot\left(\frac{\partial\delta\boldsymbol{\xi}}{\partial t} + \boldsymbol{U}\cdot\nabla\delta\boldsymbol{\xi} - \delta\boldsymbol{\xi}\cdot\nabla\boldsymbol{U} \right) \right. \\ &\quad \left. - \nabla\cdot(\rho\delta\boldsymbol{\xi})\frac{U^2}{2} + \frac{\gamma p \nabla\cdot\delta\boldsymbol{\xi}}{\gamma - 1} + \frac{\delta\boldsymbol{\xi}\cdot\nabla p}{\gamma - 1} - \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \right]. \end{aligned} \tag{86}$$

Then integration by parts shows that $\delta L = 0$ for all $\delta\boldsymbol{\xi}$ vanishing at t_0, t_1, and spatial boundaries, if and only if (2) is satisfied.

The existence of this Hamilton's principle for the MHD equations is extremely important. It can be shown to underlie most of the general results on MHD such as self-adjointness with steady flow, energy principles for stability of static equilibrium, and energy conservation (Kulsrud, 1968). Further, it has been shown that small-scale hydromagnetic waves preserve wave action, that is they can be thought of as quantized, and this also is a direct consequence of this Lagrangian approach (Dewar 1970).

This section has so far exclusively discussed the properties of the one-fluid ideal MHD equations. All of these properties are also possessed by the double adiabatic formalism if we replace p and γ by the appropriate generation. For example $p/(\gamma - 1)$ should be replaced by $p_\perp + p_\parallel/2$ in (75), (80), and (84) while $3p$ should be replaced by $2p_\perp + p_\parallel$ in (82). Similar results appear to hold for the guiding center theory, although they have so far only been effectively determined in certain limiting situations. The reader is referred to the literature for details (Bernstein et al., 1958; Kulsrud, 1962).

Flux frozen in plasma

Probably the most useful of the intuitive concepts implied by the ideal MHD equations, as well as the guiding center theory and the double adiabatic theory, is that concerning the magnetic flux lines frozen in the plasma. Precisely stated, the flux conserving theorem is as follows:

Assume that at some initial time t_0 magnetic lines of force are drawn throughout the plasma volume in such a way that their density is proportional to the field strength B, and they are everywhere tangent to $\boldsymbol{B}$. (For simplicity we take a finite but very large number of such lines so their density is not precisely determined at each point but can be defined to any desired precision by taking a sufficiently large number of such lines.) Then at time t_0 the magnetic field $\boldsymbol{B}$ is completely represented by these lines. Let the plasma flow with velocity $\boldsymbol{U}$ and let the magnetic field evolve according to (10). At the same time let the lines of force be bodily transported by this velocity U to some new configuration, just as though they were "frozen" in the plasma. Then, at any later time t, the configuration of the lines at that time will represent the magnetic field at that time both as to field strength given by line density, and direction given by the tangents to the lines.

This theorem holds true to the extent that (10) does. That is, if $\boldsymbol{B}$ deviates from the field given by (10) due to finite resistivity, it will deviate from the field given by the line configuration to exactly the same extent. Since the displacement of the lines evolves in a continuous manner, their topology must be preserved. Closed lines remain closed, ergodic lines remain ergodic, magnetic surfaces existing at time t_0 continue to exist, etc. This flux-freezing concept is often a very critical one and it is important to know under what conditions it can be broken. The plasma can occasionally be kept from reaching a state of much lower magnetic energy by this constraint alone. A change in topology which may be produced by a breakdown in (10) over a very small region, say near an X point, could conceivably lead to a large conversion of magnetic energy to kinetic energy in a plasma. This possibility is usually termed the reconnection problem and it is a problem of great interest since its resolution could conceivably lead to an explanation for certain observed violent plasma behavior such as disruption in tokomaks, solar flares, etc.

There are two mathematical ways to express the theorem of flux freezing. The first is the Lundqvist identity, while the second makes use of the Clebsch formula (Lundqvist, 1951).

The Lundqvist identity expresses the magnetic field at time t and position $\boldsymbol{r}$ in terms of its value at time t_0 and a different position $\boldsymbol{r}_0$

$$\boldsymbol{B}(\boldsymbol{r},t)/\rho = [\boldsymbol{B}(\boldsymbol{r}_0,t_0)/\rho]\cdot\nabla_0\boldsymbol{r}(\boldsymbol{r}_0,t). \tag{87}$$

In this formula $\boldsymbol{r}$ is understood to be a function of $\boldsymbol{r}_0$ and t which represents the position of the fluid element at time t that occupied the position $\boldsymbol{r}_0$ at initial time t_0. The subscript 0 on ∇_0 indicates that derivatives are to be taken with respect to $\boldsymbol{r}_0$ at fixed t. Let $\boldsymbol{B}_0$ and ρ_0 represent $\boldsymbol{B}(\boldsymbol{r}_0,t_0)$ and $\rho(\boldsymbol{r}_0,t_0)$ respectively. To establish the validity of (87) it is first shown that it satisfies (10). Making use of $(\partial\boldsymbol{r}/\partial t)_{r_0}=\boldsymbol{U}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{B}}{\rho}\right)=\frac{\boldsymbol{B}_0}{\rho_0}\cdot\nabla_0\boldsymbol{U}, \tag{88}$$

where $\mathrm{d}/\mathrm{d}t=\partial/\partial t+\boldsymbol{U}\cdot\nabla\equiv(\partial/\partial t)_{r_0}$. Also

$$\nabla\times(\boldsymbol{U}\times\boldsymbol{B})=\boldsymbol{B}\cdot\nabla\boldsymbol{U}-\boldsymbol{U}\cdot\nabla\boldsymbol{B}-\boldsymbol{B}\,\nabla\cdot\boldsymbol{U},$$

so

$$\begin{aligned}\frac{\partial\boldsymbol{B}}{\partial t}-\nabla\times(\boldsymbol{U}\times\boldsymbol{B})&=\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}-\boldsymbol{B}\cdot\nabla\boldsymbol{U}+\boldsymbol{B}\,\nabla\cdot\boldsymbol{U}\\&=\frac{\rho}{\rho_0}(\boldsymbol{B}_0\cdot\nabla_0\boldsymbol{U})+\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\boldsymbol{B}-\boldsymbol{B}\cdot\nabla\boldsymbol{U}-\frac{\boldsymbol{B}}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\\&=\frac{\rho}{\rho_0}[\boldsymbol{B}_0\cdot\nabla_0\boldsymbol{U}-(\boldsymbol{B}_0\cdot\nabla_0\boldsymbol{r})\cdot\nabla\boldsymbol{U}],\end{aligned} \tag{89}$$

where the first line follows from the definition of $\mathrm{d}/\mathrm{d}t$, the second line from (88) and (1) and the third from substitution of (87) for $\boldsymbol{B}$. The bracket in the third line of (89) vanishes because of the chain rule for differentiation. Thus, (87) satisfies (10) for the evolution of the magnetic field and is valid initially, so it remains valid for all t. Its relation to flux freezing can be seen geometrically. $(\boldsymbol{B}_0\cdot\nabla_0\boldsymbol{r})/B_0$ is the shearing of a unit line element along the initial line of force by the flow, so (87) states that $\boldsymbol{B}$ continues to be parallel to the sheared line element. Also the line has been lengthened by the same shear flow, but factor ρ/ρ_0 represents the decrease in volume. This combined with the lengthening of the line element gives the shrinking of the cross-sectional area which thus represents the amplification of the density of the lines of force.

The other alternative mathematical method for describing flux conservation involves the Clebsch formula for expressing an arbitrary divergence-free vector field such as $\boldsymbol{B}$ in terms of two scalar functions

$$\boldsymbol{B}=\nabla\alpha\times\nabla\beta. \tag{90}$$

If such α and β scalars exist, $\boldsymbol{B}$ given by (90) clearly is divergence-free. Further, dotting (90) with $\nabla\alpha$ and with $\nabla\beta$ gives

$$\boldsymbol{B}\cdot\nabla\alpha=0,\qquad \boldsymbol{B}\cdot\nabla\beta=0, \tag{91}$$

so α and β are constants along lines of force and, indeed, a general line of force can be determined by $\alpha=\alpha_0$, $\beta=\beta_0$ where α_0 and β_0 are constants. Lastly because $J=(\boldsymbol{B}\cdot\nabla\alpha\times\nabla\beta/B)=B$ is the Jacobian for a transformation from coordinates $\boldsymbol{r}$ to coordinates α, β, l, where l is arc length along the lines, we can see that $\mathrm{d}\alpha\,\mathrm{d}\beta$ represents the element of flux. That is, if we parameterize a surface S cutting the lines by α and β then $\mathrm{d}\alpha\,\mathrm{d}\beta$ is the flux through the corresponding element of area (Fig. 1.4.2). Thus, if we select the lines of force by a uniform distribution of values of α and β, their density will be proportional to the magnetic field strength B.

The above properties of α and β show how they can actually be found to satisfy (90). As in Fig. 1.4.2, choose α and β' arbitrarily on S and extend them through all space so as to satisfy (91) and $\boldsymbol{B}\cdot\nabla\beta'=0$, i.e. by keeping them constant on $\boldsymbol{B}$ lines. Then

$$\boldsymbol{B}\times(\nabla\alpha\times\nabla\beta')=\boldsymbol{B}\cdot\nabla\beta'\,\nabla\alpha-\boldsymbol{B}\cdot\nabla\alpha\,\nabla\beta'=0,$$

so

$$\boldsymbol{B}=g(\nabla\alpha\times\nabla\beta'),$$

where g is a scalar. From $\nabla\cdot\boldsymbol{B}=0$ we have

$$(\nabla\alpha\times\nabla\beta')\cdot\nabla g=(\boldsymbol{B}\cdot\nabla g)/g=0,$$

so g is constant along $\boldsymbol{B}$ lines, and thus a function of α and β', $g=g(\alpha,\beta')$. Now choose β to satisfy

$$\partial\beta/\partial\beta'=g(\alpha,\beta'). \tag{92}$$

Then for this α and β (90) is easily verified.

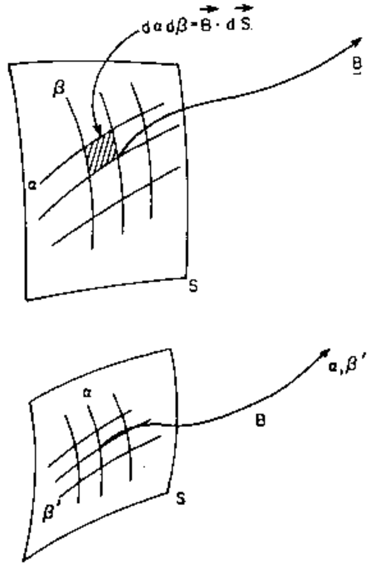

Fig. 1.4.2. Clebsch coordinates α and β.

Now α and β are clearly not unique. However, once they are chosen to represent $\boldsymbol{B}$ at some initial time t_0, they can be chosen at any later time by demanding they stay constant on any fluid element; that is, they satisfy

$$\partial\alpha/\partial t + \boldsymbol{U}\cdot\nabla\alpha = 0, \tag{92a}$$

$$\partial\beta/\partial t + \boldsymbol{U}\cdot\nabla\beta = 0. \tag{92b}$$

Then $\boldsymbol{B}$ as given by (90) satisfies (10) and thus continues to give the magnetic field. For

$$\begin{aligned}
&\frac{\partial}{\partial t}(\nabla\alpha\times\nabla\beta) - \nabla\times[\boldsymbol{U}\times(\nabla\alpha\times\nabla\beta)]\\
&= \nabla\frac{\partial\alpha}{\partial t}\times\nabla\beta + \nabla\alpha\times\nabla\frac{\partial\beta}{\beta t} - \nabla\times[\boldsymbol{U}\cdot\nabla\beta\,\nabla\alpha - \boldsymbol{U}\cdot\nabla\alpha\,\nabla\beta]\\
&= -\nabla(\boldsymbol{U}\cdot\nabla\alpha)\times\nabla\beta - \nabla\alpha\times\nabla(\boldsymbol{U}\cdot\nabla\beta)\\
&\quad - \nabla(\boldsymbol{U}\cdot\nabla\beta)\times\nabla\alpha + \nabla(\boldsymbol{U}\cdot\nabla\alpha)\times\nabla\beta = 0,
\end{aligned}$$

where the second line follows from expanding out of the triple vector product in the bracket in the first line, while the third line follows from (92) and taking the curl of the bracket in the second line.

The properties of α and β clearly correspond to those of magnetic lines in the flux conservation theorem.

A constant of the motion of considerable recent interest is the "$\boldsymbol{B}\cdot\boldsymbol{A}$ invariant" of Taylor (1974). It is closely related to the linkage of magnetic flux. Consider the

integral

$$K = \int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}\tau \tag{93}$$

where the integral is taken over a bounded fixed volume V at which $\boldsymbol{B}$ is tangential. This integral is gauge invariant. If $\boldsymbol{A}$ is replaced by $\boldsymbol{A} + \nabla\chi$, then the change induced in K is

$$\delta K = \int \boldsymbol{B}\,\nabla\chi\,\mathrm{d}\tau = \int \nabla\cdot(\boldsymbol{B}\chi)\,\mathrm{d}\tau = \int \boldsymbol{B}\cdot\boldsymbol{n}\chi\,\mathrm{d}S = 0 \tag{94}$$

since $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ on the surface. (This argument assumes the vector potential and gauge are changed throughout all space, not just in V, so that we can be certain that χ is single-valued.) Select a gauge with $\boldsymbol{E} = -(1/c)(\partial\boldsymbol{A}/\partial t)$. Then the rate of change of K with time is given by

$$\begin{aligned}
\frac{\mathrm{d}K}{\mathrm{d}t} &= \int\left(\frac{\partial\boldsymbol{A}}{\partial t}\cdot(\nabla\times\boldsymbol{A}) + \boldsymbol{A}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\right)\mathrm{d}\tau\\
&= \int\left[-\nabla\cdot\left(\frac{\partial\boldsymbol{A}}{\partial t}\times\boldsymbol{A}\right) + \frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\right]\mathrm{d}\tau\\
&= -\int \mathrm{d}S\boldsymbol{n}\cdot\left(\frac{\partial\boldsymbol{A}}{\partial t}\times\boldsymbol{A}\right) + \int 2\boldsymbol{A}\cdot\nabla\times(\boldsymbol{U}\times\boldsymbol{B})\,\mathrm{d}\tau\\
&= +c\int \mathrm{d}S(\boldsymbol{n}\times\boldsymbol{E})\cdot\boldsymbol{A} - 2\int\nabla\cdot[\boldsymbol{A}\times(\boldsymbol{U}\times\boldsymbol{B})]\,\mathrm{d}\tau + 2\int\boldsymbol{B}\cdot(\boldsymbol{U}\times\boldsymbol{B})\,\mathrm{d}\tau\\
&= \int \mathrm{d}S\cdot(\boldsymbol{n}\times\boldsymbol{A})\cdot(\boldsymbol{U}\times\boldsymbol{B}) = 0
\end{aligned} \tag{95}$$

where (10) has been employed in the third line; the surface term vanishes in the fourth line because the tangential component of $\boldsymbol{E}$ vanishes on an infinite conducting surface. Thus, K is a constant of the motion for an ideal plasma.

The physical significance of K is that it represents the amount of flux linkage of a field, for example the amount of linkage of toroidal and poloidal flux in toroidal geometry (Kruskal and Kulsrud, 1958). Thus, it is not really an independent constant of the motion but expresses a topological quantity related to line and flux conservation. However, Taylor (1974) has pointed out that K is actually not changed by certain resistive instabilities and reconnection phenomena so that it is actually a more general constant, of considerable importance.

1.4.5. An example

The guiding center formalism will be illustrated by an example which will also bring out the limitations of the double adiabatic formalism.

Consider a homogeneous, magnetized, ion–electron plasma with unequal perpendicular and parallel temperatures. Take the uniform field $\boldsymbol{B}_0$ in the z direction.

For simplicity take the equilibrium distribution to be a bi-Maxwellian with unequal perpendicular and parallel temperatures:

$$f_{0s}=\frac{n}{(2\pi m_s)^{3/2}T_{\perp s}T_{\|s}^{1/2}}\exp\left(-\frac{m_s v_\perp^2}{2T_{\perp s}}-\frac{m_s v_\|^2}{2T_{\|s}}\right). \tag{96}$$

Consider a sinusoidal perturbation of this plasma proportional to $\exp(-\mathrm{i}\omega t+\mathrm{i}k_x x+\mathrm{i}k_z z)$. Under what conditions is this perturbation unstable?

If the plasma displacement $\boldsymbol{\xi}$, with $U=-\mathrm{i}\omega\boldsymbol{\xi}$ is introduced into the fluid equations (46) and (47), then these become

$$-\rho\omega^2\boldsymbol{\xi}=-\nabla'\boldsymbol{P}_1-\frac{1}{4\pi}\nabla(\boldsymbol{B}_0\cdot\boldsymbol{B}_1)+\frac{\boldsymbol{B}_0\cdot\nabla\boldsymbol{B}_1}{4\pi}, \tag{97}$$

$$\boldsymbol{B}_1=\mathrm{i}k_z\xi_x B_0\hat{\boldsymbol{x}}-\mathrm{i}k_x\xi_x B_0\hat{\boldsymbol{z}}, \tag{98}$$

where the subscript or superscript 1 indicates perturbed quantities. From (44a) the perturbed pressure is given by:

$$\boldsymbol{P}_1=p_\perp'\boldsymbol{I}+(p_\|'-p_\perp')\boldsymbol{bb}+(p_\|-p_\perp)(\boldsymbol{b}_1\boldsymbol{b}+\boldsymbol{bb}_1). \tag{99}$$

Now from (98)

$$B_1=-\mathrm{i}k_x\xi_x B_0, \tag{100a}$$

$$\boldsymbol{b}_1=\mathrm{i}k_z\xi_x\hat{\boldsymbol{x}}, \tag{100b}$$

so

$$\nabla\cdot\boldsymbol{P}_1=\left[\mathrm{i}k_x p_\perp'-(p_\|-p_\perp)k_z^2\xi_x\right]\hat{\boldsymbol{x}}+\left[\mathrm{i}k_z p_\|'-(p_\|-p_\perp)k_x k_z\xi_x\right]\hat{\boldsymbol{z}}. \tag{101}$$

Substituting (101) in the equation of motion (97) and taking the x and z components gives two equations:

$$-\rho\omega^2\xi_x=-\mathrm{i}k_x p_\perp'+k_z^2(p_\|-p_\perp)\xi_x-(k_x^2+k_z^2)(B_0^2/4\pi)\xi_x, \tag{102a}$$

$$-\rho\omega^2\xi_z=-\mathrm{i}k_z p_\|'+k_x k_z(p_\|-p_\perp)\xi_x, \tag{102b}$$

for ξ_x and ξ_z. In order to complete the system equations of state for $p_\perp'$ and $p_\|'$ are needed.

Up to this point the double adiabatic theory and the guiding center theory coincide. They differ as to the determination of $p_\perp'$ and $p_\|'$, however. First the equations are completed by invoking the two equations of state, (56a) and (56b), of the double adiabatic theory, to express $p_\perp'$ and $p_\|'$ in terms of ξ_x and ξ_z. Since from the continuity equation (48) $\rho_1=-\mathrm{i}(k_x\xi_x+k_z\xi_z)$, then from (100a)

$$\frac{p_\perp'}{p_\perp}=\frac{\rho_1}{\rho}+\frac{B_1}{B_0}=-2\mathrm{i}k_x\xi_x-\mathrm{i}k_z\xi_z, \tag{103a}$$

$$\frac{p_\|'}{p_\|}=\frac{3\rho_1}{\rho}-\frac{2B_1}{B_0}=-\mathrm{i}k_x\xi_x-3\mathrm{i}k_z\xi_z. \tag{103b}$$

Substitution of (103a) and (103b) in (102a) and (102b) yields two equations for ξ_x and ξ_z alone. Setting the determinant of these equations to zero gives the eigenvalue equation for ω

$$\left[\rho\omega^2-\left((2k_x^2+k_z^2)p_\perp+\frac{k^2B_0^2}{4\pi}-k_z^2p_\|\right)\right](\rho\omega^2-3k_z^2p_\|)=k_x^2k_z^2p_\perp^2. \tag{104}$$

It is easy to see that the roots of ω^2 are real. We have instability if one of the roots for ω^2 is negative and the condition for this is

$$2k_x^2\left[\frac{B_0^2}{8\pi}+p_\perp\left(1-\frac{p_\perp}{6p_\|}\right)\right]+k_z^2\left(\frac{B_0^2}{4\pi}+p_\perp-p_\|\right)<0.$$

This is negative if $k_x=0$ and

$$p_\|>p_\perp+B_0^2/4\pi, \tag{105}$$

the "fire hose instability", or $k_z\to0$ (it must not vanish) and

$$p_\perp^2/6p_\|>B^2/8\pi+p_\perp, \tag{106}$$

the "mirror instability". Equations (105) and (106) are the stability results derived from double adiabatic theory.

The guiding center theory is now used to find $p_\perp'$ and $p_\|'$ and to complete (102a) and (102b). Actually $p_\perp'$ can be determined from ξ_x alone and only (102a) need be considered. $p_\perp'$ is found from f' which is given by solving (51), for example. Let $f=f_0+f_1$. Then, since B is the Jacobian of the transformation to μ, $v_\|$ variables,

$$p_\perp=\sum_s m_s\int f_s\mu B(B\,\mathrm{d}\mu)\,\mathrm{d}v_\|\,\mathrm{d}\phi,$$

and

$$p_\perp'=\sum_s m_s\int f_s'\mu B\,\mathrm{d}^3v+\frac{2B_1}{B_0}p_\perp. \tag{107}$$

Perturbing (51) and using (96) gives

$$f_{1s}=\frac{\left[-k_xk_z\xi_x(v_\perp^2/2)+(e_s/m_s)E_\|\right]}{-\mathrm{i}\omega+\mathrm{i}k_zv_\|}\frac{m_sv_\|}{T_{\|s}}f_s. \tag{108}$$

Near the marginal point for stability, ω may be neglected in the denominator [if $\omega\ll k_z(T/m)^{1/2}$] and

$$f_{1s}=\mathrm{i}k_x\xi_x\frac{m_sv_\perp^2}{2T_{\|s}}f_s-\frac{\mathrm{i}e_sE_\|}{k_zT_{\|s}}f_s. \tag{109}$$

Now from charge neutrality $E_\|$ can be determined to be

$$E_\|=\frac{k_z}{e}(k_x\xi_x)\frac{(T_\perp/T_\|)_\mathrm{i}-(T_\perp/T_\|)_\mathrm{e}}{(1/T_{\|\mathrm{i}})-(1/T_{\|\mathrm{e}})}. \tag{110}$$

For simplicity, $(T_\perp/T_\parallel)_i$ is taken to equal $(T_\perp/T_\parallel)_e$ so that $E_\parallel = 0$. Substituting (109) (with $E_\parallel = 0$) into (107) and making use of (100a) gives

$$p'_\perp = 2ik_x\xi_x \sum_s \left(\frac{T_\perp^2}{T_\parallel}\right)_s n - 2ik_x\xi_x p_\perp, \qquad (111)$$

and if, further, $T_{\perp i}$ is taken to equal $T_{\perp e}$

$$p'_\perp = 2ik_x\xi_x\left(\frac{p_\perp^2}{p_\parallel} - p_\perp\right). \qquad (112)$$

Then for sufficiently small ω (see above), from (102a).

$$\rho\omega^2 = 2k_x^2\left(p_\perp + \frac{B_0^2}{8\pi} - \frac{p_\perp^2}{p_\parallel}\right) + k_z^2\left(p_\perp + \frac{B_0^2}{4\pi} - p_\parallel\right).$$

Again we have the fire hose instability if $k_x = 0$ and (105) is satisfied. However, the condition for the mirror instability is changed to $k_z \to 0$ and

$$\left(p_\perp^2/p_\parallel\right) > p_\perp + \left(B_0^2/8\pi\right),$$

a criterion differing substantially from (106) (by a factor of 6).

The reason for the different criteria for the guiding center theory of the mirror instability and the double adiabatic theory is that ω must pass through zero so that particle communication sets in over a distance k^{-1} along the lines in a time short compared with ω^{-1}, so the condition necessary for the validity of the latter theory fails.

This example illustrates the dangers inherent in the double adiabatic theory, since the failure of the validity conditions to hold really only becomes evident after the more accurate guiding center theory is carried out. The fire hose instability theory remains valid since, as can be seen from intuitive picture of the instability, parallel heat flow plays no role.

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