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Ohm's law for mean magnetic fields

By A. H. BOOZER
Plasma Physics Laboratory, Princeton University, P.O. Box 451,
Princeton, New Jersey 08544

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The magnetic fields associated with plasmas frequently exhibit small-amplitude MHD fluctuations. It is useful to have equations for the magnetic field averaged over these fluctuations, the so-called mean field equations. Under very general assumptions, it is shown that the effect of MHD fluctuations on a force-free plasma can be represented by one parameter in Ohm's law, which is effectively the coefficient of electric current viscosity.

1. Introduction

The magnetic fields that are associated with both laboratory and astrophysical plasmas frequently exhibit small-amplitude MHD fluctuations. The field lines of the average or mean magnetic field often lie in perfect surfaces even though the lines of the exact field may stochastically cover a volume of space. Owing to the complexity of the exact description of plasmas containing fluctuating fields, it is useful to have equations which describe the behaviour of the mean magnetic field. The usefulness of such equations depends on the number of free parameters required to represent the effects of the fluctuations and the extent to which the conservation properties of the exact description survive in the mean field equations. Assuming the mean field is force-free,

$$\nabla \times \mathbf{B} = (\mu_0 j_\parallel / B)\,\mathbf{B}, \tag{1}$$

we will show that the effects of the fluctuations on the mean field can be represented by introducing an extra positive parameter λ into Ohm's law so that

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} - \frac{\mathbf{B}}{B^2} \nabla . (\lambda \nabla j_\parallel / B) \tag{2}$$

with η the classical Spitzer resistivity and $\mathbf{j} = (j_\parallel / B)\,\mathbf{B}$ the current.

Mean field equations have been discussed in the plasma physics literature under the topics of dynamo theory (Moffatt 1978), reversal maintenance in the reversed field pinch device (Strauss 1984; Schnack, Caramana & Nebel 1985) and the so-called α effect (Steenbeck, Krause & Rädler 1966; Rädler 1968; Moffatt 1978). Proponents of the α effect employ an average over the fluctuating field to obtain a mean-field Ohm's law which is of the form

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} + \alpha \mathbf{B} \tag{3}$$

with the parameter α taking positive and negative values. Unfortunately, the helicity conservation properties of a plasma are not naturally contained in the α-effect form for Ohm's law. This conservation law implies that if either the helicity

$$K = \int \mathbf{A}.\mathbf{B}\, d^3x \tag{4}$$

or the magnetic energy

$$W = \int (B^2/2\mu_0)\, d^3x \tag{5}$$

is dissipated at an enhanced rate due to the fluctuations, then the enhancement of the helicity dissipation is much less than the enhancement of the energy dissipation. The definition of enhanced dissipation and a proof are given in Appendix A.

The proposed form of Ohm's law, equation (2), obeys a helicity conservation law. Indeed, the proposed form is in essence the unique form of Ohm's law for a force-free mean field under the following assumptions:

(i) The exact magnetic field energy and helicity are closely approximated by the energy and helicity of the mean field. This is equivalent to the turbulent field being weak compared with the mean field. This assumption is needed to obtain the mean field equations in Appendix B.

(ii) The fluctuating field can cause differential transport of both field energy and helicity. By differential transport we mean that the fluxes of energy and helicity depend only on local gradients. That is, the fluctuations are assumed to couple directly only infinitesimally separated regions of the plasma. This assumption is largely one of mathematical convenience and there would be little change in the macroscopic physical effects if it were not a good approximation. Since ideal MHD conserves the helicity inside a magnetic surface containing fixed toroidal flux, MHD fluctuations can only transport helicity through regions of space in which resistive effects on the fluctuations are important. In general resistivity affects MHD fluctuations only in the vicinity of the magnetic islands produced by the fluctuations.

(iii) The fluctuating field can lead to an enhanced dissipation of field energy but not of field helicity. This is the important assumption. It is implied by the usual form for Ohm's law for the exact magnetic field (Appendix A) as well as by other models of field dissipation; so this assumption should be satisfied by any physically meaningful mean field model.

The proposed form for Ohm's law, equation (2), has been previously considered by a number of authors (Schmidt & Yoshikawa 1971; Stix 1978; Jacobson & Moses 1984). The novel feature of this paper is the demonstration in § 2 that this form for Ohm's law follows from very general assumptions. The application of the proposed Ohm's law to the study of toroidal plasmas is discussed in § 3.

2. Derivation of Ohm's law

In this section, the modified form of Ohm's Law, equation (2), is derived using the assumptions listed in the introduction. The important assumptions concern the rate of destruction of the helicity K and the energy W. Faraday's law,

$\partial\mathbf{B}/\partial t = -\nabla\times\mathbf{E}$, relates the rate of change of K and W to the electric field $\mathbf{E}$ The rate of change of the helicity and magnetic energy in a region of space bounded by a magnetic surface which contains a constant amount of toroidal flux $2\pi\psi$ is

$$\frac{dK}{dt} = 2(2\pi)^2\psi\frac{d\chi}{dt} - 2\int \mathbf{E}.\mathbf{B}\,d^3x, \tag{6}$$

$$\frac{dW}{dt} = -\frac{1}{\mu_0}\int \mathbf{E}\times\mathbf{B}.d\mathbf{a} - \int \mathbf{j}.\mathbf{E}\,d^2x, \tag{7}$$

with $\nabla\times\mathbf{B} = \mu_0\mathbf{j}/c$ and with $2\pi\,d\chi/dt$ the rate of change of poloidal magnetic flux, which is the flux contained in the central hole of the toroidal surface.

The electric field $\mathbf{E}$ in an observer's frame of reference is related to the electric field $\mathbf{R}$ in a frame moving with velocity $\mathbf{v}$ by

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{R}. \tag{8}$$

This expression is well known if $\mathbf{v}$ is a constant, but it is also valid if $\mathbf{v}$ has spatial and temporal dependence. In the standard Ohm's law, $\mathbf{v}$ is interpreted as the plasma velocity and $\mathbf{R} = \eta\mathbf{j}$ is generally called the dissipative field. In the mean field theory, we do not state precisely what moves with velocity $\mathbf{v}$; so without loss of generality we can assume $\mathbf{R}$ is parallel to $\mathbf{B}$.

The contribution $\mathbf{R}_f$ of the fluctuating magnetic field to the dissipative field can now be calculated. The field $\mathbf{R}_f$ must have a form which does not dissipate helicity. Using (6) and (8), this means than $\mathbf{R}_f$ must have a form such that $\mathbf{R}_f.\mathbf{B}$, integrated over the volume between two magnetic surfaces, is equivalent to a surface integral. This surface integral gives the transport of helicity by the fluctuations. The form of $\mathbf{R}_f$ must be

$$\mathbf{R}_f = (\mathbf{B}/B^2)\,\nabla.\mathbf{h} \tag{9}$$

with $\mathbf{h}$ the flux of helicity. It should be noted that a term of the form $f\mathbf{B}/B^2$ could be added to $\mathbf{R}_f$ with f having a zero average over the magnetic surfaces. Nevertheless, at least on the irrational magnetic surfaces, a function F exists such that $f = \mathbf{B}.\nabla F$. The expression $\mathbf{B}.\nabla F$ equals $\nabla.(F\mathbf{B})$; so that the addition of a term $f\mathbf{B}/B^2$ adds no generality.

The rate of energy dissipation by the fluctuations is determined by the volume integral of $\mathbf{j}.\mathbf{R}_f$, equations (7) and (8). Using (9) for $\mathbf{R}_f$,

$$\int \mathbf{j}.\mathbf{R}_f\,d^3x = -\int \mathbf{h}.\nabla(j_\parallel/B)\,d^3x + \int (j_\parallel/B)\,\mathbf{h}.d\mathbf{a} \tag{10}$$

with the current $\mathbf{j} = (j_\parallel/B)\,\mathbf{B}$. Although not unique, the natural interpretation of this expression is that $(j_\parallel/B)\,\mathbf{h}$ is the flux of field energy and $\mathbf{h}.\nabla(j_\parallel/B)$ is the rate of dissipation. The most general expression for $\mathbf{h}$ is

$$\mathbf{h} = -\left[\lambda - \frac{\nabla.\mathbf{s}}{|\nabla(j_\parallel/B)|^2}\right]\nabla(j_\parallel/B) + \mathbf{u}\times\nabla(j_\parallel/B) \tag{11}$$

with the vectors **s** and **u** and the scalar λ all arbitrary functions of position. Inserting this form into (10), we find

$$\int \mathbf{j}.\mathbf{R}_f\, d^3x = \int \lambda[\nabla(j_\parallel/B)]^2\, d^3x + \int [(j_\parallel/B)\,\mathbf{h}+\mathbf{s}].d\mathbf{a}. \qquad (12)$$

The assumption that the fluctuations can only enhance the rate of energy dissipation implies that **s** can always be chosen so that the coefficient λ is everywhere positive. The flux of field energy due to the fluctuations is

$$\mathbf{q} = (j_\parallel/B)\,\mathbf{h}+\mathbf{s}. \qquad (13)$$

The term $\mathbf{u}\times\nabla(j_\parallel/B)$ of (11) produces no physical effects in a force-free plasma since it neither changes the field energy nor does it move helicity from one magnetic surface to another. That is, $[\mathbf{u}\times\nabla(j_\parallel/B)].\nabla\psi$ is zero. Consequently, a term of this form, which moves helicity in the magnetic surfaces, may be chosen according to mathematical convenience.

There is no unique method of specifying the vector **s** since the only constraint is $\lambda \geqslant 0$. Unless $\nabla.\mathbf{s}$ is sufficiently positive to make $\lambda-(\nabla.\mathbf{s})/[\nabla(j_\parallel/B)]^2$ negative, one can chose λ so that **s** is zero. This additional assumption, that **s** can be chosen to be zero, will be made for simplicity.

The full Ohm's law, (2), contains the dissipative term $\eta\mathbf{j}$ in addition to the term $\mathbf{R}_f$. Since the $\eta\mathbf{j}$ term cannot be enhanced without increasing the rate of helicity dissipation, one obtains the important result that the resistivity η cannot be changed by the fluctuations. That is, the resistivity must be determined by classical collisions or other kinetic effects.

3. Application

Possibly the most important application of the simple parallel Ohm's law,

$$\mathbf{E}.\mathbf{B} = \eta\mathbf{j}.\mathbf{B}-\nabla.\left(\lambda\nabla\frac{\mathbf{j}.\mathbf{B}}{B^2}\right), \qquad (14)$$

is the simulation of the effects of tearing modes or magnetic flux reconnexion using transport codes written for axisymmetric toroidal plasmas. A number of these codes already contain the extra term in Ohm's law since this term makes the transport equations easier to integrate numerically. The extra term is known to the writers of codes as an artificial viscosity. The general effect of the new term in Ohm's law, which has the form of a viscosity, is to prevent $j_\parallel/B$ from changing on a spatial scale shorter than

$$\delta^2 \equiv \lambda/\eta B^2. \qquad (15)$$

In the artificial viscosity application, δ is chosen to be somewhat larger than the numerical grid so that unresolvable current sheets cannot arise.

If $\lambda \gg \eta B^2a^2$ with a the plasma radius, then the effect of the viscous term is to make $j_\parallel/B$ nearly uniform across the plasma. The limit of a uniform $j_\parallel/B$ is called a Taylor state (Taylor 1974). A so-called Taylor relaxation corresponds to $\lambda/\eta B^2a^2$ approaching infinity.

Since the positive coefficient λ can depend arbitrarily on position or any other parameters, the proposed form for Ohm's law can simulate the effect of tearing modes, which exist in only part of the plasma. An example of such a mode is the double tearing mode which tends to arise near surfaces on which the rotational transform has zero gradient.

If one uses the idealization of a sharp plasma–vacuum interface, then the natural boundary condition is $\mathbf{h} = 0$ on the interface. This boundary condition will be assumed.

The form of the parallel Ohm's law, (14), has an important implication on the loop voltage required to maintain a plasma current in a toroidal plasma. Inside a quasi-stationary plasma, the electric field must be curl-free, or

$$\mathbf{E} = \frac{V}{2\pi}\nabla\phi - \nabla\Phi, \tag{16}$$

where V = constant is the loop voltage, ϕ is the toroidal angle, and $\nabla\phi = \hat{\phi}/R$. Assuming the plasma is bounded by a magnetic surface containing toroidal magnetic flux $2\pi\psi$,

$$V = \frac{1}{2\pi\psi}\int \mathbf{E}\,.\,\mathbf{B}\,d^3x \tag{17}$$

with the volume integral covering the plasma. Inserting the parallel Ohm's law, (14), into (17) for the loop voltage, one finds that

$$V = \frac{1}{2\pi\psi}\int \eta\,\mathbf{j}\,.\,\mathbf{B}\,d^3x \tag{18}$$

depends on the resistivity η and the current distribution $j_{\parallel}/B$. The coefficient λ can affect the loop voltage only by modifying the current distribution. As noted at the end of § 2, the MHD fluctuations cannot affect the resistivity η. There is a related result on the total Ohmic dissipation, $\int \mathbf{E}\,.\,\mathbf{j}\,d^3x$, in a quasi-stationary plasma. Using (16) one can show the Ohmic power is IV with

$$I = \frac{1}{2\pi}\int \mathbf{j}\,.\,\nabla\phi\,d^3x \tag{19}$$

the net toroidal current in the plasma. There is experimental evidence that the loop voltage is accurately given by the Spitzer resistivity even in a reversed field pinch with strong MHD fluctuations (Shoenberg, Moses & Hagenson 1984). It should be noted that (18) is consistent with the so-called $F-\theta$ pumping method of current drive (Bevir & Gray 1981) in which the loop voltage is slowly oscillated as the toroidal flux content of the plasma $2\pi\psi$ is changed so that the time average of $V\psi$ is non-zero although the time average of V is zero.

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Appendix A

We use the Schwarz inequality to demonstrate that if either helicity K or magnetic energy W is dissipated faster than at a characteristic rate, then the enhancement of helicity dissipation is always less than that of energy dissipation (this proof was found independently by Berger (1984)). The characteristic dissipation rates $\dot{K}_c$ and $\dot{W}_c$ are so defined that they would be the exact destruction rates if the current in the region were $\mathbf{j} = (j_\parallel/B)\,\mathbf{B}$ with $(j_\parallel/B)$ a constant. They are

$$\dot{W}_c \equiv -4(W/K)^2 \int \eta B^2 \, d^3x \tag{A 1}$$

and

$$\dot{K}_c \equiv (K/W)\, \dot{W}_c. \tag{A 2}$$

The Schwarz inequality implies

$$\left(\int \eta j^2 \, d^3x\right)\left(\int \eta B^2 \, d^3x\right) \geqslant \left(\int \eta \mathbf{j} \cdot \mathbf{B} \, d^3x\right)^2, \tag{A 3}$$

which is equivalent to the desired inequality, (6) and (7),

$$\dot{W}/\dot{W}_c \geqslant (\dot{K}/\dot{K}_c)^2. \tag{A 4}$$

Appendix B

Here we consider the mean field equations. A mean field, as any other magnetic field, must satisfy

$$\nabla \cdot \mathbf{B} = 0. \tag{B 1}$$

We assume that the magnetic fluctuations are sufficiently weak that the exact magnetic energy and helicity are closely approximated by the mean field. These assumptions lead to a mean-field form of Faraday's law, a portion of Ohm's law, Ampère's law, and force balance. Although the mean-field equations may appear obvious, they do require derivation.

The form of Faraday's law follows from the zero divergence condition on $\mathbf{B}$. That is, $\partial \mathbf{B}/\partial t$ is divergence-free so that it must be the curl of some vector

$$\partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}. \tag{B 2}$$

This definition of the mean electric field, $\mathbf{E}$, automatically gives part of Ohm's law. The mean electric field $\mathbf{R}$ in a frame of reference moving with velocity $\mathbf{v}(\mathbf{x}, t)$ is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R}. \tag{B 3}$$

In the usual plasma equations, $\mathbf{v}$ is identified with the plasma velocity and $\mathbf{R} = \eta \mathbf{j}$ with η the resistivity of the plasma. Using mean magnetic fields, $\mathbf{v}$ may not be the plasma velocity nor is $\mathbf{R}$ so simply related to the resistivity. The form of (B 3) leads us to the important conclusion that only $\mathbf{R}_\parallel$, the component of $\mathbf{R}$ along the mean field $\mathbf{B}$, can change the structure of the mean magnetic field. The other components of $\mathbf{R}$ can be eliminated by a proper choice of reference frame, owing to the arbitrariness of $\mathbf{v}$.

By assumption, the mean field energy W closely approximates the exact magnetic field energy with

$$W = \int \frac{B^2}{2\mu_0} d^3x. \tag{B 4}$$

Differentiating W with respect to time,

$$\frac{\partial W}{\partial t} = -\int \left[\mathbf{j} . \mathbf{E} + \nabla . \left(\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \right) \right] d^3x. \tag{B 5}$$

The mean current is defined by

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{B 6}$$

which is Ampère's law. In the cases of interest, $\mathbf{R}_\parallel$ is very small compared with a characteristic value of $|\mathbf{v} \times \mathbf{B}|$ with $\mathbf{v}$ the sound speed. With the assumption that $\mathbf{R}_\parallel$ is small, one can use $\mathbf{E} = \mathbf{v} \times \mathbf{B}$, to show that $\mathbf{j} \times \mathbf{B}$ is the force on a current-carrying medium which is moving with velocity $\mathbf{v}$.

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