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A weak turbulence theory for incompressible magnetohydrodynamics

S. GALTIER,[1,2] S. V. NAZARENKO,[1] A. C. NEWELL[1] and A. POUQUET[2]

[1]Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

[2]Observatoire de la Côte d'Azur, CNRS UMR 6529, BP 4229, 06304 Nice Cedex 4, France

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Abstract. We derive a weak turbulence formalism for incompressible magnetohydrodynamics. Three-wave interactions lead to a system of kinetic equations for the spectral densities of energy and helicity. The kinetic equations conserve energy in all wavevector planes normal to the applied magnetic field $B_0\,\hat{\mathbf{e}}_\parallel$. Numerically and analytically, we find energy spectra $E^\pm \sim k_\perp^{n_\pm}$, such that $n_+ + n_- = -4$, where $E^\pm$ are the spectra of the Elsässer variables $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$ in the two-dimensional case ($k_\parallel = 0$). The constants of the spectra are computed exactly and found to depend on the amount of correlation between the velocity and the magnetic field. Comparison with several numerical simulations and models is also made.

1. Introduction and general discussion

Magnetohydrodynamic (MHD) turbulence plays an important role in many astrophysical situations (Parker 1994), ranging from the solar wind (Marsch and Tu 1994), to the Sun (Priest 1982), the interstellar medium (Heiles et al. 1993) and beyond (Zweibel and Heiles 1997), as well as in laboratory devices such as tokamaks (see e.g. Wild et al. 1981; Taylor 1986, 1993; Gekelman and Pfister 1988). A very instrumental step in recognizing some of the features that distinguished MHD turbulence from hydrodynamic turbulence was taken independently in the early 1960s by Iroshnikov (1963) and Kraichnan (1965) (hereinafter IK). They argued that the destruction of phase coherence by Alfvén waves travelling in opposite directions along local large-eddy magnetic fields introduces a new time scale and a slowing down of energy transfer to small scales. They pictured the scattering process as being principally due to three-wave interactions. Assuming three-dimensional isotropy, dimensional analysis then leads to the prediction of a $k^{-3/2}$ Kolmogorov finite energy flux spectrum.

However, it is clear, and it has been a concern to Kraichnan and others throughout the years, that the assumption of local 3D isotropy is troublesome. Indeed, numerical simulations and experimental measurements both indicate that the presence of strong magnetic fields makes MHD turbulence strongly anisotropic. Anisotropy is manifested in a two-dimensionalization of the turbulence spectrum in a plane transverse to the locally dominant magnetic field and in inhibiting spectral energy transfer along the direction parallel to the field (Montgomery and Turner 1981; Montgomery and Matthaeus 1995; Matthaeus et al. 1996; Kinney and McWilliams 1998). Replacing the 3D isotropy assumption by a 2D one, and retaining the rest of the IK picture, leads to the dimensional analysis prediction of a $k_\perp^{-2}$ spectrum ($\mathbf{B}_0 = B_0\,\hat{\mathbf{e}}_\|$, the applied magnetic field, $k_\| = \mathbf{k}\cdot\hat{\mathbf{e}}_\|$, $\mathbf{k}_\perp = \mathbf{k} - k_\|\,\hat{\mathbf{e}}_\|$, $k_\perp = |\mathbf{k}_\perp|$) (Goldreich and Sridhar 1997; Ng and Bhattacharjee 1997).

A major controversy in the debate over the universal features of MHD turbulence was introduced by Sridhar and Goldreich (1994) (hereinafter SG). Following IK, they assumed that small-scale MHD turbulence can be described as a large ensemble of weakly interacting Alfvén waves within the framework of weak turbulence theory. However, SG challenged that part of the IK thinking that viewed Alfvén-wave scattering as a three-wave interaction process, an assumption implicit in the IK derivation of the $k^{-3/2}$ spectrum. SG argue that, in the inertial range where amplitudes are small, significant energy exchange between Alfvén waves can only occur for resonant three-wave interactions. Moreover, their argument continues, because one of the fluctuations in such a resonant triad has zero Alfvén frequency, the three-wave coupling is empty. They conclude therefore that the long-time dynamics of weak MHD fields are determined by four-wave resonant interactions.

This conclusion is false. In this paper, we shall show that resonant three-wave interactions are non-empty (see also Montgomery and Matthaeus 1995; Ng and Bhattacharjee 1996) and lead to a relaxation to universal behaviour and significant spectral energy redistribution. Moreover, weak turbulence theory provides a set of closed kinetic equations for the long-time evolution of the eight power spectra (to be defined below, in (21) and (22)), corresponding to total energy $e^s(\mathbf{k})$, poloidal energy $\Phi^s(\mathbf{k})$, and magnetic and pseudomagnetic helicities $R^s(\mathbf{k})$ and $I^s(\mathbf{k})$ constructed from the Elsässer fields $\mathbf{z}^s = \mathbf{v}+s\mathbf{b}$, $s = \pm 1$, where $\mathbf{v}$ and $\mathbf{b}$ are the fluctuating velocity and Alfvén velocity respectively. The latter is defined such that $\mathbf{b} = \mathbf{B}(\mu_0\rho_0)^{-1/2}$, where ρ_0 is the uniform density and μ_0 the magnetic permeability. We shall also show that a unique feature of Alfvén-wave weak turbulence is the existence of additional conservation laws. One of the most important of these is the conservation of energy on all wavevector planes perpendicular to the applied field $\mathbf{B}_0$. There is no energy transfer between planes. This extra symmetry means that relaxation to universal behaviour only takes place as a function of $k_\perp$, so that, in the inertial range (or window of transparency), $e^s(\mathbf{k}) = f(k_\|)k_\perp^{-p}$ where $f(k_\|)$ is non-universal.

Because weak turbulence theory for Alfvén waves is not straightforward and because of the controversy raised by SG, it is important to discuss carefully and understand clearly some of the key ideas before outlining the main results. We therefore begin by giving an overview of the theory for the statistical initial-value problem for weakly nonlinear MHD fields.

1.1. *Alfvén weak turbulence: kinematics, asymptotic closure and some results*

Weak turbulence theory is an approach that is widely familiar to the plasma physics community (see e.g. Vedenov 1967, 1968; Sagdeev and Galeev 1969; Tsytovich 1970; Kuznetsov 1972, 1973; Zakharov 1974, 1984; Akhiezer et al. 1975; McIvor 1977; Achterberg 1979; Zakharov et al. 1992). This approach considers statistical states that can be viewed as large ensembles of weakly interacting waves and that can be described by a kinetic equation for the wave energy. Recall that the IK theory considers large ensembles of weakly interacting Alfvén waves, but IK do not derive a kinetic equation and they restrict themselves to phenomenology based on the dimensional argument. Ng and Bhattacharjee (1996) developed a theory of weakly interacting Alfvén wavepackets that takes into account anisotropy and leads to certain predictions for the turbulence spectra based on some additional phenomenological assumptions and bypassing derivation of the weak turbulence kinetic equations. To date, there exists no rigorous theory of weak Alfvén turbulence in incompressible MHD, and the derivation of such a theory via a systematic asymptotic expansion in powers of small nonlinearity is the main goal of the present paper. It is interesting that the kinetic equations were indeed derived (in some limits) for the Alfvén waves for the cases when effects such as finite Larmor radius (Mikhailovskii et al. 1989) or compressibility (Kaburaki and Uchida 1971; Kuznetsov 1973) cause these waves to be dispersive. Perhaps the main reason why such a theory has not been developed for non-dispersive Alfvén waves in incompressible magnetofluids was a general understanding within the 'weak turbulence' community that a consistent asymptotic expansion is usually impossible for non-dispersive waves. The physical reason for this is that all wavepackets propagate with the same group velocity even if their wavenumbers are different. Thus, no matter how weak the nonlinearity is, the energy exchanged between the wavepackets will be accumulated over a long time and it may not be considered small, as would be required in weak turbulence theory. As we shall show in this paper, the Alfvén waves represent a unique exception to this rule. This arises because the nonlinear interaction coefficient for co-propagating waves is null, whereas the counter-propagating wavepackets pass through each other in a finite time and exchange only small amounts of energy, which makes the weak turbulence approach applicable in this case. Because of this property, the theory of weak Alfvén turbulence that is going to be developed in this paper possesses a novel and interesting mathematical structure that is quite different from the classical weak turbulence theory of dispersive waves.

The starting point in our derivation is a kinematic description of the fields. We assume that the Elsässer fields $\mathbf{z}^s(\mathbf{x}, t)$ are random, homogeneous, zero-mean fields in the three spatial coordinates $\mathbf{x}$. This means that the n-point correlation functions between combinations of these variables estimated at $\mathbf{x}_1, ..., \mathbf{x}_n$ depend only on the relative geometry of the spatial configuration. We also assume that, for large separation distances $|\mathbf{x}_i - \mathbf{x}_j|$ along *any* of the three spatial directions, fluctuations are statistically independent. We shall also discuss the case of strongly two-dimensional fields for which there is significant correlation along the direction of the applied magnetic field. We choose to use cumulants rather than moments, to which the cumulants are related by a one-to-one map. The choice is made for two reasons. The first is that they are exactly those combinations of moments that are asymptotically zero for all large separations. Therefore they have well-defined and (at least initially before long-distance correlations can be built up by nonlinear couplings) smooth Fourier transforms. We shall be particularly interested in the spectral densities

$$q^{ss'}_{jj'}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \langle z^s_j(\mathbf{x}) z^{s'}_{j'}(\mathbf{x}+\mathbf{r})\rangle \, e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \tag{1}$$

of the two-point correlations. (Remember that $z^s_j(\mathbf{x})$ has zero mean, so that the second-order cumulants and moments are the same.) The second reason for the choice of cumulants as dependent variables is that, for joint Gaussian fields, all cumulants above second order are identically zero. Moreover, because of linear wave propagation, initial cumulants of order three and higher decay to zero on a time scale $(b_0 k_\|)^{-1}$, where $\mathbf{b}_0 = \mathbf{B}_0(\mu_0\rho_0)^{-1/2}$ is the Alfvén velocity ($b_0 = |\mathbf{b}_0|$) and $k_\|^{-1}$ a dominant parallel length scale in the initial field. This is a simple consequence of the Riemann–Lebesgue lemma; all Fourier space cumulants become multiplied by fast non-vanishing oscillations because of linear wave properties, and these oscillations give rise to cancellations upon integration. Therefore the statistics approaches a state of joint Gaussianity. The amount by which it differs, and the reason for a non-trivial relaxation of the dynamics, is determined by the long-time cumulative response generated by nonlinear couplings of the waves. The special manner in which third and higher cumulants are regenerated by nonlinear processes leads to a natural asymptotic closure of the statistical initial-value problem.

Basically, because of the quadratic interactions, third-order cumulants (equal to third-order moments) are regenerated by fourth-order cumulants and binary products of second-order ones. But the only long-time contributions arise from a subset of the second-order products that lie on certain resonant manifolds defined by zero divisors. It is exactly these terms that appear in the kinetic equations describing the evolution of the power spectra of second-order moments over time scales $(\epsilon^2 b_0 k_\parallel)^{-1}$. Here ϵ is a measure of the strength of the nonlinear coupling. Likewise, higher-order cumulants are nonlinearly regenerated principally by products of lower-order cumulants. Some of these small-divisor terms contribute to frequency renormalization, and others contribute to further (e.g. four-wave resonant interactions) corrections of the kinetic equations over longer times.

What are the resonant manifolds for three-wave interactions and, in particular, what are they for Alfvén waves ? They are defined by the divisors of a system of weakly coupled wavetrains $a_j e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_s(k_j)t)}$, with $\omega_s(k_j)$ the linear wave frequency and s its level of degeneracy, which undergo quadratic coupling. One finds that triads $\mathbf{k}$, $\boldsymbol{\kappa}$, $\mathbf{L}$ which lie on the resonant manifold defined for some choice of s, s', s'' by

$$\mathbf{k} = \boldsymbol{\kappa} + \mathbf{L}, \tag{2a}$$

$$\omega_s(\mathbf{k}) = \omega_{s'}(\boldsymbol{\kappa}) + \omega_{s''}(\mathbf{L}), \tag{2b}$$

interact strongly (cumulatively) over long times $(\epsilon^2 \omega_0)^{-1}$, ω_0 being a typical frequency. For Alfvén waves, $\omega_s(\mathbf{k}) = s\mathbf{b}_0 \cdot \mathbf{k} = sb_0 k_\parallel$ when $s = \pm 1$ (Alfvén waves of a given wavevector can travel in one of two directions) and b_0, the Alfvén velocity, is the strength of the applied field. Given the dispersion relation $\omega = s\mathbf{b_0} \cdot \mathbf{k}$, one might ask why there is any weak turbulence for Alfvén waves at all, because, for $s = s' = s''$, (2) is satisfied for all triads. Furthermore, in that case, the fast oscillations multiplying the spectral cumulants of order $N + 1$ in the evolution equation for the spectral cumulant of order N disappear, so that there is no cancellation (phase mixing) and therefore no natural asymptotic closure. However, the MHD wave equations have the property that the coupling coefficient for this interaction is identically zero, and therefore the only interactions of importance occur between oppositely travelling waves, where $s' = -s$, $s'' = s$. In this case, (2) becomes

$$2s\mathbf{b}_0 \cdot \boldsymbol{\kappa} = 2sb_0\kappa_\parallel = 0. \tag{3}$$

The third wave in the triad interaction is a fluctuation with zero Alfvén frequency. SG conclude incorrectly that the effective amplitude of this zero mode is zero and that therefore the resonant three-wave interactions are null.

Although some of the kinetic equations will involve a principal-value integral (PVI) with denominator $s\omega(\mathbf{k}) + s\omega(\boldsymbol{\kappa}) - s\omega(\mathbf{k} - \boldsymbol{\kappa}) = 2sb_0\kappa_\parallel$, whose meaning we discuss later, the majority of the terms contain the Dirac delta functions of this quantity. The equation for the total energy density contains only the latter, implying that energy exchange takes place by resonant interactions. Both the resonant delta functions and PVI arise from taking long-time limits $t \to \infty$, with $\epsilon^2 t$ finite, of integrals of the form

$$\int F(k_\|, \epsilon^2 t)(e^{(2isb_0k_\| t)} - 1)(2isb_0k_\|)^{-1}\, dk_\|$$
$$\sim \int F(k_\|, \epsilon^2 t)\left[\pi \mathrm{sgn}(t)\, \delta(2sb_0k_\|) + i\mathscr{P}(\frac{1}{2sb_0k_\|})\right] dk_\| . \quad (4)$$

Therefore implicit in the derivation of the kinetic equations is the assumption that $F(k_\|, \epsilon^2 t)$ is relatively smooth near $k_\| = 0$, so that $F(k_\|, \epsilon^2 t)$ remains nearly constant for $k_\| \sim \epsilon^2$. In particular, the kinetic equation for the total energy density

$$e^s(\mathbf{k}_\perp, k_\|) = \sum_{j=1}^{3} q_{jj}^{ss}(\mathbf{k}_\perp, k_\|) \quad (5)$$

is the integral over $\boldsymbol{\kappa}_\perp$ of a product of a combination of $q_{jj'}^{ss}(\mathbf{k}_\perp - \boldsymbol{\kappa}_\perp, k_\|)$ with $Q^{-s}(\boldsymbol{\kappa}_\perp, 0) = \Sigma_{p,m} k_p k_m\, q_{pm}^{-s-s}(\boldsymbol{\kappa}_\perp, 0)$. Three observations (O1,2,3) and two questions (Q1,2) arise from this result.

O1. Unlike the cases for most systems of dispersive waves, the resonant manifolds for Alfvén waves *foliate* wavevector space. For typical dispersion relations, a wavevector $\boldsymbol{\kappa}$, lying on the resonant manifold of the wavevector $\mathbf{k}$, will itself have a different resonant manifold, and members of that resonant manifold will again have different resonant manifolds. Indeed the union of all such manifolds will fill $\mathbf{k}$ space so that energy exchange occurs throughout $\mathbf{k}$ space.

O2. In contrast, for Alfvén waves, the kinetic equations for the total energy density contain $k_\|$ as a parameter that identifies which wavevector plane perpendicular to $\mathbf{B}_0$ we are on. Thus the resonant manifold for all wavevectors of a given $\bar{k}_\|$ is the plane $k_\| = \bar{k}_\|$. The resonant manifolds foliate $\mathbf{k}$ space.

O3. Further, conservation of total energy holds for each $k_\|$ plane. There is energy exchange between energy densities having the same $k_\|$ value, but not between those having different $k_\|$ values. Therefore relaxation towards a universal spectrum with constant transverse flux occurs in wavevector planes perpendicular to the applied magnetic field. The dependence of the energy density on $k_\|$ is non-universal, and is inherited from the initial distribution along $k_\|$.

Q1. If the kinetic equation describes the evolution of power spectra for values of $k_\|$ outside a band of order ϵ^ξ, $\xi < 2$, then how does one define the evolution of the quantities contained in $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$ so as to close the system in $k_\|$?

Q2. Exactly what is $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$? Could it be effectively zero, as SG surmise? Could it be possibly singular with singular support located near $k_\| = 0$, in which case the limit (4) is suspect?

To answer the crucially important question Q2, we begin by considering the simpler example of a one-dimensional, stationary random signal $u(t)$ of zero mean. Its power spectrum is $f(\omega)$, the limit of the sequence

$$f_L(\omega) = \frac{1}{2\pi} \int_{-L}^{L} \langle u(t)u(t+\tau)\rangle e^{-i\omega\tau}\, d\tau,$$

which exists because the integrand decays to zero as $\tau \to \pm\infty$. Ergodicity and the stationarity of $u(t)$ allow us to estimate the average $R(\tau) = \langle u(t)u(t+\tau)\rangle$ by the biased estimator

$$R_L(\tau) = \frac{1}{2L}\int_{-L+|\tau|/2}^{L-|\tau|/2} u(t-\tau/2)u(t+\tau/2)\,dt,$$

with mean $E\{R_L(\tau)\} = (1-\tau/2L)R(\tau)$. Taking L sufficiently large and assuming a sufficiently rapid decay so that we can take $R_L(\tau) = 0$ for $|\tau| > 2L$ means that $R_L(\tau)$ is simply the convolution of the signal with itself. Furthermore, the Fourier transform $S_L(\omega)$ can then be evaluated as

$$\begin{aligned} S_L(\omega) &= \frac{1}{2\pi}\int_{-2L}^{2L} R_T(\tau)e^{-i\omega\tau}\,d\tau \\ &= \frac{1}{4\pi L}\left|\int_{-L}^{L} u(t)e^{-i\omega t}\,dt\right|^2. \end{aligned}$$

For sufficiently large L, the expected value of $S_L(\omega)$ is $S(\omega)$, the Fourier transform of $R(\tau)$, although the variance of this estimate is large. Nevertheless, $S_L(\omega)$, and in particular $S_L(0)$, is generally non-zero and measures the power in the low-frequency modes. To make the connection with Fourier space, we can think of replacing the signal $u(t)$ by the periodic extension of the truncated signal $\tilde{u}_L(t) = u(t)$ for $|t| < L$: $\tilde{u}_L(t+2L)$ for $|t| > L$. The zero mode of the Fourier transform

$$a_L(0) = \frac{1}{2L}\int_{-L}^{L} \tilde{u}(t)\,dt$$

is a non-zero random variable, and, while its expected value (for large L) is zero, the expected value of its square is certainly not zero. Indeed, the expected value of $S_L(0) = 2L\,a_L^2(0)$ has a finite non-zero value that, as $L \to \infty$, is independent of L, since $a_L(0)$ has zero mean and a standard deviation proportional to $(2L)^{-1/2}$. Likewise, for Alfvén waves, the power associated with the zero mode $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$ is non-zero, and, furthermore, for the class of three-dimensional fields in which correlations decay in all directions, $Q^{-s}(\boldsymbol{\kappa}_\perp, k_\parallel)$ is smooth near $k_\parallel = 0$. Therefore, for these fields, we may consider $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$ as a limit of $Q^{-s}(\boldsymbol{\kappa}_\perp, k_\parallel)$ as $k_\parallel/k_{\perp 0} \to 0$ and $\epsilon^{-2}k_\parallel/k_{\perp 0} \to \infty$. Here $k_{\perp 0}$ is some wavenumber near the energy-containing part of the inertial range. Therefore, in this case, we solve first the nonlinear kinetic equation for $\lim_{k_\parallel \to 0} e_k^s(k_\perp, k_\parallel)$, namely for very oblique Alfvén waves, and, having found the asymptotic time behaviour of $e^s(k_\perp, 0)$, then return to solve the equation for $e^s(k_\perp, k_\parallel)$ for finite $k_\parallel$.

Assuming isotropy in the transverse $k_\perp$ plane, we find universal spectra $c_n^s k_\perp^{n_s}$ for $E^s(k_\perp)$ ($\int E^s(k_\perp, 0)\,dk_\perp = \int e^s(k_\perp)\,d\mathbf{k}_\perp$), corresponding to finite fluxes of energy from low to high transverse wavenumbers. Then $e^s(k_\perp, k_\parallel) = f^s(k_\parallel)c_n^s k_\perp^{n_s - 1}$, where $f^s(k_\parallel)$ is not universal. These solutions each correspond to energy conservation. We find that convergence of all integrals is guaranteed for $-3 < n_\pm < -1$ and that

$$n_+ + n_- = -4, \tag{6}$$

which means that, for no directional preference, $n_+ = n_- = -2$. These solutions have finite energy, i.e. $\int E\,dk_\perp$ converges. If we interpret them as being set up by a constant flux of energy from a source at low $k_\perp$ to a sink at high $k_\perp$ then, since they have finite capacity and can only absorb a finite amount of energy, they must be set up in finite time. When we searched numerically for the evolution of initial states to the final state, we found a remarkable result that we do not yet fully understand. Each $E^s(k_\perp)$ behaves as a propagating front in the form $E^s(k_\perp) = (t_0 - t)^{1/2} E_0(k_\perp(t_0 - t)^{3/2})$ and $E_0(l) \sim l^{-7/3}$ as $l \to +\infty$. This means that for $t < t_0$, the $E^s(k_\perp)$ spectrum had a tail for $k_\perp < (t_0 - t)^{-3/2}$ with stationary form $k_\perp^{-7/3}$ joined to $k_\perp = 0$ through a front $E_0(k_\perp(t_0 - t)^{3/2})$. The 7/3 spectrum is steeper than the +2 spectrum. Amazingly, as t approached very closely to t_0, disturbances in the high-$k_\perp$ part of the $k_\perp^{-7/3}$ solution propagated back along the spectrum, rapidly turning it into the finite-energy flux spectrum $k_\perp^{-2}$. We understand neither the origin nor the nature of this transition solution, nor do we understand the conservation law involved with the second equilibrium solution of the kinetic equations. Once the connection to infinity is made, however, the circuit between source and sink is closed and the finite flux energy spectrum takes over.

Up to this point, we have explained how MHD turbulent fields for which correlations decay in all directions relax to quasiuniversal spectra via the scattering of high-frequency Alfvén waves with very oblique, low-frequency ones. But there is another class of fields that it is also important to consider. There are homogeneous, zero-mean random fields that have the anisotropic property that correlations in the direction of applied magnetic field do not decay with increasing separation $\mathbf{B}_0 \cdot (\mathbf{x}_1 - \mathbf{x}_2)$. For this case, we may think of decomposing the Elsässer fields as

$$z_j^s(\mathbf{x}_\perp, x_\parallel) = \bar{z}_j^s(\mathbf{x}_\perp) + \hat{z}_j^s(\mathbf{x}_\perp, x_\parallel), \tag{7}$$

where the $\hat{z}_j^s(\mathbf{x}_\perp, x_\parallel)$ have the same properties as the fields considered heretofore, but where the average of $z_j^s(\mathbf{x}_\perp, x_\parallel)$ over $x_\parallel$ is non-zero. The total average of z_j^s is still zero when one averages $\bar{z}_j^s(\mathbf{x}_\perp)$ over $\mathbf{x}_\perp$. In this case, it is not hard to show that correlations

$$\langle z_j^s(\mathbf{x}_\perp, x_\parallel)\, z_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp, x_\parallel + r_\parallel)\rangle$$

divide into two parts

$$\langle \bar{z}_j^s(\mathbf{x}_\perp)\, \bar{z}_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp)\rangle + \langle \hat{z}_j^s(\mathbf{x}_\perp, x_\parallel)\, \hat{z}_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp, x_\parallel + r_\parallel)\rangle,$$

with Fourier transforms

$$q_{jj'}^{ss'}(\mathbf{k}) = \delta(k_\parallel)\, \bar{q}_{jj'}^{ss'}(k_\perp) + \hat{q}_{jj'}^{ss'}(k_\perp, k_\parallel), \tag{8}$$

when $\bar{q}$ is smooth in $\mathbf{k}_\perp$ and $\hat{q}$ is smooth in both $\mathbf{k}_\perp$ and $k_\parallel$. The former is simply the transverse Fourier transform of the two-point correlations of the $x_\parallel$-averaged field. Likewise, all higher-order cumulants have delta-function multipliers $\delta(k_\parallel)$ for each $\mathbf{k}$ dependence. For example,

$$\bar{q}_{jj'j''}^{ss's''}(\mathbf{k}, \mathbf{k}') = \delta(k_\parallel)\, \delta(k'_\parallel)\, \bar{q}_{jj'j''}^{ss's''}(\mathbf{k}_\perp, \mathbf{k}'_\perp)$$

is the Fourier transform of

$$\langle \bar{z}_j^s(\mathbf{x}_\perp)\, \bar{z}_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp)\, \bar{z}_{j''}^{s''}(\mathbf{x}_\perp + \mathbf{r}'_\perp)\rangle.$$

Such a singular dependence of the Fourier-space cumulants has a dramatic effect on the dynamics, especially since the singularity is supported precisely on the resonant manifold. Indeed, the hierarchy of cumulant equations for $\bar{q}^{(n)}$ simply loses the fast (Alfvén) time dependence altogether and becomes fully nonlinear MHD turbulence in two dimensions with time t replaced by ϵt. Let us imagine, then, that the initial fields are dominated by this 2D component and that the fields have relaxed on the time scale $t \sim \epsilon^{-1}$ to their equilibrium solutions of finite energy flux, for which $\bar{E}(k_\perp)$ is the initial Kolmogorov finite energy flux spectrum $k_\perp^{-5/3}$

for $k_\perp > k_0$, k_0 some input wavenumber and $\bar{E}(k_\perp) \sim k_\perp^{-1/3}$, corresponding to the inverse flux of the squared magnetic vector potential ($\mathscr{A}$, with $\mathbf{b} = \boldsymbol{\nabla} \times \mathscr{A}$). $\hat{\mathscr{A}}(\mathbf{k})$, the spectral density of $\langle \mathscr{A}^2 \rangle$, behaves as $k_\perp^{-7/3}$. These are predicted from phenomenological arguments and supported by numerical simulations.

Let us then ask: how do Alfvén waves (Bragg) scatter off this 2D turbulent field? To answer this question, one should of course redo all the analysis, taking proper account of the $\delta(k_\parallel)$ factors in $\bar{q}^{(n)}$. However, there is a simpler way. Let us imagine that the power spectra for the $\hat{z}_j^s$ fields are supported at finite $k_\parallel$ and have much smaller integrated power over an interval $0 \leqslant k_\parallel < \beta \ll 1$ than do the 2D fields. Let us replace the $\delta(k_\parallel)$ multiplying $q_{jj'}^{ss'}(\mathbf{k}_\perp)$ by a function of finite width β and height β^{-1}. Then the kinetic equation is linear and describes how the power spectra, and in particular $\hat{e}^s(\mathbf{k})$, of the $\hat{z}_j^s$ fields interact with the power spectra of the 2D field $\bar{z}_j^s$. Namely, the $Q^{-s}(\mathbf{k}_\perp, 0)$ field in the kinetic equation is determined by the 2D field and taken as known. The time scale of the interaction is now $\beta\epsilon^{-2}$, because the strength of the interaction is increased by β^{-1}, and is faster than that of pure Alfvén-wave scattering. But the equilibrium of the kinetic equation will retain the property that $n_{-s} + n_s = -4$, where now n_{-s} is the phenomenological exponent associated with 2D MHD turbulence and n_s the exponent of the Alfvén waves. Note that when $n_{-s} = -5/3$, n_s is $-7/3$, which is the same exponent (perhaps accidentally) as for the temporary spectrum observed in the finite-time transition to the $k_\perp^{-2}$ spectrum.

We now proceed to a detailed presentation of our results.

2. The derivation of the kinetic equations

The purpose in this section is to obtain closed equations for the energy and helicity spectra of weak MHD turbulence, using the fact that, in the presence of a strong uniform magnetic field, only Alfvén waves of opposite polarities propagating in opposite directions interact.

2.1. The basic equations

We shall use the weak turbulence approach, the ideas of which are described in great detail in the book of Zakharov et al. (1992). There are several different ways to derive the weak turbulence kinetic equations. We follow here the technique that can be found in Benney and Newell (1969). We write the 3D incompressible MHD equations for the velocity $\mathbf{v}$ and the Alfvén velocity $\mathbf{b}$:

$$(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\boldsymbol{\nabla} P_* + \mathbf{b} \cdot \boldsymbol{\nabla}\, \mathbf{b} + \nu \nabla^2 \mathbf{v}, \tag{9}$$

$$(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{b} = \mathbf{b} \cdot \boldsymbol{\nabla} \mathbf{v} + \eta \nabla^2 \mathbf{b}, \tag{10}$$

where P_* is the total pressure, ν the viscosity, η the magnetic diffusivity, and $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{b} = 0$. In the absence of dissipation, these equations have three quadratic invariants in dimension three, namely the total energy $E^T = \frac{1}{2}\langle v^2 + b^2 \rangle$, the cross-correlation $E^C = \langle \mathbf{v} \cdot \mathbf{b} \rangle$ and the magnetic helicity $H^M = \langle \mathscr{A} \cdot \mathbf{b} \rangle$ (Woltjer 1958).

The Elsässer variables $\mathbf{z}^s = \mathbf{v} + s\mathbf{b}$ with $s = \pm 1$ give these equations a more symmetrized form, namely

$$(\partial_t + \mathbf{z}^{-s} \cdot \boldsymbol{\nabla})\, \mathbf{z}^s = -\boldsymbol{\nabla} P_*, \tag{11}$$

where we have dropped the dissipative terms, which pose no particular closure problems. The first two invariants are then simply written as $2E^s = \langle |\mathbf{z}^s|^2 \rangle$.

We now assume that there is a strong uniform magnetic induction field $\mathbf{B}_0$ along the unit vector $\hat{\mathbf{e}}_\parallel$ and non-dimensionalize the equations with the corresponding magnetic induction $\mathbf{B}_0$, where the z^s fields have an amplitude proportional to ϵ ($\epsilon \ll 1$) assumed small compared with b_0. Linearizing the equations leads to

$$(\partial_t - s b_0 \partial_\parallel) z_j^s = -\epsilon \partial_{x_m} z_m^{-s} z_j^s - \partial_{x_j} P_*, \tag{12}$$

where $\partial_\parallel$ is the derivative along $\hat{\mathbf{e}}_\parallel$. The frequency of the modes at a wavevector $\mathbf{k}$ is $\omega(\mathbf{k}) = \omega_k = \mathbf{b}_0 \cdot \mathbf{k} = b_0 k_\parallel$. We Fourier transform the wave fields using the interaction representation,

$$\begin{aligned} z_j^s(\mathbf{x}, t) &= \int A_j^s(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} \\ &= \int a_j^s(\mathbf{k}, t)\, e^{i(\mathbf{k}\cdot\mathbf{x} + s\omega_k t)}\, d\mathbf{k}, \end{aligned} \tag{13}$$

where $a_j^s(\mathbf{k}, t)$ varies slowly in time because of the weak nonlinearities; hence

$$\partial_t a_j^s(\mathbf{k}, t) = -i\epsilon k_m P_{jn} \int a_m^{-s}(\boldsymbol{\kappa})\, a_n^s(\mathbf{L})\, e^{i(-s\omega_k - s\omega_\kappa + s\omega_L)t} \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}} \tag{14}$$

with $d_{\boldsymbol{\kappa}\mathbf{L}} = d\boldsymbol{\kappa}\, d\mathbf{L}$ and $\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} = \delta(\mathbf{k} - \boldsymbol{\kappa} - \mathbf{L})$; finally, $P_{jn}(k) = \delta_{jn} - k_j k_n k^{-2}$ is the usual projection operator keeping the $\mathbf{A}^s(\mathbf{k})$ fields transverse to $\mathbf{k}$ because of incompressibility. The exponentially oscillating term in (14) is essential: its exponent should not vanish when $\mathbf{k} - \boldsymbol{\kappa} - \mathbf{L} = \mathbf{0}$, i.e. the waves should be dispersive for the closure procedure to work. In that sense, incompressible MHD can be coined 'pseudo'-dispersive because, although $\omega_k \sim k$, the fact that waves of one s-polarity interact *only* with the opposite polarity has the consequence that the oscillating factor is non-zero except at resonance; indeed, with $\omega_k = b_0 k_\parallel$, one immediately sees that $-s\omega_k - s\omega_\kappa + s\omega_L = s(-k_\parallel - \kappa_\parallel + L_\parallel) = -2s\kappa_\parallel$, using the convolution constraint between the three waves in interaction. In fact, Alfvén waves may have a particularly weak form of interactions, since such interactions take place only when two waves propagating in opposite directions along the lines of the uniform magnetic field meet. As will be seen later (see Sec. 3), this has the consequence that the transfer in the direction parallel to $\mathbf{B}_0$ is zero, rendering the dynamics two-dimensional, as is well known (see e.g. Montgomery and Turner 1981; Shebalin et al. 1983). Technically, we note that there are two types of waves that propagate in opposite directions, so that the classical criterion (Zakharov et al. 1992) for resonance to occur, namely $\omega'' > 0$, does not apply here.

2.2. *Toroidal and poloidal fields*

The divergence-free condition implies that only two scalar fields are needed to describe the dynamics; following classical works in anisotropic turbulence, they are taken as (Craya 1958; Herring 1974; Riley et al. 1981)

$$\begin{aligned} \mathbf{z}^s &= \mathbf{z}_1^s + \mathbf{z}_2^s \\ &= \boldsymbol{\nabla} \times (\psi^s \hat{\mathbf{e}}_\parallel) + \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\phi^s \hat{\mathbf{e}}_\parallel)], \end{aligned} \tag{15}$$

which in Fourier space gives

$$A_j^s(\mathbf{k}) = i\mathbf{k} \times \hat{\mathbf{e}}_\parallel\, \hat{\psi}^s(\mathbf{k}) - \mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{e}}_\parallel)\, \hat{\phi}^s(\mathbf{k}). \tag{16}$$

We elaborate somewhat on the significance of the $\hat{\psi}^s$ and $\hat{\phi}^s$ fields, since they are the basic fields with which we shall deal. Note that $\mathbf{z}_1^s$ are 2D fields with no parallel component and thus with only a vertical vorticity component ('vertical' means parallel to $\mathbf{B}_0$), whereas the $\mathbf{z}_2^s$ fields have zero vertical vorticity ; such a decomposition is used as well for stratified flows (see Lesieur 1990, and references therein). Indeed, rewriting the double cross-product in (16) leads to

$$\mathbf{A}^s(\mathbf{k}) = i\mathbf{k} \times \hat{\mathbf{e}}_\parallel \, \hat{\psi}^s(\mathbf{k}) - \mathbf{k}\, k_\parallel \hat{\phi}^s(\mathbf{k}) + \hat{\mathbf{e}}_\parallel \, k^2 \hat{\phi}^s(\mathbf{k}), \tag{17}$$

or, using $\mathbf{k} = \mathbf{k}_\perp + k_\parallel \hat{\mathbf{e}}_\parallel$,

$$\mathbf{A}^s(\mathbf{k}) = i\mathbf{k} \times \hat{\mathbf{e}}_\parallel \, \hat{\psi}^s(\mathbf{k}) - \mathbf{k}_\perp k_\parallel \, \hat{\phi}^s(\mathbf{k}) + \hat{\mathbf{e}}_\parallel \, k_\perp^2 \, \hat{\phi}^s(\mathbf{k}). \tag{18}$$

The above equations indicate the relationships between the two orthogonal systems (with $\mathbf{p} = \mathbf{k} \times \hat{\mathbf{e}}_\parallel$ and $\mathbf{q} = \mathbf{k} \times \mathbf{p}$) made of the triads $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ and $(\hat{\mathbf{e}}_\parallel, \mathbf{p}, \mathbf{k}_\perp)$ and the system $(\hat{\mathbf{e}}_\parallel, \mathbf{p}, \mathbf{k})$. In terms of the decomposition used in Waleffe (1992), with

$$\mathbf{h}_\pm = \mathbf{p} \times \mathbf{k} \pm i\mathbf{p}, \tag{19}$$

and writing $\mathbf{z}^s = A_+^s \mathbf{h}_+ + A_-^s \mathbf{h}_-$, it can easily be shown that $\psi^s = A_+^s - A_-^s$ and $\phi^s = A_+^s + A_-^s$. In these latter variables, the s-energies E^s are proportional to $\langle |A_+^s|^2 + |A_-^s|^2 \rangle$ and the s-helicities $\langle \mathbf{z}^s \cdot \boldsymbol{\nabla} \times \mathbf{z}^s \rangle$ are proportional to $\langle |A_+^s|^2 - |A_-^s|^2 \rangle$. Note that E^s is not a scalar : when going from a right-handed to a left-handed frame of reference, E^s changes into E^{-s}.

2.3. *Moments and cumulants*

We now seek a closure for the energy tensor $q_{jj'}^{ss'}(\mathbf{k})$, defined as

$$\langle a_j^s(\mathbf{k})\, a_{j'}^{s'}(\mathbf{k}') \rangle \equiv q_{jj'}^{ss'}(\mathbf{k}')\, \delta(\mathbf{k} + \mathbf{k}'), \tag{20}$$

in terms of second order moments of the two scalar fields $\hat{\psi}^s(\mathbf{k})$ and $\hat{\phi}^s(\mathbf{k})$. Simple manipulations lead, with the restriction $s = s'$ (it can be shown that correlations with $s' = -s$ have no long-time influence and therefore are, for convenience of exposition, omitted), to

$$q_{11}^{ss}(\mathbf{k}') = k_2^2 \Psi^s(\mathbf{k}) - k_1 k_2 k_\parallel I^s(\mathbf{k}) + k_\parallel^2 k_1^2 \Phi^s(\mathbf{k}), \tag{21a}$$

$$q_{22}^{ss}(\mathbf{k}') = k_1^2 \Psi^s(\mathbf{k}) + k_1 k_2 k_\parallel I^s(\mathbf{k}) + k_\parallel^2 k_2^2 \Phi^s(\mathbf{k}), \tag{21b}$$

$$q_{12}^{ss}(\mathbf{k}') + q_{21}^{ss}(\mathbf{k}') = -2k_1 k_2 \Psi^s(\mathbf{k}) + k_\parallel (k_1^2 - k_2^2) I^s(\mathbf{k}) + 2k_1 k_2 k_\parallel^2 \Phi^s(\mathbf{k}), \tag{21c}$$

$$q_{1\parallel}^{ss}(\mathbf{k}') + q_{\parallel 1}^{ss}(\mathbf{k}') = k_2 k_\perp^2 \, I^s(\mathbf{k}) - 2k_1 k_\parallel k_\perp^2 \, \Phi^s(\mathbf{k}), \tag{21d}$$

$$q_{2\parallel}^{ss}(\mathbf{k}') + q_{\parallel 2}^{ss}(\mathbf{k}') = -k_1 k_\perp^2 \, I^s(\mathbf{k}) - 2k_2 k_\parallel k_\perp^2 \, \Phi^s(\mathbf{k}), \tag{21e}$$

$$q_{\parallel\,\parallel}^{ss}(\mathbf{k}') = k_\perp^4 \, \Phi^s(\mathbf{k}), \tag{21f}$$

$$\begin{aligned} \frac{1}{k_1}[q_{2\parallel}^{ss}(\mathbf{k}') - q_{\parallel 2}^{ss}(\mathbf{k}')] &= \frac{1}{k_2}[q_{\parallel 1}^{ss}(\mathbf{k}') - q_{1\parallel}^{ss}(\mathbf{k}')] \\ &= \frac{1}{k_\parallel}[q_{12}^{ss}(\mathbf{k}') - q_{21}^{ss}(\mathbf{k}')] \\ &= -ik_\perp^2 \, R^s(\mathbf{k}), \end{aligned} \tag{21g}$$

where the following correlators involving the toroidal and poloidal fields have been introduced:

$$\langle\hat{\psi}^s(\mathbf{k})\hat{\psi}^s(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\Psi^s(\mathbf{k}'), \tag{22a}$$

$$\langle\hat{\phi}^s(\mathbf{k})\hat{\phi}^s(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\Phi^s(\mathbf{k}'), \tag{22b}$$

$$\langle\hat{\psi}^s(\mathbf{k})\hat{\phi}^s(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\Pi^s(-\mathbf{k}), \tag{22c}$$

$$\langle\hat{\phi}^s(\mathbf{k})\hat{\psi}^s(\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\Pi^s(\mathbf{k}), \tag{22d}$$

$$R^s(\mathbf{k}) = \Pi^s(-\mathbf{k}) + \Pi^s(\mathbf{k}), \tag{22e}$$

$$I^s(\mathbf{k}) = i[\Pi^s(-\mathbf{k}) - \Pi^s(\mathbf{k})], \tag{22f}$$

and where

$$k_\perp^2 = k_1^2 + k_2^2, \qquad k^2 = k_\perp^2 + k_\parallel^2.$$

Note that $\Sigma_s R^s$ is the only pseudoscalar, linked to the lack of symmetry of the equations under plane reversal, i.e. to a non-zero helicity.

The density energy spectrum can be written as

$$e^s(\mathbf{k}) = \sum_j q_{jj}^{ss}(\mathbf{k}) = \mathbf{k}_\perp^2[\Psi^s(\mathbf{k}) + k^2\Phi^s(\mathbf{k})]. \tag{23}$$

Note that it can easily be shown that the kinetic and magnetic energies $\frac{1}{2}\langle u^2\rangle$ and $\frac{1}{2}\langle b^2\rangle$ are equal in the context of the weak turbulence approximation. Similarly, expressing the magnetic induction as a combination of $\mathbf{z}^\pm$ and thus of $\hat{\psi}^\pm$ and $\hat{\phi}^\pm$, we had the following symmetrized cross-correlator of magnetic helicity (where the Alfvén velocity is used for convenience) and its Fourier transform:

$$\tfrac{1}{2}\langle\hat{\mathscr{A}}_j(\mathbf{k})\,\hat{b}_j(\mathbf{k}')\rangle + \tfrac{1}{2}\langle\hat{\mathscr{A}}_j(\mathbf{k}')\,\hat{b}_j(\mathbf{k})\rangle = \tfrac{1}{4}\mathbf{k}_\perp^2\sum_s R^s(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}'), \tag{24}$$

where the correlations between the + and − variables are ignored because they are exponentially damped in the approximation of weak turbulence. Similarly to the case of energy, there is equivalence between the kinetic and magnetic helical variables in that approximation; hence the kinetic helicity defined as $\langle\mathbf{u}\cdot\boldsymbol{\omega}\rangle$ can be written simply in terms of its spectral density $H^V(\mathbf{k})$:

$$H^V(\mathbf{k}) = k^2 H^M(\mathbf{k}) = \tfrac{1}{4}k^2\mathbf{k}_\perp^2\sum_s R^s(\mathbf{k}). \tag{25}$$

In summary, the eight fundamental spectral density variables for which we seek a weak turbulence closure are the energy $e^s(\mathbf{k})$ of the three components of the $\mathbf{z}^s$ fields, the energy density along the direction of the uniform magnetic field $\Phi^s(\mathbf{k})$, the correlators related to the off-diagonal terms of the spectral energy density tensor $I^s(\mathbf{k})$ and finally the only helicity-related pseudoscalar correlators, namely $R^s(\mathbf{k})$.

The main procedure that leads to a closure of weak turbulence for incompressible MHD is outlined in the Appendix. It leads to the equations (A 8) giving the temporal evolution of the components of the spectral tensor $q_{jj'}^{ss'}(\mathbf{k})$ just defined. The last technical step consists in transforming (A 8) in terms of the eight correlators that we have defined above. This leads us to the final set of equations, constituting the kinetic equations for weak MHD turbulence.

2.4. The kinetic equations

In the general case, the kinetic equations for weak MHD turbulence are

$$\begin{aligned}\partial_t e^s(\mathbf{k}) = \frac{\pi\varepsilon^2}{b_0}\int \Bigg[& \left(L_\perp^2 - \frac{X^2}{k^2}\right)\mathbf{\Psi}^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{L^2}\right)\mathbf{\Psi}^s(\mathbf{k}) \\ & + \left(L_\perp^2 L^2 - \frac{k_\parallel^2 W^2}{k^2}\right)\mathbf{\Phi}^s(\mathbf{L}) - \left(k_\perp^2 k^2 - \frac{k_\parallel^2 Y^2}{L^2}\right)\mathbf{\Phi}^s(\mathbf{k}) \\ & + \frac{k_\parallel XY}{L^2} I^s(\mathbf{k}) - \frac{k_\parallel XW}{k^2} I^s(\mathbf{L})\Bigg] Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}, \end{aligned} \tag{26}$$

$$\begin{aligned}\partial_t[k_\perp^2 k^2 \mathbf{\Phi}^s(\mathbf{k})] = \frac{\pi\varepsilon^2}{b_0}\int \Bigg\{ & k_\parallel^2 X^2 \left[\frac{\mathbf{\Psi}^s(\mathbf{L})}{k_\perp^2 k^2} - \frac{\mathbf{\Phi}^s(\mathbf{k})}{L_\perp^2}\right] \\ & + (k_\parallel^2 Z + k_\perp^2 L_\perp^2)^2 \left[\frac{\mathbf{\Phi}^s(\mathbf{L})}{k_\perp^2 k^2} - \frac{\mathbf{\Phi}^s(\mathbf{k})}{L_\perp^2 L^2}\right] + \frac{k_\parallel X}{k_\perp^2 k^2}(k_\parallel^2 Z + k_\perp^2 L_\perp^2) I^s(\mathbf{L}) \\ & + \frac{k_\parallel XY}{2L^2} I^s(\mathbf{k})\Bigg\} Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}} \\ & - \frac{\varepsilon^2}{b_0} s R^s(\mathbf{k}) \mathscr{P}\int \frac{X}{2\kappa_\parallel L^2}(k_\parallel Z - L_\parallel k_\perp^2) Q_k^{-s}(\boldsymbol{\kappa})\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}}, \end{aligned} \tag{27}$$

$$\begin{aligned}\partial_t\left[k_\perp^2 R^s(\mathbf{k})\right] = -\frac{\pi\varepsilon}{b_0}\int \Bigg\{ & L_\perp^2 \frac{Z + k_\parallel^2}{k^2} R^s(\mathbf{L}) \\ & + \frac{k_\perp^2}{2}\left[1 + \frac{(Z + k_\parallel^2)^2}{k^2 L^2}\right] R^s(\mathbf{k})\Bigg\} Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}} \\ & + \frac{\varepsilon^2}{b_0} s\mathscr{P}\int \Big\{ 2X(k_\parallel Z - L_\parallel k_\perp^2)[\mathbf{\Psi}^s(\mathbf{k}) + k^2\mathbf{\Phi}^s(\mathbf{k})] \\ & + \left[(k_\parallel Z - L_\parallel k_\perp^2)^2 - k^2 X^2\right] I^s(\mathbf{k})\Big\} \frac{Q_k^{-s}(\boldsymbol{\kappa})}{2\kappa_\parallel k^2 L^2}\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}}, \end{aligned} \tag{28}$$

$$\begin{aligned}\partial_t[k_\perp^2 k^2 I^s(\mathbf{k})] = \frac{\pi\varepsilon^2}{b_0}\int \Bigg\{ & \left[L_\perp^2 Z + \frac{k_\parallel^2}{k_\perp^2}(Z^2 - X^2)\right] I^s(\mathbf{L}) \\ & + \left(\frac{k_\parallel^2 Y^2}{2L^2} - k_\perp^2 k^2 + \frac{k^2 X^2}{2L^2}\right) I^s(\mathbf{k}) + \frac{k_\parallel XY}{L^2}[\mathbf{\Psi}^s(\mathbf{k}) + k^2\mathbf{\Phi}^s(\mathbf{k})] \\ & + \frac{2k_\parallel X}{k_\perp^2}[Z\mathbf{\Psi}^s(\mathbf{L}) - (k_\parallel^2 Z + k_\perp^2 L_\perp^2)\mathbf{\Phi}^s(\mathbf{L})]\Bigg\} Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}} \\ & - \frac{\varepsilon^2}{b_0} s R^s(\mathbf{k})\mathscr{P}\int \frac{1}{2\kappa_\parallel L^2}\left[(k_\parallel Z - L_\parallel k_\perp^2)^2 - k^2 X^2\right] Q_k^{-s}(\boldsymbol{\kappa})\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}}, \end{aligned} \tag{29}$$

with

$$\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} = \delta(\mathbf{L} + \boldsymbol{\kappa} - \mathbf{k}),$$

$$d_{\boldsymbol{\kappa}\mathbf{L}} = d\boldsymbol{\kappa}\, d\mathbf{L},$$

and

$$\begin{aligned} Q_k^{-s}(\boldsymbol{\kappa}) &= k_m k_p\, q_{p\ m}^{-s-s}(\boldsymbol{\kappa}) \\ &= X^2\Psi^{-s}(\boldsymbol{\kappa}) + X(k_\| \kappa_\perp^2 - \kappa_\| Y) I^{-s}(\boldsymbol{\kappa}) + (\kappa_\| Y - k_\| \kappa_\perp^2)^2 \phi^{-s}(\boldsymbol{\kappa}). \qquad (30) \end{aligned}$$

Note that Q_k^{-s} does not involve the spectral densities $R^s(\mathbf{k})$, because of the symmetry properties of the equations. The geometrical coefficients appearing in the kinetic equations are

$$X = (\mathbf{k}_\perp \times \boldsymbol{\kappa}_\perp)_z = k_\perp \kappa_\perp \sin\theta, \qquad (31a)$$

$$Y = \mathbf{k}_\perp \cdot \boldsymbol{\kappa}_\perp = k_\perp \kappa_\perp \cos\theta, \qquad (31b)$$

$$\begin{aligned} Z &= \mathbf{k}_\perp \cdot \mathbf{L}_\perp = k_\perp^2 - k_\perp \kappa_\perp \cos\theta \\ &= k_\perp^2 - Y, \qquad (31c) \end{aligned}$$

$$\begin{aligned} W &= \boldsymbol{\kappa}_\perp \cdot \mathbf{L}_\perp = k_\perp^2 - L_\perp^2 - k_\perp \kappa_\perp \cos\theta \\ &= Z - L_\perp^2, \qquad (31d) \end{aligned}$$

where θ is the angle between $\mathbf{k}_\perp$ and $\boldsymbol{\kappa}_\perp$, and with

$$d\boldsymbol{\kappa}_\perp = \kappa_\perp d\kappa_\perp d\theta = \frac{L_\perp}{k_\perp \sin\theta}\, d\kappa_\perp dL_\perp, \qquad (32)$$

$$\cos\theta = \frac{\kappa_\perp^2 + k_\perp^2 - L_\perp^2}{2\kappa_\perp k_\perp}. \qquad (33)$$

In (27)–(29), $\mathscr{P}\int$ means the Cauchy principal value of the integral in question.

3. General properties of the kinetic equations

3.1. Dynamical decoupling in the direction parallel to $\mathbf{B}_0$

The integral on the right-hand side of the kinetic equation (26) contains a delta function of the form $\delta(\kappa_\|)$, the integration variable corresponding to the parallel component of one of the wavenumbers in the interacting triad. This delta function arises because of the three-wave frequency resonance condition. Thus, in any resonantly interacting wave triad ($\mathbf{k}$, $\boldsymbol{\kappa}$, $\mathbf{L}$), there is always one wave that corresponds to a purely 2D motion – having no dependence on the direction parallel to the uniform magnetic field – whereas the other two waves have equal parallel components of their corresponding wavenumbers, namely $L_\| = k_\|$. This means that the parallel components of the wavenumber enter in the kinetic equation of the total energy $e^s(\mathbf{k})$ as an external parameter and that the dynamics is decoupled at each level of $k_\|$. In other words, there is no transfer associated with the three-wave resonant interaction along the $k_\|$ direction in $\mathbf{k}$ space for the total energy. This result, using the exact kinetic equations developed here, corroborates what has already been found in Montgomery and Turner (1981) using a phenomenological analysis of the basic MHD equations, in Ng and Bhattacharjee (1996, 1997) in the framework of a model of weak MHD turbulence using individual wave packets, and in Kinney and McWilliams (1998) with a reduced MHD (RMHD) approach.

As for the kinetic equation (26), the other kinetic equations (27)–(29) have integrals containing delta functions of the form $\delta(\kappa_\|)$. But, in addition, they have PVIs that can, a priori, contribute to a transfer in the parallel direction. The eventual contributions of these PVIs are discussed in Sec. 3.4.

3.2. Detailed energy conservation

Detailed conservation of energy for each interacting triad of waves is a usual property in weak turbulence theory. This property is closely related to the frequency resonance condition

$$\omega_{\mathbf{k}} = \omega_{\mathbf{L}} + \omega_{\boldsymbol{\kappa}},$$

because ω can be interpreted as the energy of one wave 'quantum'. For Alfvén waves, the detailed energy-conservation property is even stronger because one of the waves in any resonant triad belongs to the 2D state with frequency equal to zero:

$$\omega_{\boldsymbol{\kappa}} \propto \kappa_\| = 0.$$

Thus, for every triad of Alfvén waves $\mathbf{k}, \mathbf{L}$, $\boldsymbol{\kappa}$ (such that $\kappa_\| = 0$), the energy is conserved within two copropagating waves having wavevectors $\mathbf{k}$ and $\mathbf{L}$. Mathematically, this corresponds to the symmetry of the integrand in the equation for e^s with respect to changing $\mathbf{k} \leftrightarrow \mathbf{L}$ (and correspondingly $\boldsymbol{\kappa} = \mathbf{k} - \mathbf{L} \to -\boldsymbol{\kappa}$).

As we have said, energy is conserved $k_\|$ plane by $k_\|$ plane, so that, for each $k_\|$, it can be shown from (26) that

$$\frac{\partial}{\partial t} \int e^s(\mathbf{k}_\perp, k_\|)\, d\mathbf{k}_\perp = 0. \tag{34}$$

3.3. The magnetic and pseudomagnetic helicities

Since the work of Woltjer (1958), we have known that the magnetic helicity is an invariant of the MHD equations. However, Stribling et al. (1994) showed that in the presence of a mean magnetic field $\mathbf{B}_0$, the part of the magnetic helicity associated with fluctuations is not conserved separately (whereas the total magnetic helicity, which takes into account a term proportional to $\mathbf{B}_0$, is, of course, an invariant). It is then interesting to know if, in the context of weak turbulence, the integral of the spectral density of fluctuations of the magnetic helicity is conserved, i.e.

$$\int H^M(\mathbf{k})\, d\mathbf{k} = \text{constant}, \tag{35}$$

with

$$H^M(\mathbf{k}) = \tfrac{1}{4} k^2 \mathbf{k}_\perp^2 \sum_s R^s(\mathbf{k}). \tag{36}$$

To investigate this point, we define in physical space the total magnetic helicity as

$$H_T^M = \langle \mathscr{A}_T \cdot \mathbf{b}_T \rangle, \tag{37}$$

where $\mathbf{b}_T = \boldsymbol{\nabla} \times \mathscr{A}_T$ and $\mathbf{b}_T = \mathbf{b}_0 + \mathbf{b}$. The magnetic induction equation

$$\partial_t \mathbf{b}_T = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{b}_T) \tag{38}$$

implies that (Stribling et al. 1994)

$$\partial_t H_T^M = \partial_t H^M + 2\mathbf{b}_0 \cdot \partial_t \langle \mathscr{A} \rangle, \tag{39}$$

where H^M is the magnetic helicity associated with fluctuations ($H^M = \langle \mathscr{A} \cdot \mathbf{b} \rangle$) and $\mathbf{b} = \nabla \times \mathscr{A}$. Direct numerical simulations (Stribling et al. 1994) show that the second term on the right-hand side of (39) makes a non-zero contribution to the total magnetic helicity, but in the context of weak turbulence the situation is different. Indeed, the magnetic induction equation leads also to the relation

$$\partial_t \langle \mathscr{A} \rangle = \tfrac{1}{2} \langle \mathbf{z}^- \times \mathbf{z}^+ \rangle. \tag{40}$$

Therefore the temporal evolution of the magnetic potential of fluctuations is proportional to the cross-product between z-fields of opposite polarities. As we have already pointed out, in the framework of weak turbulence, this kind of correlation has no long-time influence, and thus the magnetic helicity associated with fluctuations appears to be an invariant of the weak turbulence equations. We leave for the future the investigation of this point: in particular, it will be helpful to make numerical computations to show, at least at this level, the invariance of the magnetic helicity.

The correlators $R^s(\mathbf{k})$ and $I^s(\mathbf{k})$ have been defined in the previous section as the real part and the imaginary part of $\mathbf{\Pi}^s(\mathbf{k})$, the cross-correlator of the toroidal field $\hat{\psi}^s(\mathbf{k})$ and of the poloidal field $\hat{\phi}^s(\mathbf{k})$. Then $\mathbf{k}_\perp^2 \Sigma_s R^s(\mathbf{k})$ appears as the spectral density of the magnetic helicity. On the other hand, $I^s(\mathbf{k})$, which we shall call the anisotropy correlator (or pseudo magnetic helicity), is neither a conserved quantity nor a positive definite quantity. Although R^s and I^s evolve according to their own kinetic equations (29) and (28), the range of values they can take that is bounded by $\mathbf{\Psi}^s$ and $\mathbf{\Phi}^s$, with the bounds being a simple consequence of the definition of these quantities. Two realizability conditions (see also Cambon and Jacquin 1989; Cambon et al. 1997) between the four correlators $\mathbf{\Psi}^s$, $\mathbf{\Phi}^s$, I^s and R^s can be obtained from

$$\langle |\hat{\psi}^s(\mathbf{k}) \pm k\hat{\phi}^s(\mathbf{k})|^2 \rangle \geqslant 0 \tag{41}$$

and

$$\langle |\hat{\psi}^s(\mathbf{k})|^2 \rangle \langle |\hat{\phi}^s(\mathbf{k})|^2 \rangle \geqslant |\langle \hat{\psi}^s(\mathbf{k}) \hat{\phi}^s(-\mathbf{k}) \rangle|^2. \tag{42}$$

These conditions are found to be respectively

$$\mathbf{\Psi}^s(\mathbf{k}) + k^2 \mathbf{\Phi}^s(\mathbf{k}) \geqslant |kR^s(\mathbf{k})| \tag{43}$$

and

$$4\mathbf{\Psi}^s(\mathbf{k}) \mathbf{\Phi}^s(\mathbf{k}) \geqslant R^{s2}(\mathbf{k}) + I^{s2}(\mathbf{k}). \tag{44}$$

Note that the combination

$$\mathscr{Z} = \tfrac{1}{2} k_\perp^2 [k^2 \mathbf{\Phi}(\mathbf{k}) - \mathbf{\Psi}(\mathbf{k}) - i|k| I(\mathbf{k})] \tag{45}$$

is called 'polarization anisotropy' in Cambon and Jacquin (1989). The consequences of the realizability conditions are explained below.

3.4. Purely 2D modes and two-dimensionalization of 3D spectra

The first consequence of the fact that there is no transfer of the total energy in the $k_\parallel$ direction in $\mathbf{k}$ space is an asymptotic two-dimensionalization of the energy spectrum $e^s(\mathbf{k})$. Namely, the 3D initial spectrum spreads over the transverse wavenumbers $\mathbf{k}_\perp$, but remains of the same size in the $k_\parallel$ direction, and the support of the spectrum becomes very flat (pancake-like) for large time. Two-dimensionalization of weak MHD turbulence has been observed in laboratory experiments (Robinson and Rusbridge 1971), in solar wind data (Bavassano et al. 1982; Matthaeus et al.

1990; Horbury et al. 1995; Bieber et al. 1996), and in many direct numerical simulations of the 3D MHD equations (Oughton et al. 1994) or of the RMHD equations (Kinney and McWilliams 1998).

From a mathematical point of view, the two-dimensionalization of the total energy means that, for large time, the energy spectrum $e^s(\mathbf{k})$ is supported on a volume of wavenumbers such that for most of them $k_\perp \gg k_\parallel$. This implies that $\mathbf{\Psi}^s(\mathbf{k})$ and $\mathbf{\Phi}^s(\mathbf{k})$ are also supported on the same anisotropic region of wavenumbers, because both of them are non-negative. This, in turn, implies that both R^s and I^s will also be non-zero only for the same region in the $\mathbf{k}$ space as $e^s(\mathbf{k})$, $\mathbf{\Psi}^s(\mathbf{k})$ and $\mathbf{\Phi}^s(\mathbf{k})$, as follows from the bounds (43) and (44). This fact allows one to expand the integrands in the kinetic equations in powers of small $k_\parallel/k_\perp$. At leading order in $k_\parallel/k_\perp$, one obtains

$$\partial_t\left[k_\perp^2\mathbf{\Psi}^s(\mathbf{k})\right] = \frac{\pi\varepsilon^2}{b_0}\int\left[\left(L_\perp^2 - \frac{X^2}{k_\perp^2}\right)\mathbf{\Psi}^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{L_\perp^2}\right)\mathbf{\Psi}^s(\mathbf{k})\right]X^2\mathbf{\Psi}^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\,d_{\boldsymbol{\kappa}\mathbf{L}}, \tag{46}$$

$$\partial_t\left[k_\perp^4\mathbf{\Phi}^s(\mathbf{k})\right] = \frac{\pi\varepsilon^2}{b_0}\int[L_\perp^4\mathbf{\Phi}^s(\mathbf{L}) - k_\perp^4\mathbf{\Phi}^s(\mathbf{k})]X^2\mathbf{\Psi}^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\,d_{\boldsymbol{\kappa}\mathbf{L}}, \tag{47}$$

$$\partial_t\left[k_\perp^2 R^s(\mathbf{k})\right] = -\frac{\pi\varepsilon^2}{b_0}\int\left[\frac{L_\perp^2 Z}{k_\perp^2}R^s(\mathbf{L}) + \left(\frac{k_\perp^2}{2} + \frac{Z^2}{2L_\perp^2}\right)R^s(\mathbf{k})\right]X^2\mathbf{\Psi}^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\,d_{\boldsymbol{\kappa}\mathbf{L}}, \tag{48}$$

$$\partial_t\left[k_\perp^2 I^s(\mathbf{k})\right] = \frac{\pi\varepsilon^2}{b_0}\int\left[\frac{L_\perp^2 Z}{k_\perp^2}I^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{2L_\perp^2}\right)I^s(\mathbf{k})\right]X^2\mathbf{\Psi}^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\,d_{\boldsymbol{\kappa}\mathbf{L}}. \tag{49}$$

Note that the principal-value terms drop out of the kinetic equations at leading order. This property means that there is no transfer of any of the eight correlators in the $k_\parallel$ direction in $\mathbf{k}$ space.

One can see from the above that the equations for the toroidal and poloidal energies decouple for large time. These equations describe the shear-Alfvén and pseudo-Alfvén waves respectively. An energy exchange between $\mathbf{\Psi}^s(\mathbf{k})$ and $\mathbf{\Phi}^s(\mathbf{k})$ is, however, possible in an initial phase, i.e. before the two-dimensionalization of the spectra. A preliminary investigation shows that this exchange is actually essentially generated by the magnetic helicity through the principal-value terms: the magnetic helicity plays the role of a catalyst that transfers toroidal energy into poloidal energy. On the other hand, in the large-time limit, the magnetic helicity $\mathbf{\Sigma}_s R^s$ and the pseudo magnetic helicity I^s are also described by equations that are decoupled from each other and from the toroidal and poloidal energies. It is interesting that the kinetic equation for the shear-Alfvén waves (i.e. for $\mathbf{\Psi}^s(\mathbf{k})$) can be obtained also from the RMHD equations, which have been derived under the same conditions of quasi-two-dimensionality (see e.g. Strauss 1976).

An important consequence of the dynamical decoupling at different values of $k_\parallel$ within the kinetic equation formalism is that the set of purely 2D modes (corresponding to $k_\parallel = 0$) evolve independently of the 3D part of the spectrum (with

$k_\| \neq 0$) and can be studied separately. One can interpret this fact as a neutral stability of the purely 2D state with respect to 3D perturbations. As we mentioned in Sec. 1 the kinetic equations themselves are applicable to a description of $k_\| = 0$ modes only if the correlations of the dynamical fields decay in all directions, so that their spectra are sufficiently smooth for all wavenumbers, including those with $k_\| = 0$. To be precise, the characteristic value of $k_\|$ above which the spectra can experience significant changes must be greater than ϵ^2. A study of such 2D limits of the 3D spectrum will be presented in Sec. 4. It is possible, however, that in some physical situations, the correlations decay slowly along the magnetic field owing to a (hypothetical) energy condensation at the $k_\| = 0$ modes. In this case, the modes with $k_\| = 0$ should be treated as a separate component, a condensate, that modifies the dynamics of the 3D modes in a manner somewhat similar to the superfluid condensate, as described by Bogoliubov (see Landau and Lifshitz 1968). We leave this problem for future study.

3.5. Asymptotic solution of the 3D kinetic equations

The parallel wavenumber $k_\|$ enters (46)–(49) only as an external parameter. In other words, the wavenumber space is foliated into the dynamically decoupled planes $k_\| = 0$. Thus the large-time asymptotic solution can be found in the following form:

$$\boldsymbol{\Psi}^s(\mathbf{k}_\perp, k_\|) = f_1(k_\|)\boldsymbol{\Psi}^s(\mathbf{k}_\perp, 0), \tag{50}$$

$$\boldsymbol{\Phi}^s(\mathbf{k}_\perp, k_\|) = f_2(k_\|)\boldsymbol{\Phi}^s(\mathbf{k}_\perp, 0), \tag{51}$$

$$R^s(\mathbf{k}_\perp, k_\|) = f_3(k_\|)R^s(\mathbf{k}_\perp, 0), \tag{52}$$

$$I^s(\mathbf{k}_\perp, k_\|) = f_4(k_\|)I^s(\mathbf{k}_\perp, 0), \tag{53}$$

where f_i ($i = 1, 2, 3, 4$) are some arbitrary functions of $k_\|$ satisfying the conditions $f_i(0) = 1$ (and such that the bounds (43) and (44) are satisfied). Substituting these formulae into (46)–(49), one can readily see that the functions f_i drop out of the problem, and the solution of the 3D equations is reduced to solving a 2D problem for $\boldsymbol{\Psi}^s(\mathbf{k}_\perp, 0), \boldsymbol{\Phi}^s(\mathbf{k}_\perp, 0), R^s(\mathbf{k}_\perp, 0)$ and $I^s(\mathbf{k}_\perp, 0)$, which will be described in the next section.

4. Two-dimensional problem

Let us consider Alfvén-wave turbulence that is axially symmetric with respect to the external magnetic field. Then $I^s(k_\perp, 0) = 0$ because of the condition $I^s(-\mathbf{k}) = -I^s(\mathbf{k})$. In the following, we shall consider only solutions with $R^s = 0$. (One can easily see that R^s will remain zero if it is zero initially.) The remaining equations to be solved are

$$\frac{\partial E_\perp^s(k_\perp, 0)}{\partial t} = \frac{\pi\varepsilon^2}{b_0}\int (\hat{\mathbf{e}}_L \cdot \hat{\mathbf{e}}_k)^2 \sin\theta \frac{k_\perp}{\kappa_\perp} E_\perp^{-s}(\kappa_\perp, 0) \times[k_\perp E_\perp^s(L_\perp, 0) - L_\perp E_\perp^s(k_\perp, 0)]d\kappa_\perp\, dL_\perp, \tag{54}$$

$$\frac{\partial E_\|^s(k_\perp, 0)}{\partial t} = \frac{\pi\varepsilon^2}{b_0}\int \sin\theta \frac{k_\perp}{\kappa_\perp} E_\perp^{-s}(\kappa_\perp, 0) \times[k_\perp E_\|^s(L_\perp, 0) - L_\perp E_\|^s(k_\perp, 0)]d\kappa_\perp\, dL_\perp, \tag{55}$$

where $\hat{\mathbf{e}}_k$ and $\hat{\mathbf{e}}_L$ are the unit vectors along $\mathbf{k}_\perp$ and $\mathbf{L}_\perp$ respectively and

$$E^s_\perp(k_\perp, 0) = k^3_\perp \Psi^s(k_\perp, 0), \tag{56}$$

$$E^s_\|(k_\perp, 0) = k^5_\perp \Phi^s(k_\perp, 0) \tag{57}$$

are the horizontal and vertical components of the energy density. Thus we have reduced the original 3D problem to one of finding a solution for the purely 2D state. It may seem unusual that strongly turbulent 2D vortices (no waves for $k_\| = 0$!) are described by the kinetic equations obtained for weakly turbulent waves. Implicitly, this fact relies on continuity of the 3D spectra near $k_\| = 0$, so that one could take the limit $k_\| \to 0$. In real physical situations, such continuity results from the fact that the external magnetic field is not perfectly unidirectional, and therefore there is a natural smoothing of the spectrum over a small range of angles.

Equation (54) corresponds to the evolution of the shear-Alfvén waves, for which the energy fluctuations are transverse to $\mathbf{B}_0$ whereas (55) describes the pseudo-Alfvén waves, for which the fluctuations are along $\mathbf{B}_0$. Both waves propagate along $\mathbf{B}_0$ at the same Alfvén speed. Equation (54) describes the interaction between two shear-Alfvén waves, $E^\pm_\perp$, propagating in opposite directions. On the other hand, the evolution of the pseudo-Alfvén waves depend on their interactions with the shear-Alfvén waves. The detailed energy conservation of (54) implies that there is no exchange of energy between the two different kinds of waves. The physical picture in this case is that the shear-Alfvén waves interact only among themselves and evolve independently of the pseudo-Alfvén waves. The pseudo-Alfvén waves scatter from the shear-Alfvén waves without amplification or damping, and they do not interact with each other.

Using a standard two-point closure of turbulence (see e.g. Lesieur 1990) in which the characteristic time of transfer of energy is assumed known and written a priori, namely the eddy-damped quasinormal Markovian (EDQNM) closure, Goldreich and Sridhar (1995) derived a variant of the kinetic equation (54), but for strong anisotropic MHD turbulence. In their analysis, the ensuing energy spectrum, which depends (as is well known) on the phenomenological evaluation of the characteristic transfer time, thus differs from our result, where the dynamics is self-consistent, closure being obtained through the assumption of weak turbulence.

It can easily be verified that the geometrical coefficient appearing in the closure equation in Goldreich and Sridhar (1995) is identical to the one we find for the $E^s_\perp(\mathbf{k}_\perp, k_\|)$ spectrum in the 2D case. However, the two formulations, beyond the above discussion on characteristic time scales, differ in a number of ways:

(i) We choose to let the flow variables be non-mirror-symmetric, whereas helicity is not taken into account in Goldreich and Sridhar (1995), who implicitly assumed $R^s \equiv 0$.

(ii) However, because of the anisotropy introduced by the presence of a uniform magnetic field, one must take into account the coupled dynamics of the energy of the shear-Alfvén waves, the pseudo-Alfvén waves and the pseudo-magnetic helicity I^s; indeed, even if initially $I^s \equiv 0$, it is produced by wave coupling and is part of the dynamics.

(iii) In three dimensions, all geometrical coefficients that depend on $k^2 = k^2_\perp + k^2_\|$ have a $k_\|$ dependence that is a function of initial conditions and again is part of the dynamics.

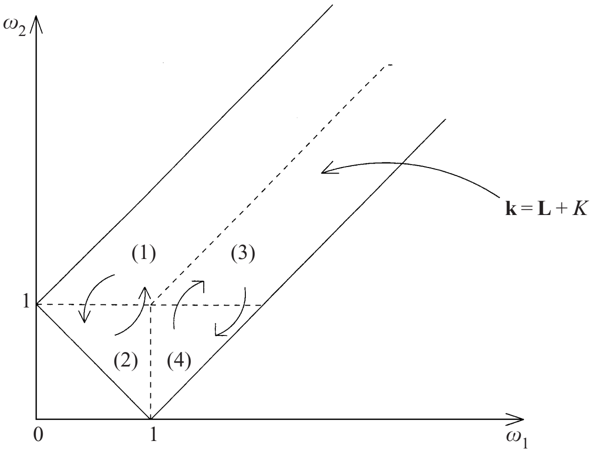

Figure 1. Geometrical representation of the Zakharov transformation. The rectangular region, corresponding to the triad interaction $\mathbf{k}_\perp = \mathbf{L} + \boldsymbol{\kappa}$, is decomposed into four different regions called (1), (2), (3) and (4); ω_1 and ω_2 are respectively the dimensionless variables $\kappa_\perp/k_\perp$ and $L_\perp/k_\perp$. The Zakharov transformation applied to the collision integral consists in exchanging regions (1) and (2), and regions (3) and (4).

4.1. Kolmogorov spectra

4.1.1. The Zakharov transformation. The symmetry of the previous equations allows us to perform a conformal transformation, called the Zakharov transformation (also used in the modelling of strong turbulence; see Kraichnan 1967), in order to find the exact stationary solutions of the kinetic equations as power laws (Zakharov et al. 1992). This operation (see Fig. 1) consists in writing the kinetic equations in dimensionless variables $\omega_1 = \kappa_\perp/k_\perp$ and $\omega_2 = L_\perp/k_\perp$, setting $E^\pm_\perp$ by $k_\perp^{n_\pm}$, and then rearranging the collision integral by the transformation

$$\omega_1' = \frac{\omega_1}{\omega_2}, \tag{58}$$

$$\omega_2' = \frac{1}{\omega_2}. \tag{59}$$

The new form of the collision integral, resulting from the summation of the integrand in its primary form and after the Zakharov transformation, is

$$\partial_t E^s_\perp \sim \int \left(\frac{\omega_2^2 + 1 - \omega_1^2}{2\omega_2}\right)^2 \left[1 - \left(\frac{\omega_1^2 + 1 - \omega_2^2}{2\omega_1}\right)^2\right]^{1/2} \omega_1^{n_{-s}-1}\omega_2 \times(\omega_2^{n_s-1} - 1)(1 - \omega_2^{-n_s-n_{-s}-4})\, d\omega_1\, d\omega_2.$$

The collision integral can be null for specific values of $n_\pm$. The exact solutions, called the Kolmogorov spectra, correspond to these values, which satisfy

$$n_+ + n_- = -4. \tag{60}$$

It is important to understand that the Zakharov transformation is not an identity transformation, and it can lead to spurious solutions. The necessary and sufficient condition for a spectrum obtained by the Zakharov transformation to be a solution of the kinetic equation is that the right-hand side integral in (54) (i.e. before the Zakharov transformation) converges. This condition is called locality of the spectrum, and leads to the following restriction on the spectral indices in our case:

$$-3 < n_{\pm} < -1. \tag{61}$$

A detailed study of the locality of the Kolmogorov spectrum will be given in Sec. 6.

In the particular case of zero cross-correlation, one has $E^+_\perp = E^-_\perp = E_\perp \sim k^n_\perp$, with only one solution

$$n = -2.$$

Note that the thermodynamic equilibrium, corresponding to the equipartition state for which the flux of energy is zero instead of being finite as in the above spectral forms, corresponds to the choices $n_+ = n_- = 1$ for both the perpendicular and the parallel components of the energy.

4.1.2. The Kolmogorov constants $C_K(n_s)$ and $C'_K(n_s)$. The final expression for the Kolmogorov-like spectra found above as a function of the Kolmogorov constant (generalized to MHD) $C_K(n_s)$ and of the flux of energy $P^s_\perp(k_\perp)$ can be obtained in the following way. For a better understanding, the demonstration will be done in the simplified case of a zero cross-correlation. The generalization to the correlated case ($E^+ \neq E^-$) is straightforward. Using the definition of the flux,

$$\partial_t E_\perp(k_\perp, 0) = -\partial_{k_\perp} P_\perp(k_\perp), \tag{62}$$

one can write the flux of energy as a function of the collision integral (with the new form of the integrand) depending on n. Then the limit $n \to -2$ is taken in order to have a constant flux $P_\perp$ with no longer a dependence in $k_\perp$, as is expected for a stationary spectrum in the inertial range. Here we have considered an infinite inertial range in order to use the Zakharov transformation. Whereas the collision integral tends to zero when $n \to -2$, the limit with which we are concerned is not zero because of the presence of a denominator proportional to $2n+4$, and which is a signature of the dimension in wavenumber of the flux. Finally, L'Hôpital's rule gives the value of $P_\perp$ from which it is possible to write the Kolmogorov spectrum of the shear-Alfvén waves,

$$E_\perp(k_\perp, 0) = P_\perp^{1/2}\, C_K(-2)\, k_\perp^{-2}, \tag{63}$$

with the Kolmogorov constant

$$C_K(n) = \left[\frac{-2b_0}{\pi\epsilon^2 J_1(n)}\right]^{1/2}, \tag{64}$$

and with the following form for the integral $J_1(n)$:

$$J_1(n) = 2^{n+3}\int_{x=1}^{\infty}\int_{y=-1}^{1}\frac{[(x^2-1)(1-y^2)]^{1/2}(xy+1)^2}{(x-y)^{n+6}(x+y)^{2-n}} \times [2^{1-n} - (x+y)^{1-n}]\ln[\tfrac{1}{2}(x+y)]\,dx\,dy. \tag{65}$$

As expected, the calculation gives a negative value for the integral $J_1(n)$, and for the particular value $n = -2$, we obtain $C_K(-2) \approx 0.585$. Note that the integral $J_1(n)$ converges only for $-3 < n < -1$.

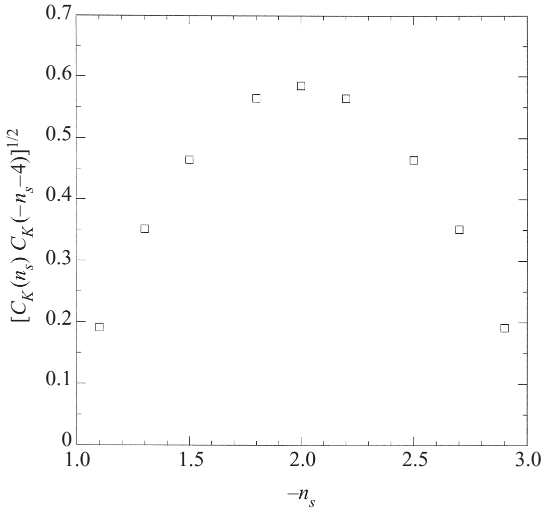

Figure 2. Variation of $[C_K(n_s)\,C_K(-n_s-4)]^{1/2}$ as a function of $-n_s$. Notice the symmetry around the value $-n_s = 2$ corresponding to the case of zero velocity–magnetic-field correlation.

The generalization to the case of non-zero cross-correlation gives the relations

$$\begin{aligned} E^+_\perp(k_\perp,0)\; E^-_\perp(k_\perp,0) &= P^+_\perp \; C^2_K(n_s)\; k_\perp^{-4} \\ &= P^-_\perp \; C^2_K(-n_s-4)\; k_\perp^{-4} \\ &= (P^+_\perp P^-_\perp)^{1/2}\, C_K(n_s)\, C_K(-n_s-4)\, k_\perp^{-4}, \end{aligned} \tag{66}$$

where the second formulation is useful to show the symmetry with respect to s. The computation of the Kolmogorov constant C_K as a function of $-n_s$ is given in Fig. 2. An asymmetric form is observed, which means that the ratio $P^+_\perp/P^-_\perp$ is not constant, as we can see in Fig. 3, where we plot this ratio as a function of $-n_s$. We see that for any ratio $P^+_\perp/P^-_\perp$ there corresponds a unique value of n_s, between the singular ratios $P^+_\perp/P^-_\perp = \infty$ for $n_s = -3$ and $P^+_\perp/P^-_\perp = 0$ for $n_s = -1$. Thus a larger flux of energy P^+ corresponds to a steeper slope of the energy spectra $E^+_\perp(k_\perp, 0)$, in agreement with the physical image that a larger flux of energy implies a faster energy cascade.

In the zero-cross-correlation case, a similar demonstration for the pseudo-Alfvén waves $E^s_\parallel(k_\perp, 0)$ leads to the relation

$$E_\parallel(k_\perp,0) = P_\parallel P_\perp^{-1/2}\; C'_K(-2)\; k_\perp^{-2}, \tag{67}$$

with the general form of the Kolmogorov constant

$$C'_K(n) = \left[\frac{-2b_0 J_1(n)}{\pi\epsilon^2 J_2(n) J_2(-n-4)}\right]^{1/2}, \tag{68}$$

where the integral $J_2(n)$ is

$$\begin{aligned} J_2(n) = 2^{n+3} \int_{x=1}^{\infty}\int_{y=-1}^{1} &\frac{[(x^2-1)(1-y^2)]^{1/2}}{(x-y)^{n+6}\,(x+y)^{2-n}} \\ &\times [2^{1-n} - (x+y)^{1-n}] \ln[\tfrac{1}{2}(x+y)]\, dx\, dy. \end{aligned} \tag{69}$$

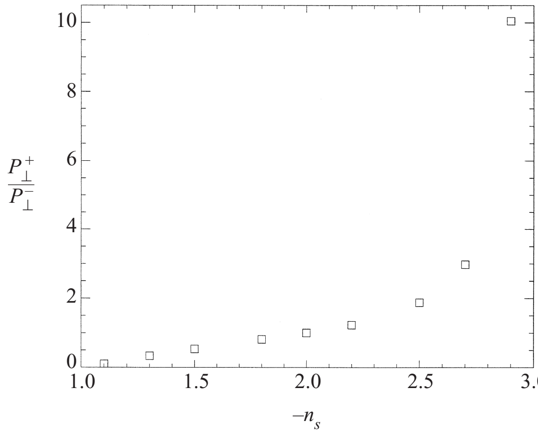

Figure 3. Variation of $P_\perp^+/P_\perp^-$, the ratio of fluxes of energy, as a function of $-n_s$. For the zero-cross-correlation case, the ratio is 1.

Note that the integral $J_2(n)$ converges only for $-3 < n < -1$. The presence of the flux $P_\perp$ in the Kolmogorov spectrum is linked to the presence of $E_\perp$ in the kinetic equation for $E_\parallel$. A numerical evaluation gives $C'_K(-2) \approx 0.0675$, whereas the generalization for non-zero cross-correlation is

$$\begin{aligned} E_\parallel^+(k_\perp, 0)\ E_\parallel^-(k_\perp, 0) &= \frac{P_\parallel^+ P_\parallel^-}{P_\perp^+}\ C_K'^2(n_s)\ k_\perp^{-4} \\ &= \frac{P_\parallel^+ P_\parallel^-}{P_\perp^-}\ C_K'^2(-n_s - 4)\ k_\perp^{-4} \\ &= \frac{P_\parallel^+ P_\parallel^-}{(P_\perp^+ P_\perp^-)^{1/2}}\ C'_K(n_s)\ C'_K(-n_s - 4)\ k_\perp^{-4}, \end{aligned} \tag{70}$$

where the last formulation shows the symmetry with respect to s. The power laws of the spectra $E_\parallel^s$ have the same indices as those of $E_\perp^s$, and the Kolmogorov constant C' is related to C by

$$\frac{C'_K(n_s)}{C'_K(-n_s - 4)} = \frac{C_K(-n_s - 4)}{C_K(n_s)}. \tag{71}$$

Therefore the choice of the ratio $P_\perp^+/P_\perp^-$ determines not only $C_K(n_s)$ but also $C'_K(n_s)$, allowing for free choices of the dissipative rates of energy $P_\parallel^\pm$.

The result of the numerical evaluation of $C'_K(n_s)$ is shown in Fig. 4. An asymmetrical form is also visible; notice also that the values of $C'_K(n_s)$ (i.e. the constant in front of the parallel energy spectra) are smaller by an order of magnitude than those of $C_K(n_s)$ for the perpendicular spectra.

4.2. Temporal evolution of the kinetic equations

4.2.1. Numerical method. Equations (54) and (55) can be integrated numerically with a standard method, as for example presented in Leith and Kraichnan (1972). Since the energy spectrum varies smoothly with k, it is convenient to use a logarithmic subdivision of the k axis,

$$k_i = \delta k\ 2^{i/F}, \tag{72}$$

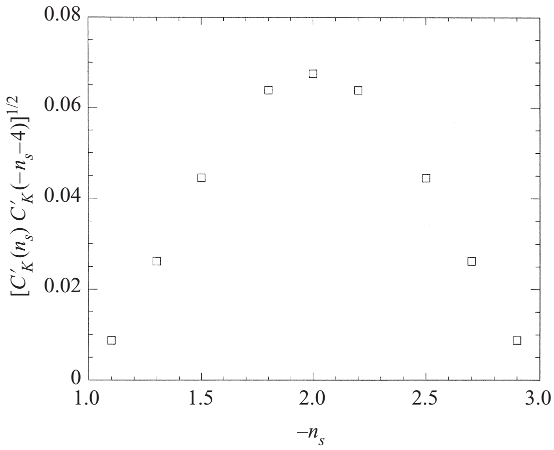

Figure 4. Variation of $[C'_K(n_s)\ C'_K(-n_s-4)]^{1/2}$ as a function of $-n_s$. Notice the symmetry around the value $-n_s = 2$ corresponding to the zero-cross-correlation case.

where i is a non-negative integer; δk is the minimum wavenumber in the computation and F is the number of wave numbers per octave. F defines the refinement of the 'grid', and in particular it is easily seen that a given value of F introduces a cutoff in the degree of non-locality of the nonlinear interactions included in the numerical computation of the kinetic equations. But since the solutions are local, a moderate value of F can be used (namely, we take $F = 4$). Tests have nevertheless been performed with $F = 8$, and we show that no significant changes occur in the results to be described below.

This technique allows us to reach Reynolds numbers much greater than in direct numerical simulations. In order to regularize the equations at large k, we have introduced dissipative terms, which were omitted in the derivation of the kinetic equations. We take the magnetic Prandtl number ν/η to be unity. For example, with $\delta k = 2^{-3}$, $F = 8$, $i_{\max} = 225$; this corresponds to a ratio of scales $2^{28}/2^{-3}$. Taking a wave energy U_0^2 and an integral scale L_0 both of order unity initially, and a kinematic viscosity of $\nu = 3.3\times10^{-8}$, the Reynolds number of such a computation is $\mathscr{R}_e = U_0\ L_0/\nu \sim 10^8$. All numerical simulations to be reported here have been computed on an Alpha Server 8200 located at the Observatoire de la Côte d'Azur (SIVAM).

4.2.2. Shear-Alfvén waves. In this paper, we only consider decaying turbulence. As a first numerical simulation, we have integrated (54) in the zero-cross-correlation case ($E^+ = E^-$) and without forcing. Figure 5(a) shows the temporal evolution of the total energy $E_\perp(t)$ with, by definition

$$E_\perp(t) = \int_{k_{\min}}^{k_{\max}} E_\perp(k_\perp, 0)\, dk_\perp, \tag{73}$$

where $k_{\min}$ and $k_{\max}$ have the values given in the previous section. The total energy is conserved up to a time $t_0 \approx 1.55$, after which it decreases because of the dissipative effects linked to mode coupling, whereas the enstrophy $\int k^2 E_\perp(k)\, d^2\mathbf{k}$ increases sharply (Fig. 5b). The energy spectra at different times are displayed in Fig. 6. As we approach the time t_0, the spectra spread out to reach the smallest scales (i.e. the

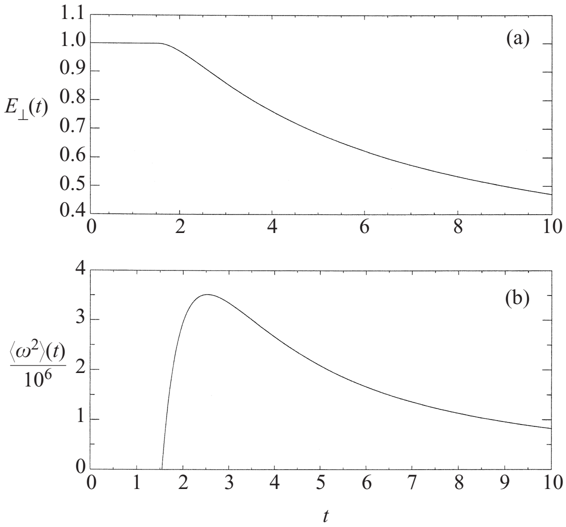

Figure 5. Temporal evolution of (a) the energy $E_\perp(t)$ and (b) the enstrophy $\langle\omega^2(t)\rangle$. Notice the conservation of the energy up to the time $t_0 \approx 1.55$.

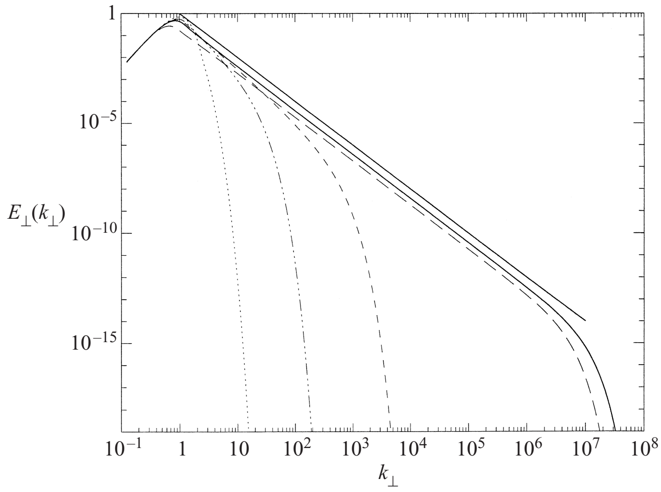

Figure 6. Energy spectra $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves in the zero-cross-correlation case for the times $t = 0$ (dots), 1.0 (dash–dots), 1.5 (short dashes), 1.6 (solid) and 10.0 (long dashes); the straight line follows a $k_\perp^{-2}$ law.

largest wavenumbers). For $t > t_0$, a constant energy flux spectrum $k_\perp^{-2}$ is obtained (indicated by the straight line). For times t significantly greater than t_0, we have a self-similar energy decay, in what constitutes the turbulent regime.

4.2.3. Shear-Alfvén versus pseudo-Alfvén waves. In a second numerical computation we have studied the system (54), (55) with an initial normalized cross-correlation of 80%. The following parameters have been used: $\delta k = 2^{-3}$, $F = 4$, $i_{\max} = 105$ and $\nu = 6.4 \times 10^{-8}$. Figure 7(a) shows the temporal evolution of energies for the four

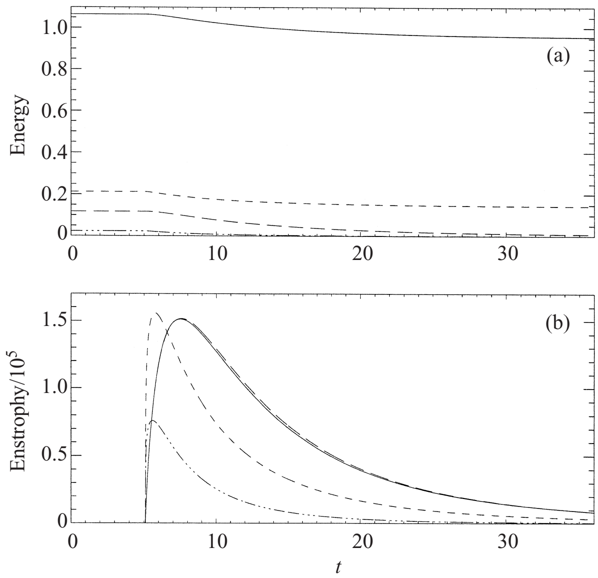

Figure 7. (a) Temporal evolution of the energies $E_\perp^+$ (solid), $E_\perp^-$ (long dashes), $E_\parallel^+$ (short dashes) and $E_\parallel^-$ (dash–dots). (b) The same notation is used for the enstrophies. Notice that the energies are conserved until the time $t_0' \approx 5$.

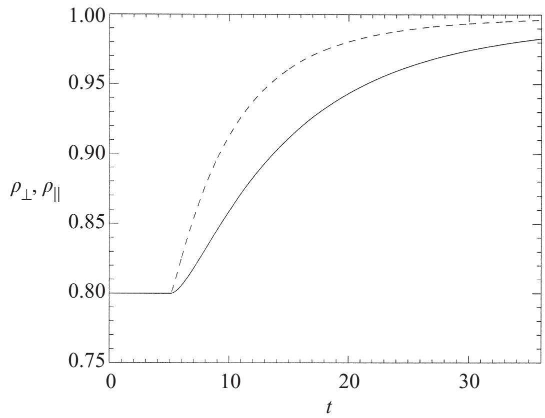

Figure 8. Temporal evolution of the cross-correlations $\rho_\perp$ (solid) and $\rho_\parallel$ (dashes). These quantities are conserved up to the time t_0'.

different waves ($E_\perp^\pm$ and $E_\parallel^\pm$). The same behaviour as that of Fig. 5(a) is observed, with conservation of energy up to the time $t_0' \approx 5$, and decay afterwards; this decay is nevertheless substantially weaker than when the correlation is zero, since, in the presence of a significant amount of correlation between the velocity and the magnetic field, it is easily seen from the primitive MHD equations that the nonlinearities are strongly reduced. On the other hand, the temporal evolution of (Fig 7b) shows that the maxima for these four types of waves are reached at different times: the pseudo-Alfvén waves are the fastest to reach their maxima at $t \approx 5.5$,

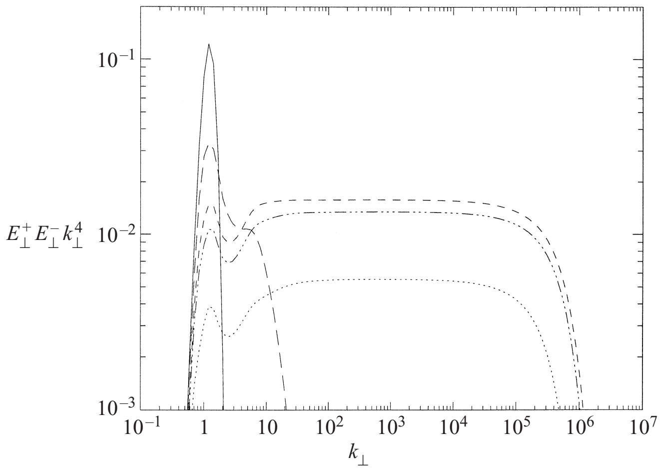

Figure 9. Compensated spectra $E_\perp^+ E_\perp^- k_\perp^4$ at times $t = 0$ (solid), 4 (long dashes), 6 (short dashes), 8 (dash–dots) and 20 (dots).

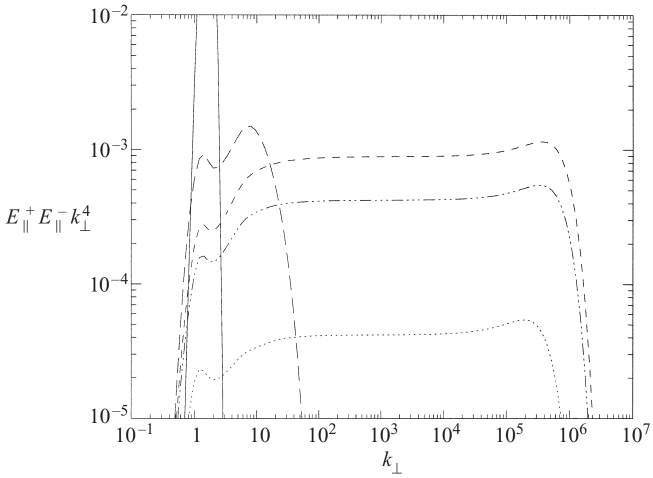

Figure 10. Compensated spectra $E_\parallel^+ E_\parallel^- k_\perp^4$ for the same times and with the same symbols as in Fig. 9.

versus $t \approx 7.5$ for the shear-Alfvén waves. Figure 8 corresponds to the temporal evolution of another conserved quantity, the cross-correlation ρ_x defined as

$$\rho_x = \frac{E_x^+ - E_x^-}{E_x^+ + E_x^-}, \tag{74}$$

where x symbolizes either $\perp$ or $\|$. As expected, ρ_x is constant during an initial period (until $t = t_0'$) and then tends asymptotically to one, but in a faster way for the pseudo-Alfvén waves. This growth of correlation is well documented in the isotropic case (Matthaeus and Montgomery 1980), and is seen to hold as well here in the weak turbulence regime. Figures 9 and 10 give the compensated spectra $E_\perp^+ E_\perp^- k_\perp^4$ and $E_\parallel^+ E_\parallel^- k_\perp^4$ respectively at different times. In both cases, from $t = 6$

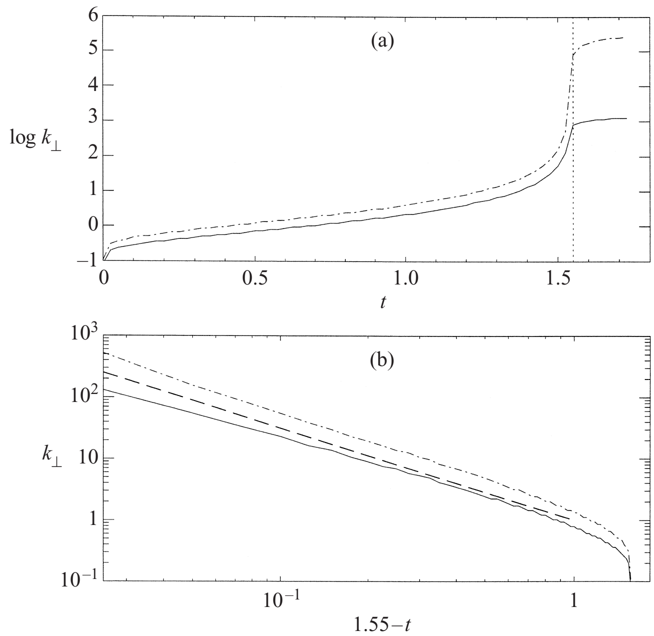

Figure 11. (a) Temporal evolution, in linear–log coordinates, of the front of energy propagating to small scales. The solid line and the dash–dotted line correspond respectively to energies of 10^{-16} and 10^{-25}. An abrupt change is visible at time $t_0 \approx 1.55$ (vertical dotted line). (b) The plot of $k_\perp$ as a function of $1.55 - t$, in log–log coordinates, displays a power-law behaviour in $k_\perp \sim (1.55 - t)^{-1.5}$ (long-dashed line).

onwards, a plateau is observed over almost four decades, and remains flat for long times; this illustrates nicely the theoretical predictions (66) and (70).

5. Front propagation

The numerical study of the transition between the initial state and the final state, where the $k_\perp^{-2}$ spectrum is reached, shows two remarkable properties illustrated by Figs 11 and 12.

We show in Fig. 11(a), in linear–log coordinates, the progression with time of the front of energy propagating to small scales; more precisely, we give the wavenumber at time t with energies of respectively 10^{-25} (dash–dotted line) and 10^{-16} (solid line). Note that both curves display an abrupt change at $t_0 \approx 1.55$, after which the growth is considerably slowed down. Using this data, Fig. 11(b) gives $k_\perp$ as a function of $1.55 - t$, in log–log coordinates the lines having the same meaning as in (a); the long-dashed line represents a power law $k_\perp \sim (1.55 - t)^{-1.5}$. Hence the small scales, in this weak turbulence formalism, are reached in a finite time, i.e. in a catastrophic way. This is also seen on the temporal evolution of the enstrophy (see Fig. 5b), with a catastrophic growth ending at $t \approx 2.5$, after which the decay of energy begins.

Figure 12 shows the temporal evolution of the energy spectrum $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves around the catastrophic time t_0. We see that before t_0, evaluated here with a better precision to be equal to 1.544, the energy spectrum propagates to small scales following a stationary $k_\perp^{-7/3}$ spectrum rather than a $k_\perp^{-2}$ spectrum.

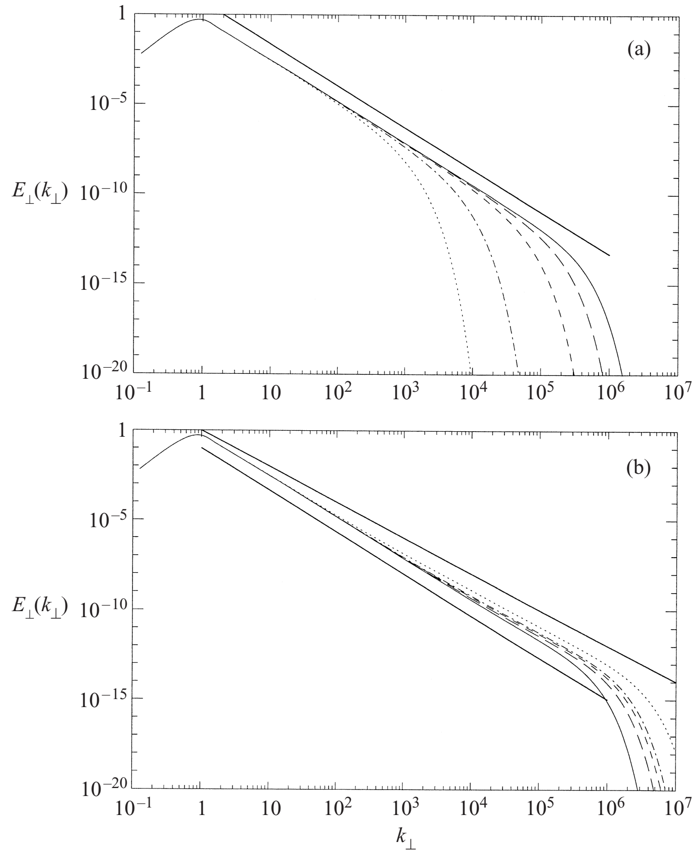

Figure 12. Temporal evolution of the energy spectrum $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves around the catastrophic time $t_0 \approx 1.544$. (a) For $t < t_0$, with $t = 1.50$ (dots), 1.53 (dash–dots), 1.54 (short dashes), 1.542 (long dashes) and 1.543 (solid), a $k_\perp^{-7/3}$ spectrum is observed. (b) For $t \geqslant t_0$, with $t = 1.544$ (solid), 1.546 (long dashes), 1.548 (short dashes), 1.55 (dash–dots) and 1.58 (dots), a fast change of the slope appears to give finally a $k_\perp^{-2}$ spectrum. Note that this change propagates from small scales to large scales. In both cases, straight lines follow either a $k_\perp^{-7/3}$ or a $k_\perp^{-2}$ law.

It is only when the dissipative scale is reached, at t_0, that a remarkable effect is observed: in a very fast time, the $k_\perp^{-7/3}$ solution turns into the finite energy flux spectrum $k_\perp^{-2}$, with a change of the slope propagating from small scales to large scales.

Note that this picture is different from the scenario proposed by Falkovich and Shafarenko (1991, hereinafter FS) for the finite-capacity spectra. In an example considered by FS, the Kolmogorov spectrum forms right behind the propagating front, whereas in our case it forms only after the front reaches infinite wavenumbers (i.e. the dissipative region). The front propagation can be described in terms of self-similar solutions having the form (Falkovich and Shafarenko 1991; Zakharov et al. 1992):

$$E_\perp(k_\perp, 0) = \frac{1}{\tau^a} E_0\left(\frac{k_\perp}{\tau^b}\right), \tag{75}$$

where $\tau = t_0 - t$. Substituting (75) into the kinetic equation (54), we have

$$\partial_\tau \left[\frac{1}{\tau^a} E_0\left(\frac{k_\perp}{\tau^b}\right)\right] \sim \tau^{-a} E_0\left(\frac{\kappa_\perp}{\tau^b}\right)\left[\tau^{b-a} E_0\left(\frac{L_\perp}{\tau^b}\right) - \tau^{b-a} E_0\left(\frac{k_\perp}{\tau^b}\right)\right]\tau^{2b},$$

which leads to the relation

$$1 + 3b = a. \tag{76}$$

If E_0 is stationary and has a power-law form $E_0 \sim k^m$ then we have another relation between a and b:

$$a + mb = 0 \tag{77}$$

Eliminating a from (76) and (77), we have $1+(3+m)b = 0$. In our case, this condition is satisfied because $b = -3/2$ and $m = -7/3$, which confirms that the front solution is of self-similar type.

6. Locality of power-law spectra

As we mentioned above, a Kolmogorov-type spectrum obtained via the Zakharov transform is a solution to the kinetic equations if and only if the original collision integral in this equation (before the Zakharov transformation) converges on it – a property called locality of the spectrum. Bearing in mind that the front propagation spectrum is also of a power-law type, let us study locality of power-law spectra of the general form

$$E_\perp^s(k_\perp, 0) = k_\perp^{m_s}, \tag{78a}$$

$$E_\perp^{-s}(k_\perp, 0) = k_\perp^{m_{-s}}, \tag{78b}$$

where the indices m_s and m_{-s} are arbitrary numbers. Recall that the collision integral in (54) is to be taken over the semi-infinite strip shown in Fig. 1. It may be singular only at the following three points:

(p1) $\kappa_\perp = L_\perp = \infty$;
(p2) $\kappa_\perp = k_\perp, \quad L_\perp = 0$;
(p3) $\kappa_\perp = 0, \quad L_\perp = k_\perp$;

i.e. the corners and infinity of the integration area shown in Fig. 1. To study convergence at the point (p1), it is convenient to change variables:

$$\kappa_\perp + L_\perp = r_+, \qquad \kappa_\perp - L_\perp = r_-, \qquad k_\perp < r_+ < \infty, \qquad -k_\perp < r_- < k_\perp. \tag{79}$$

Taking the limit $r_+ \to \infty$ in the integrand (which corresponds to (p1)) and retaining the largest terms, we obtain the following conditions for convergence:

$$m_{-s} + m_s < 0, \qquad m_{-s} < -1. \tag{80}$$

In the vicinity of the point (p2), it is convenient to use the polar coordinates

$$\kappa_\perp = k_\perp(1 + r\cos\theta), \quad L_\perp = k_\perp r\sin\theta, \quad -\tfrac{1}{4}\pi < \theta < \tfrac{1}{4}\pi, \quad -k_\perp < r < k_\perp. \tag{81}$$

Considering the limit $r \to 0$ and integrating over θ, one can see that the collision integral converges if and only if $m_s > -3$. Similarly, one obtains the convergence condition at the point (p3) which is $m_{-s} > -3$.

All of the convergence conditions in the kinetic equation for E^{-s} are, of course, symmetric to the case of E^s; one simply has to exchange m_{-s} and m_s in these conditions. Summarizing, one can write the conditions for simultaneous convergence for both E^s and E^{-s}:

$$-3 < m_{\pm} < -1. \tag{82}$$

The Kolmogorov spectral exponents lie on the line $m_+ + m_- = -4$, and the locality interval $(-3, -1)$ for one of them maps exactly onto the same interval for another exponent. In particular, the symmetric -2 Kolmogorov spectum is local. One can also see that the front solution with index $-7/3$ is local according to the above locality condition.

7. Fokker–Planck approximation

In the previous section, we established the fact that both the Kolmogorov -2 and the front $-7/3$ spectra are local. However, during the initial phase of the turbulence decay, turbulence may be very non-local. Namely, the nonlinear interaction for short waves will be dominated by triads that involve a long wave corresponding to the initial large-scale turbulence. Our locality analysis suggests that this will happen when the slope of the large-scale part of the spectrum is still steeper than -3; i.e. neither a $-7/3$ nor a -2 small-scale tail has grown strong enough in amplitude yet for the local interaction to take over. Further, the locality analysis suggests that the dominant contribution to the collision integral in this case will come from small vicinities of the points (p2) and (p3), both of which involve one small wavenumber: L and κ respectively. Thus one can expand the integrand of the collision integral in powers of these wavenumbers and reduce the kinetic equation to a second order Fokker–Planck equation, similarly to the way it was done for the Rossby three-wave process in Balk et al. (1990a,b). Below, we shall derive such an equation, considering contributions of the points (p2) and (p3) separately.

Below in this section, we shall consider only the two-dimensional symmetric case, and therefore we omit the superscript s in E^s and the subscript $\perp$ for the wavevectors. The kinetic equation (54) can be rewritten as

$$\frac{\partial E(k)}{\partial t} = 2 \int F(k, \kappa, L)\, d\kappa\, dL, \tag{83}$$

where

$$F(k, \kappa, L) = \frac{\pi \varepsilon^2}{2b_0} (\cos\phi)^2\, |\sin\phi|\, \frac{k^2 L^2}{\kappa^2}\, E(\kappa) \left[\frac{E(L)}{L} - \frac{E(k)}{k} \right], \tag{84}$$

and ϕ is the angle between wavevectors $\mathbf{k}$ and $\boldsymbol{L}$, so that

$$\cos\phi = \frac{k^2 + L^2 - \kappa^2}{2kL}. \tag{85}$$

In a small vicinity of (p2), one can expand F in powers of small L and $h = k - \kappa = O(L)$. Taking into account that $\cos^2\phi = (h/L)^2 + O(L)$, we have

$$F(k, \kappa, L) = \frac{\pi \varepsilon^2}{2b_0} \left(\frac{h}{L}\right)^2 \left| 1 - \left(\frac{h}{L}\right)^2 \right|^{1/2} L\, E(k) E(L). \tag{86}$$

Substituting this expression into (83) and integrating over h from $-L$ to L, we have the following contribution from the point (p2):

$$\left(\frac{\partial E(k)}{\partial t}\right)_2 = 2 D E(k), \tag{87}$$

where the constant

$$D = \frac{\pi^2 \epsilon^2}{16 b_0} \int_0^\infty L^2 E(L)\, dL. \tag{88}$$

Let us now consider the contribution to the collision integral that comes from the vicinity of the point (p3). Introducing the new variable l so that $L = k + l$ and applying the Zakharov transformation (simply $l \to -l$ near the point (p3)), we rewrite (83) as

$$\frac{\partial E(k)}{\partial t} = \int_0^\infty d\kappa \int_{-\kappa}^{\kappa} dl\, [F(k, \kappa, k + l) + F(k, \kappa, k - l)]. \tag{89}$$

Assuming that κ and l are small and that they are of the same order near the point (p3), we have

$$\frac{E(k + l)}{k + l} - \frac{E(k)}{k} = l \left[\frac{\partial (E(k')/k')}{\partial k'} \right]_{k'=k+l/2} + O(l^3), \tag{90}$$

$$k^2 L^2 = k^2 (k + l)^2 = (k + l/2)^4 + O(l^2), \tag{91}$$

$$\cos\phi = 1 + O(l^2), \tag{92}$$

$$|\sin\phi| = |l^2 - \kappa^2|^{1/2} [(k + l/2)^{-1} + O(l^2)]. \tag{93}$$

Substituting these expressions into (89) and further Taylor-expanding the integrand and integrating over l, we have the following main-order contribution from the point (p3):

$$\left(\frac{\partial E(k)}{\partial t} \right)_3 = D \frac{\partial}{\partial k} \left\{ k^3 \frac{\partial}{\partial k} \left[\frac{E(k)}{k} \right] \right\}, \tag{94}$$

where the diffusion constant D is given by (88). Combining (87) and (94), we have the following kinetic Fokker–Planck equation:

$$\frac{\partial E(k)}{\partial t} = D \frac{\partial}{\partial k} \left\{ k^3 \frac{\partial}{\partial k} \left[\frac{E(k)}{k} \right] \right\} + 2D E(k). \tag{95}$$

The first term on the right-hand side of this equation conserves the energy of the short waves because the three-wave interaction near the point (p3) does not transfer any energy to (or from) the large-scale component. The second term is not conservative: it describes a direct non-local energy transfer from the long waves to the short ones. According to (95), the total energy of the short waves grows exponentially. Indeed, one can rewrite this equation in the form of a local conservation law for $N = e^{-2Dt} E$ as follows:

$$\frac{\partial N}{\partial t} = D \frac{\partial}{\partial k} \left[k^3 \frac{\partial}{\partial k} \left(\frac{N}{k} \right) \right]. \tag{96}$$

It is interesting that this equation can also be rewritten in the form of a conservation law for N/k^2:

$$\frac{\partial (N/k^2)}{\partial t} = D \frac{\partial}{\partial k} \left[k^{-1} \frac{\partial}{\partial k} (kN) \right]. \tag{97}$$

Further, there are two independent power-law solutions to (96) and (97): $N \sim k$ and $N \sim 1/k$. The first of these solutions corresponds to the equipartition of N and a constant flux of N/k^2, whereas the second corresponds to the equipartition of N/k^2 and a constant flux of N. This property of the Fokker–Planck equations to have two independent integrals of motion such that the constant flux of one of them corresponds to the equipartition of another one (and vice versa) was recently noticed by Nazarenko and Laval (2000) in the context of the problem of passive scalars. Note, however, that one could expect solutions $N \sim k^{\pm 1}$ only in a very idealized situation when short-wave turbulence is generated by a source separated from the intense long waves by a spectral gap, and only for a limited time until the $-7/3$ tail growing from the large-scale side will fill this spectral gap. In general, the dynamics given by the Fokker–Planck equation (95) describes more complex combination of the instability and diffusion processes, with an energy influx from the initial large scales.

8. MHD turbulence without an external magnetic field

We have considered up to now turbulence of Alfvén waves that arises in the presence of a strong uniform magnetic field. Following Kraichnan (1965), one can assume that the results obtained for turbulence in a strong external magnetic field are applicable to MHD turbulence at small scales that experience the magnetic field of the large-scale component as a quasi-uniform external field. Furthermore, the large-scale magnetic field is much stronger than the one produced by the small scales themselves, because most of the MHD energy is condensed at large scales owing to the decreasing distribution of energy among modes as the wavenumbers grow. In this case, therefore, the small-scale dynamics consists again of a large number of weakly interacting Alfvén waves. Using such a hypothesis and applying a dimensional argument, Kraichnan derived the $k^{-3/2}$ energy spectrum for MHD turbulence. However, Kraichnan did not take into account the local anisotropy associated with the presence of this external field. In Ng and Bhattacharjee (1997) (see also Goldreich and Sridhar 1997), the dimensional argument of Kraichnan is modified in order to take into account the anisotropic dependence of the characteristic time associated with Alfvén waves on the wavevector by simply writing

$$\tau \sim \frac{1}{b_0 k_\parallel}. \tag{98}$$

In that way, one obtains a $k_\perp^{-2}$ energy spectrum, which agrees with the analytical and numerical results of the present paper for the spectral dependence on $k_\perp$. On the other hand, the dependence of the spectra on $k_\parallel$, as we have shown earlier in this paper, is not universal, because of the absence of energy transfer in the $k_\parallel$ direction, although it is shown in Kinney and McWilliams (1998) that, for a quasi-uniform field as considered in this section, there is some transfer in the quasiparallel direction. In the strictly uniform case, this spectral dependence is determined only by the dependence on $k_\parallel$ of the driving and/or initial conditions.

For large time, the spectrum is almost two-dimensional. The characteristic width of the spectrum in $k_\parallel$ (described by the function $f_1(k_\parallel)$) is much less than its width in $k_\perp$, so that approximately one can write

$$e^s(\mathbf{k}) = C\, k_\perp^{-3}\, \delta(k_\parallel), \tag{99}$$

where C is a constant. The factor $k_\perp^{-3}$ corresponds to the $E_\perp \propto k_\perp^{-2}$ Kolmogorov-like spectrum found in this paper (the physical dimensions of e^s and $E_\perp^s$ being different). In the context of MHD turbulence, this spectrum is valid only locally, that is for distances smaller than the length-scale of the magnetic field associated with the energy-containing part of MHD turbulence. Let us average this spectrum over the large energy-containing scales, that is over all possible directions of $\mathbf{B}_0$. Writing $k_\perp = |\mathbf{k} \times \mathbf{B}_0|/|\mathbf{B}_0|$ and $k_\parallel = |\mathbf{k} \cdot \mathbf{B}_0|/|\mathbf{B}_0|$ and assuming that $\mathbf{B_0}$ takes all possible directions in 3D space with equal probability, we have for the averaged spectrum

$$\begin{aligned}\langle e^s \rangle &= \int e^s(\mathbf{k}, \mathbf{x})\, d\sigma(\boldsymbol{\zeta}) \\ &= \int C\delta(\boldsymbol{\zeta} \cdot \mathbf{k})\, |\boldsymbol{\zeta} \times \mathbf{k}|^{-3} d\sigma(\boldsymbol{\zeta}),\end{aligned} \tag{100}$$

where $\boldsymbol{\zeta} = (\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta)$ is a unit vector in the coordinate space and $d\sigma = \sin\theta\, d\theta\, d\phi$ is the surface element on the unit sphere. Choosing θ to be the angle between $\mathbf{k}$ and $\mathbf{B}_0$ and ϕ to be the angle in the plane transverse to $\mathbf{k}$, we have

$$\begin{aligned}\langle e^s \rangle &= C \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\delta(\cos\theta)}{|\mathbf{k}|} \frac{(\sin\theta)^{-3}}{|\mathbf{k}|^3}\, \sin\theta\, d\theta \\ &= 2\pi C k^{-4}.\end{aligned} \tag{101}$$

This isotropic spectrum represents the averaged energy density in 3D wavevector space. By averaging over all possible directions of the wavevector, we get the following density of the energy distribution over the absolute value of the wavevector:

$$E_k = 8\pi^2 C\, k^{-2}. \tag{102}$$

As we see, taking into account the local anisotropy and subsequent averaging over the isotropic energy-containing scales results in an isotropic energy spectrum k^{-2}. This result is different from the $k^{-3/2}$ spectrum derived by Kraichnan without taking into account the local anisotropy of small scales. The difference in spectral indices may also arise from the fact that the approach here is that of weak turbulence, whereas in the strong turbulence case, isotropy is recovered on average and a different spectrum – that of Kraichnan – obtains.

Solar-wind data (Matthaeus and Goldstein 1982) indicates that the isotropic spectrum scales as $k^{-\alpha}$, with $\alpha \approx 1.67$, close to the Kolmogorov value for neutral fluids (without intermittency corrections, which are known to occur); hence it could be interpreted as well as being a Kraichnan-like spectrum steepened by intermittency effects, which are known to occur in strong MHD turbulence as well in the form of current and vorticity filaments, ribbons and sheets, and magnetic flux tubes. However, in the context of the interstellar medium (ISM), data show that the velocity dispersion is correlated with the size of the region observed (Larson 1981; Scalo 1984; Falgarone et al. 1992). These correlations are approximately of power-law form such that the corresponding energy spectrum scales with a spectral index ranging from $\alpha \approx 1.6$ to $\alpha \approx 2$. Then the weak turbulence approach could explain the steepening of the spectrum. But the variety of physical processes in the ISM, such as shocks and dispersive effects, do not allow a definitive answer, but rather raise the question: by how much is the energy spectrum of a turbulent medium affected by such physical processes? A formalism that incorporates dispersive effects in MHD, for example the Hall current for a strongly ionized plasma, as in the magnetosphere or in the vicinity of protostellar jets or the ambipolar drift in the weakly ionized plasma of the interstellar medium at large, will be useful, but is left for future work. So is the incorporation of compressibility.

9. Conclusions

We have obtained in this paper the kinetic equations for weak Alfvénic turbulence in the presence of correlations between the velocity and the magnetic field, and taking into account the non-mirror invariance of the MHD equations leading to non-zero helical terms. These equations, contrary to what is stated in Sridhar and Goldreich (1994), hold at the level of three-wave interactions (see also Montgomery and Matthaeus 1995; Ng and Bhattacharjee 1996).

In this anisotropic medium, a new spectral tensor must be taken into account in the formalism when compared with the isotropic case (which can include terms proportional to the helicity); this new spectral tensor I^s is linked to the anisotropy induced by the presence of a strong uniform magnetic field, and we can also study its dynamics. This purely anisotropic correlator was also analysed in the case of neutral fluids in the presence of rotation (Cambon and Jacquin 1989).

We have obtained an asymptotic two-dimensionalization of the spectra: indeed, the evolution of the turbulent spectra at each $k_\parallel$ is determined only by the spectra at the same $k_\parallel$ and by the purely 2D state characterized by $k_\parallel = 0$. This property of two-dimensionalization was previously obtained theoretically from an analysis of the linearized MHD equations (Montgomery and Turner 1981) and using phenomenological models (Ng and Bhattacharjee 1996), and as well as numerically (Oughton et al. 1994; Kinney and McWilliams 1998), whereas it has been obtained in our paper from the rigorously derived kinetic equations. Note that the strong field $\mathbf{B}_0$ has no structure (it is a $\mathbf{k} = \mathbf{0}$ field), whereas the analysis performed in Kinney and McWilliams (1998) considers a strong quasi-uniform magnetic field of characteristic wavenumber $k_L \neq 0$, in which case the authors find that two-dimensionalization obtains as well for large enough wavenumbers.

We have considered three-dimensional turbulence in the asymptotic regime of large time, when the spectrum tends to a quasi-2D form. This is the same regime for which the RMHD approach is valid (Strauss 1976). However, in addition to the shear-Alfvén waves described by the RMHD equations, the kinetic equations also describe the dynamics of the so-called pseudo-Alfvén waves, which are decoupled from the shear-Alfvén waves in this case, from the magnetic helicity and from the pseudo-helicity. Finding the Kolmogorov solution for the 3D case is technically very similar to the case of 2D turbulence. This leads to the spectrum $f(k_\parallel)\, k_\perp^{-2}$, where $f(k_\parallel)$ is an arbitrary function, which is to be fixed by matching in the forcing region at small wavenumbers. The $k_\parallel$ dependence is non-universal, and depends on the form of the forcing because of the property that there is no energy transfer between different values of $k_\parallel$.

The $f(k_\parallel)\, k_\perp^{-2}$ energy spectrum is well verified by numerics, which also show that this spectrum is reached in a singular fashion, with small scales developing in a finite time. We have also obtained a family of Kolmogorov solutions with different values of spectra for different wave polarities, and we have shown that the sum of the spectral exponents of these spectra is equal to -4. The dynamics of both the shear-Alfvén waves and the pseudo-Alfvén waves have been obtained. Finally, the small-scale spectrum of isotropic 3D MHD turbulence in the case when there is no external field has also been derived.

The weak turbulence regime remains valid as long as the Alfvén characteristic time $(k_\parallel b_0)^{-1}$ remains small compared with the transfer time (which can be evaluated from (26)), that is to say for $\epsilon^2 E^s_\perp(k_\perp, 0) k_\perp^2 / B_0^2 \ll k_\parallel / k_\perp$. Using the exact scaling law found in this paper, $E^s_\perp(k_\perp, 0) \propto k_\perp^{-2}$, we see that the condition for weak turbulence is less well satisfied for large $k_\perp$, or small $k_\parallel$. This means that we have a non-uniform expansion in B_0.

The dynamo problem in the present formalism reduces to its simplest expression: in the presence of a strong uniform magnetic field $\mathbf{B}_0$, to a first approximation (closing the equations at the level of second-order correlation tensors), one obtains immediate equipartition between the kinetic and magnetic wave energies, corresponding to an instantaneous efficiency of the dynamo. Of course, one may ask about the origin of $\mathbf{B}_0$ itself, in which case one may resort to standard dynamo theories (see e.g. Parker 1994). We see no tendency towards condensation.

In view of the ubiquity of turbulent conducting flows embedded in strong quasi-uniform magnetic fields, the present derivation should be of some use when studying the dynamics of such media, even though compressibility effects have been ignored. It can be argued (Goldreich and Sridhar 1995) that this incompressible approximation may be sufficient because of the damping of the fast magnetosonic wave by plasma kinetic effects. Note, however, that Bhattacharjee et al. (1998) found that, in the presence of spatial inhomogeneities, there are significant departures from incompressibility at the leading order of an asymptotic theory that assumes that the Mach number of the turbulence is small. Finally, the wave energy may not remain negligible for all times, in which case resort to phenomenological models for strong MHD turbulence is required. Also desirable is an exploration of such complex flows through the analysis of laboratory and numerical experiments, and through detailed observations like those stemming from satellite data for the solar wind, from the THEMIS instrument for the Sun looking at the small-scale magnetic structures of the photosphere, and from the planned large-array instrumentation (LSA–ALMA) to observe the interstellar medium in detail.

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Appendix

From the dynamical equations (14), one writes successively for the second- and third-order moments of the $\mathbf{z}^s$ fields

$$\partial_t[a_j^s(\mathbf{k})a_{j'}^{s'}(\mathbf{k}')] = -i\epsilon k_m P_{jn}(\mathbf{k}) \int [a_m^s(\boldsymbol{\kappa})a_n^s(\mathbf{L})a_{j'}^{s'}(\mathbf{k}')] e^{-2is\omega_\kappa t}\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}}\, d_{\boldsymbol{\kappa}\mathbf{L}} \\ -i\epsilon k'_m P_{j'm}(\mathbf{k}') \int [a_m^{-s'}(\boldsymbol{\kappa})a_n^{s'}(\mathbf{L})a_j^s(\mathbf{k})] e^{-2is'\omega_\kappa t}\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}'}\, d_{\boldsymbol{\kappa}\mathbf{L}} \quad \text{(A 1)}$$

and

$$\partial_t[a_j^s(\mathbf{k})a_{j'}^{s'}(\mathbf{k}')a_{j''}^{s''}(\mathbf{k}'')] = \\ -i\epsilon k_m P_{jn}(\mathbf{k}) \int [a_m^{-s}(\boldsymbol{\kappa})a_n^s(\mathbf{L})a_{j'}^{s'}(\mathbf{k}')a_{j''}^{s''}(\mathbf{k}'')] e^{-2is\omega_\kappa t}\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}}\, d_{\boldsymbol{\kappa}\mathbf{L}} \\ + \left\{[\mathbf{k}, s, j] \to [\mathbf{k}', s', j'] \to [\mathbf{k}'', s'', j''] \to [\mathbf{k}, s, j]\right\}. \quad \text{(A 2)}$$

Asymptotic closure for the leading-order contributions to each of the cumulants follows from the following procedure or algorithm. Cumulants are in one-to-one correspondence with the moments: in the zero-mean case, the second and third moments are the second and third cumulants; the fourth-order moment is the sum of cumulants where each decomposition of the moment is taken once, namely the sum of the fourth-order cumulant plus products of second-order ones. One attempts to solve the hierarchy of cumulant equations perturbatively. The asymptotic expansions generated in this way are not uniform in time, because of the presence of small divisors (mainly, but not totally, due to resonances). In order to restore the uniform validity of the asymptotic expansions, we must allow the leading-order contributions to each of the cumulants to vary slowly in time and choose that time dependence to achieve that goal. Where necessary, and where the notion of well-orderedness does not make sense in Fourier space, one must look at the corresponding asymptotic expansions in physical space. The result is another set of asymptotic expansions that include both the kinetic equations for the second-order moments, combinations of which give the parallel and total energies and helicity densities, and similar equations for higher-order cumulants, which can be collectively solved by a common frequency renormalization. The kinetic equations are valid for time scales of the order of ϵ^{-2} and possibly longer, depending on how degenerate the resonant manifolds are. Success in obtaining asymptotic closure depends on two ingredients. The first is the degree to which the linear waves interact to randomize phases, and the second is the fact that the nonlinear regeneration of the third-order moment by the fourth-order moment in (A 2) depends more on the product of the second-order moments than it does on the fourth-order cumulant. We now give details of the calculations.

The fourth-order moment in the above equation, $\langle \boldsymbol{\kappa}\mathbf{L}\mathbf{k}'\mathbf{k}''\rangle$ in short-hand notation, decomposes into the sum of three products of second-order moments, and a fourth-order cumulant $\{mnj'j''\}$. The latter does not contribute to secular behaviour, and, of the remaining terms, one is absent as well in the kinetic equations because it involves the combination of wavenumbers $\langle \boldsymbol{\kappa}\mathbf{L}\rangle\langle \mathbf{k}'\mathbf{k}''\rangle$: it introduces, because of homogeneity, a $\delta(\boldsymbol{\kappa}+\mathbf{L})$ factor that combined with the convolution integral, leads to a zero contribution for $\mathbf{k}=\mathbf{0}$. Hence the time evolution of the third-order cumulants leads to six terms that read

$$\begin{aligned}\partial_t\left[a_j^s(\mathbf{k})a_{j'}^{s'}(\mathbf{k}')a_{j''}^{s''}(\mathbf{k}'')\right] = &-i\epsilon k_m P_{jn}(\mathbf{k})q_{mj'}^{-ss'}(\mathbf{k}')q_{nj''}^{ss''}(\mathbf{k}'')e^{2is\omega' t}\\ &-i\epsilon k_m P_{jn}(\mathbf{k})q_{mj''}^{-ss''}(\mathbf{k}'')q_{nj'}^{ss'}(\mathbf{k}')e^{2is\omega'' t}\\ &-i\epsilon k'_m P_{j'n}(\mathbf{k}')q_{mj''}^{-s's''}(\mathbf{k}'')q_{nj}^{s's}(\mathbf{k})e^{2is'\omega'' t}\\ &-i\epsilon k'_m P_{j'n}(\mathbf{k}')q_{mj}^{-s's}(\mathbf{k})q_{nj''}^{s's''}(\mathbf{k}'')e^{2is'\omega t}\\ &-i\epsilon k''_m P_{j''n}(\mathbf{k}'')q_{mj}^{-s''s}(\mathbf{k})q_{nj'}^{s''s'}(\mathbf{k}')e^{2is''\omega t}\\ &-i\epsilon k''_m P_{j''n}(\mathbf{k}'')q_{mj'}^{-s''s'}(\mathbf{k}')q_{nj}^{s''s}(\mathbf{k})e^{2is''\omega' t}.\end{aligned} \quad \text{(A 3)}$$

It can be shown that, of these six terms, only the fourth and fifth make non-zero contributions to the kinetic equations. Defining

$$\omega_{k,\kappa L} = \omega_k - \omega_\kappa - \omega_L, \quad \text{(A 4)}$$

and integrating (A 3) over time, the exponential terms lead to

$$\begin{aligned}\Delta(\omega_{k,\kappa L}) &= \int_0^t \exp\left(it\omega_{k,\kappa L}\right)dt\\ &= \frac{\exp\left(i\omega_{k,\kappa L}\right)-1}{i\omega_{k,\kappa L}}.\end{aligned} \quad \text{(A 5)}$$

Substituting these expressions into (A 3), only the terms that have arguments in the Δ functions that cancel exactly with the arguments in the exponential appearing in (A 1) will contribute. We then obtain the fundamental kinetic equations for the energy tensor:

$$\begin{aligned}\partial_t[q^{ss'}_{jj'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')] = &-\epsilon^2 k_m P_{jn}(\mathbf{k})\int k_{2p}P_{nq}(\mathbf{L})q^{-s-s}_{pm}(\boldsymbol{\kappa})q^{ss'}_{qj'}(\mathbf{k}')\Delta(-2s\omega_1)\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}}\,d_{\boldsymbol{\kappa}\mathbf{L}}\\ &-\epsilon^2 k_m P_{jn}(\mathbf{k})\delta_{ss'}\int k'_p P_{j'q}(\mathbf{k}')q^{-s'-s}_{pm}(\boldsymbol{\kappa})q^{s's}_{qn}(\mathbf{L})\Delta(-2s\omega_1)\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}}\,d_{\boldsymbol{\kappa}\mathbf{L}}\\ &-\epsilon^2 k'_m P_{j'n}(\mathbf{k}')\int k_{2p}P_{nq}(\mathbf{L})q^{-s'-s'}_{pm}(\boldsymbol{\kappa})q^{s's}_{qj}(\mathbf{k})\Delta(-2s'\omega_1)\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}'}\,d_{\boldsymbol{\kappa}\mathbf{L}}\\ &-\epsilon^2 k'_m P_{j'n}(\mathbf{k}')\delta_{ss'}\int k_p P_{nq}(\mathbf{k})q^{-s-s'}_{pm}(\boldsymbol{\kappa})q^{ss'}_{qn}(\mathbf{L})\Delta(-2s\omega_1)\delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}'}\,d_{\boldsymbol{\kappa}\mathbf{L}}. \qquad \text{(A 6)}\end{aligned}$$

We now perform an integration over the Δs, and, taking the limit $t\to\infty$, we find

$$\begin{aligned}\partial_t[q^{ss'}_{jj'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')] = -\epsilon^2\iiint\Bigg\{&Q^{-s}_k(\boldsymbol{\kappa})P_{jn}(\mathbf{k})P_{nl}(\mathbf{L})[q^{ss'}_{jj'}(\mathbf{k}')\left[\frac{\pi}{2}\delta(\kappa_\parallel)-i\mathscr{P}\left(\frac{1}{2s\kappa_\parallel}\right)\right]\\ &+Q^{-s'}_k(\boldsymbol{\kappa})P_{j'n}(\mathbf{k})P_{nl}(\mathbf{L})[q^{s's}_{lj}(\mathbf{k})\left[\frac{\pi}{2}\delta(\kappa_\parallel)+i\mathscr{P}\left(\frac{1}{2s'\kappa_\parallel}\right)\right]\\ &-\pi\delta_{ss'}Q^{-s}_k(\boldsymbol{\kappa})P_{j'l}(\mathbf{k})P_{jn}(\mathbf{k}')q^{ss}_{ln}(\mathbf{L})\delta(\kappa_\parallel)\Bigg\}d\kappa_1 d\kappa_2 d\kappa_\parallel, \qquad \text{(A 7)}\end{aligned}$$

where $\mathscr{P}$ stands for the principal value of the integral.

In the case $s=s'$, which is of interest here because the cross-correlators between z fields of opposite polarities decay to zero in that approximation, the above equations simplify to

$$\begin{aligned}\frac{2}{\pi}\partial_t[q^{ss}_{jj'}(\mathbf{k}')\pm q^{ss}_{j'j}(\mathbf{k}')] = 2\epsilon^2&\int P_{jn}(\mathbf{k})P_{j'q}(\mathbf{k})[q^{ss}_{qn}(\mathbf{L})\pm q^{ss}_{nq}(\mathbf{L})]Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel\\ &-\epsilon^2\int P_{jn}(\mathbf{k})P_{nq}(\mathbf{L})[q^{ss}_{qj'}(\mathbf{k}')\pm q^{ss}_{j'q}(\mathbf{k}')]Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel\\ &-\epsilon^2\int P_{j'n}(\mathbf{k})P_{nq}(\mathbf{L})[q^{ss}_{jq}(\mathbf{k}')\pm q^{ss}_{qj}(\mathbf{k}')]Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel\\ &+\frac{i\epsilon^2}{s\pi}\mathscr{P}\int P_{jn}(\mathbf{k})P_{nq}(\mathbf{L})[q^{ss}_{qj'}(\mathbf{k}')\mp q^{ss}_{j'q}(\mathbf{k}')]Q^{-s}_k(\boldsymbol{\kappa})\frac{d\kappa_1 d\kappa_2 d\kappa_\parallel}{\kappa_\parallel}\\ &-\frac{i\epsilon^2}{s\pi}\mathscr{P}\int P_{j'n}(\mathbf{k})P_{nq}(\mathbf{L})[q^{ss}_{jq}(\mathbf{k}')\mp q^{ss}_{qj}(\mathbf{k}')]Q^{-s}_k(\boldsymbol{\kappa})\frac{d\kappa_1 d\kappa_2 d\kappa_\parallel}{\kappa_\parallel}. \qquad \text{(A 8)}\end{aligned}$$

To derive the kinetic equations, we now need to develop

$$\partial_t e^s(\mathbf{k}) = \partial_t[q^{ss}_{11}(\mathbf{k})+q^{ss}_{22}(\mathbf{k})+q^{ss}_{\parallel\parallel}(\mathbf{k})], \qquad \text{(A 9)}$$

$$\partial_t\Phi^s(\mathbf{k}) = k_\perp^{-4}\partial_t q^{ss}_{\parallel\parallel}(\mathbf{k}), \qquad \text{(A 10)}$$

$$\partial_t R^s(\mathbf{k}) = \frac{1}{-ik_\parallel k_\perp^2}\partial_t[q^{ss}_{12}(\mathbf{k})-q^{ss}_{21}(\mathbf{k})], \qquad \text{(A 11)}$$

$$\partial_t I^s(\mathbf{k}) = k_\perp^{-4}\partial_t\{k_2[q^{ss}_{1\parallel}(\mathbf{k})+q^{ss}_{\parallel 1}(\mathbf{k})]-k_1[q^{ss}_{2\parallel}(\mathbf{k})+q^{ss}_{\parallel 2}(\mathbf{k})]\} \qquad \text{(A 12)}$$

in terms of (A 7) and (A 8). This leads to

$$\partial_t e^s(\mathbf{k}) = \frac{\pi\epsilon^2}{2}\int\bigg\{2[q^{ss}_{11}(\mathbf{L}) + q^{ss}_{22}(\mathbf{L}) + q^{ss}_{\parallel\parallel}(\mathbf{L}) - q^{ss}_{11}(\mathbf{k}) - q^{ss}_{22}(\mathbf{k}) + q^{ss}_{\parallel\parallel}(\mathbf{k})] + \frac{L_nL_l}{L^2}[q^{ss}_{ln}(\mathbf{k}) + q^{ss}_{nl}(\mathbf{k})] - \frac{k_nk_l}{k^2}[q^{ss}_{ln}(\mathbf{L}) + q^{ss}_{nl}(\mathbf{L})]\bigg\}Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel, \tag{A 13}$$

$$\partial_t \Phi^s(\mathbf{k}) = \frac{\pi\epsilon^2}{2\mathbf{k}_\perp^4}\int\bigg\{2q^{ss}_{\parallel\parallel}(\mathbf{L}) - 2\frac{k_\parallel k_l}{k^2}[q^{ss}_{l\parallel}(\mathbf{L}) + q^{ss}_{\parallel l}(\mathbf{L})] + \frac{k_\parallel^2}{\mathbf{k}^4}k_nk_l[q^{ss}_{ln}(\mathbf{L}) + q^{ss}_{nl}(\mathbf{L})]\bigg\}$$
$$+\bigg\{-2q^{ss}_{\parallel\parallel}(\mathbf{k}) + \left(\frac{L_\parallel L_l}{L^2} + \frac{k_\parallel k_l}{k^2} - \frac{k_\parallel \mathbf{k}\cdot\mathbf{L}}{k^2L^2}L_l\right)[q^{ss}_{l\parallel}(\mathbf{k}) + q^{ss}_{\parallel l}(\mathbf{k})]\bigg\}$$
$$\times Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)\,d\kappa_1\,d\kappa_2\,d\kappa_\parallel$$
$$+\frac{i\epsilon^2 s}{2}\mathscr{P}\int\bigg\{-\left(\frac{L_\parallel L_l}{L^2} + \frac{k_\parallel k_l}{k^2} - \frac{k_\parallel \mathbf{k}\cdot\mathbf{L}}{k^2L^2}L_l\right)[q^{ss}_{l\parallel}(\mathbf{k}) - q^{ss}_{\parallel l}(\mathbf{k})]\bigg\}$$
$$\times Q^{-s}_k(\boldsymbol{\kappa})\frac{d\kappa_1 d\kappa_2 d\kappa_\parallel}{\kappa_\parallel}, \tag{A 14}$$

$$\partial_t R^s(\mathbf{k}) = \frac{i\pi\epsilon^2}{2k_\parallel k_\perp^2}\bigg\{2\int\bigg\{[q^{ss}_{21}(\mathbf{L}) - q^{ss}_{12}(\mathbf{L})] - \frac{k_2k_l}{k^2}[q^{ss}_{l1}(\mathbf{L}) - q^{ss}_{1l}(\mathbf{L})]$$
$$-\frac{k_1k_n}{k^2}[q^{ss}_{2n}(\mathbf{L}) - q^{ss}_{n2}(\mathbf{L})] + \frac{k_1k_2k_nk_l}{k^4}[q^{ss}_{ln}(\mathbf{L}) - q^{ss}_{nl}(\mathbf{L})]\bigg\}$$
$$\times Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel - \int\bigg\{[q^{ss}_{12}(\mathbf{k}) - q^{ss}_{21}(\mathbf{k})] - \frac{L_1L_l}{L^2}[q^{ss}_{l2}(\mathbf{k}) - q^{ss}_{2l}(\mathbf{k})]$$
$$-\frac{k_1k_l}{k^2}[q^{ss}_{l2}(\mathbf{k}) - q^{ss}_{2l}(\mathbf{k})] + \frac{k_1\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}[q^{ss}_{l2}(\mathbf{k}) - q^{ss}_{2l}(\mathbf{k})]\bigg\}Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel$$
$$-\int\bigg\{[q^{ss}_{12}(\mathbf{k}) - q^{ss}_{21}(\mathbf{k})] - \frac{L_2L_l}{L^2}[q^{ss}_{1l}(\mathbf{k}) - q^{ss}_{l1}(\mathbf{k})] - \frac{k_2k_l}{k^2}[q^{ss}_{1l}(\mathbf{k}) - q^{ss}_{l1}(\mathbf{k})]$$
$$+\frac{k_2\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}[q^{ss}_{1l}(\mathbf{k}) - q^{ss}_{l1}(\mathbf{k})]\bigg\}Q^{-s}_k(\boldsymbol{\kappa})\delta(\kappa_\parallel)d\kappa_1 d\kappa_2 d\kappa_\parallel$$
$$+\frac{is\mathscr{P}}{\pi}\int\bigg\{-\frac{L_1L_l}{L^2}[q^{ss}_{l2}(\mathbf{k}) + q^{ss}_{2l}(\mathbf{k})] - \frac{k_1k_l}{k^2}[q^{ss}_{l2}(\mathbf{k}) + q^{ss}_{2l}(\mathbf{k})]$$
$$+\frac{k_1\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}[q^{ss}_{l2}(\mathbf{k}) + q^{ss}_{2l}(\mathbf{k})] + \frac{L_2L_l}{L^2}[q^{ss}_{1l}(\mathbf{k}) + q^{ss}_{l1}(\mathbf{k})] + \frac{k_2k_l}{k^2}[q^{ss}_{1l}(\mathbf{k}) + q^{ss}_{l1}(\mathbf{k})]$$
$$-\frac{k_2\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}[q^{ss}_{1l}(\mathbf{k}) + q^{ss}_{l1}(\mathbf{k})]\bigg\}\,Q^{-s}_k(\boldsymbol{\kappa})\frac{d\kappa_1 d\kappa_2 d\kappa_\parallel}{\kappa_\parallel}\bigg\}, \tag{A 15}$$

$$
\begin{aligned}
\partial_t I^s(\mathbf{k}) = \frac{\pi\epsilon^2 k_2}{2k_\perp^4}\Bigg\{ & 2\int\Bigg\{[q^{ss}_{1\|}(\mathbf{L}) + q^{ss}_{\|1}(\mathbf{L})] - \frac{k_1 k_l}{k^2}[q^{ss}_{l\|}(\mathbf{L}) + q^{ss}_{\|l}(\mathbf{L})] \\
& -\frac{k_\| k_n}{k^2}[q^{ss}_{1n}(\mathbf{L}) + q^{ss}_{n1}(\mathbf{L})] + \frac{k_\| k_1 k_n k_l}{k^4}[q^{ss}_{ln}(\mathbf{L}) + q^{ss}_{nl}(\mathbf{L})]\Bigg\} \\
& \times Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\|)d\kappa_1 d\kappa_2 d\kappa_\| - \int\Bigg\{[q^{ss}_{\|1}(\mathbf{k}) + q^{ss}_{1\|}(\mathbf{k})] \\
& +\left(-\frac{L_\| L_l}{L^2} - \frac{k_\| k_l}{k^2} + \frac{k_\|\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{l1}(\mathbf{k}) + q^{ss}_{1l}(\mathbf{k})]\Bigg\}Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\|)d\kappa_1 d\kappa_2 d\kappa_\| \\
& -\int\Bigg\{[q^{ss}_{\|1}(\mathbf{k}) + q^{ss}_{1\|}(\mathbf{k})] + \left(-\frac{L_1 L_l}{L^2} - \frac{k_1 k_l}{k^2} + \frac{k_1\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{\|l}(\mathbf{k}) + q^{ss}_{l\|}(\mathbf{k})]\Bigg\} \\
& \times Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\|)d\kappa_1 d\kappa_2 d\kappa_\| \\
& +\frac{is\mathscr{P}}{\pi}\int\Bigg\{\left(-\frac{L_\| L_l}{L^2} - \frac{k_\| k_l}{k^2} + \frac{k_\|\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{l1}(\mathbf{k}) - q^{ss}_{1l}(\mathbf{k})] \\
& +\left(-\frac{L_1 L_l}{L^2} - \frac{k_1 k_l}{k^2} + \frac{k_1\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{\|l}(\mathbf{k}) - q^{ss}_{l\|}(\mathbf{k})]\Bigg\}Q_k^{-s}(\boldsymbol{\kappa})\frac{d\kappa_1 d\kappa_2 d\kappa_\|}{\kappa_\|}\Bigg\} \\
-\frac{\pi\epsilon^2 k_1}{2k_\perp^4}\Bigg\{ & 2\int\Bigg\{[q^{ss}_{\|2}(\mathbf{L}) + q^{ss}_{2\|}(\mathbf{L})] - \frac{k_\| k_l}{k^2}[q^{ss}_{l2}(\mathbf{L}) + q^{ss}_{2l}(\mathbf{L})] \\
& -\frac{k_2 k_n}{k^2}[q^{ss}_{\|n}(\mathbf{L}) + q^{ss}_{n\|}(\mathbf{L})] + \frac{k_2 k_\| k_n k_l}{k^4}[q^{ss}_{ln}(\mathbf{L}) + q^{ss}_{nl}(\mathbf{L})]\Bigg\} \\
& \times Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\|)d\kappa_1 d\kappa_2 d\kappa_\| \\
& -\int\Bigg\{\left(-\frac{L_2 L_l}{L^2} - \frac{k_2 k_l}{k^2} + \frac{k_2\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{l\|}(\mathbf{k}) + q^{ss}_{\|l}(\mathbf{k})] \\
& +\left(-\frac{L_\| L_l}{L^2} - \frac{k_\| k_l}{k^2} + \frac{k_\|\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{2l}(\mathbf{k}) + q^{ss}_{l2}(\mathbf{k})]\Bigg\}Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\|)d\kappa_1 d\kappa_2 d\kappa_\| \\
& +\frac{is\mathscr{P}}{\pi}\int\Bigg\{\left(-\frac{L_2 L_l}{L^2} - \frac{k_2 k_l}{k^2} + \frac{k_2\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{l\|}(\mathbf{k}) - q^{ss}_{\|l}(\mathbf{k})] \\
& +\left(-\frac{L_\| L_l}{L^2} - \frac{k_\| k_l}{k^2} + \frac{k_\|\mathbf{k}\cdot\mathbf{L}L_l}{k^2L^2}\right)[q^{ss}_{2l}(\mathbf{k}) - q^{ss}_{l2}(\mathbf{k})]\Bigg\}Q_k^{-s}(\boldsymbol{\kappa})\frac{d\kappa_1 d\kappa_2 d\kappa_\|}{\kappa_\|}\Bigg\}.
\end{aligned}
\tag{A 16}
$$

The final step, which leads to the kinetic equations (26)–(29), consists in substituting the expressions (21) into (A 13)–(A 16). For this computation, it is useful to note that

$$X^2 + Y^2 = \kappa_\perp^2 k_\perp^2, \tag{A 17}$$

$$X^2 + Z^2 = k_\perp^2 L_\perp^2, \tag{A 18}$$

$$Z^2 - X^2 = (L_1^2 - L_2^2)(k_1^2 - k_2^2) + 4k_1k_2L_1L_2, \tag{A 19}$$

$$XZ = L_1L_2(k_2^2 - k_1^2) + k_1k_2(L_1^2 - L_2^2), \tag{A 20}$$

$$XY = k_\perp^2(k_2L_1 - k_1L_2) + L_1L_2(k_1^2 - k_2^2) + k_1k_2(L_2^2 - L_1^2). \tag{A 21}$$

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