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A New Look at the Instability of a Stratified Horizontal Magnetic Field

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Although the undular instabilities of a stratified horizontal magnetic field have been studied in a number of contexts we believe that the physical mechanism responsible for the instability has not been fully explained. In this paper we present a new explanation of why these instabilities occur, considering in detail the differing cases of two-dimensional and three-dimensional motions.

KEY WORDS: Undular instability, interchange instability, magnetic buoyancy.

1. INTRODUCTION

The pressure distribution within an electrically conducting gas is affected by the presence of a magnetic field. One particular consequence of this is that a horizontal field which increases with depth is able to support more mass against gravity than would be possible in its absence. An equilibrium state with such a field is therefore top-heavy, to some degree, and consequently may be unstable, the instability mechanism being known as "magnetic buoyancy". Instabilities of magnetic fields due to magnetic buoyancy are of consi-

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derable importance in astrophysics, being at least partially responsible for both the structure of the interstellar magnetic field [Parker (1966) and several subsequent papers—see especially Parker (1979)] and also the escape of magnetic flux from stellar interiors (Gilman, 1970; Acheson, 1978, 1979; Hughes, 1985a, b). In this paper we shall study the linear stability of an electrically conducting gas at rest with a horizontal magnetic field, $B(z)\hat{\mathbf{y}}$, with all diffusivities taken to be zero. Instability can then occur only via an exchange of stabilities.

There are two distinct unstable modes. One is the interchange mode in which the perturbed magnetic field lines remain in the y-direction, the other is the undular mode in which the perturbed field lines are wavy. Interchange modes can only be unstable if a flux tube displaced upwards finds itself less dense than its surroundings, the condition for the atmosphere to be so top-heavy being given by $(g/c^2)d[\ln(B/\rho)]/dz > N^2/V_A^2$, where gravity is taken to be acting in the *positive* z-direction, c is the adiabatic sound speed, N is the Brunt–Väisälä frequency and V_A is the Alfvén speed (see Tayler, 1973; Moffatt, 1978; Acheson, 1979). Thus these modes are directly analogous to the familiar Rayleigh–Taylor instabilities arising from a heavy gas overlying a lighter one.

The most interesting feature concerning the undular modes is that they can be unstable when the interchange modes are stable, essentially relying on an increase of B (rather than B/ρ) with depth. Consequently the instability of these modes *cannot* be explained simply as a Rayleigh–Taylor phenomenon—something more subtle is taking place and it is the aim of this paper to present a new explanation of the physical processes responsible for the instability.

Undular modes for a layer of magnetic gas were first studied by Newcomb (1961), who used the energy principle of Bernstein *et al.* (1958) to show that a necessary and sufficient condition for instability is that $d\rho/dz < \rho^2 g/\gamma p$ somewhere in the gas. Since the static distribution of pressure and density are affected by the magnetic field this criterion has an implicit dependence on the magnetic field and, as shown by Thomas and Nye (1975) and Acheson (1979), may alternatively be expressed as $(g/c^2)\,d[\ln B]/dz > N^2/V_A^2$. Gilman (1970) considered these modes under the assumption of infinite thermal conductivity, thereby annihilating conventional buoyancy effects, whilst the more general problem, incorporating magnetic and thermal diffusion, together with viscosity, has been studied by Acheson (1978, 1979), using a local analysis, and by Hughes (1985a), using the magneto-Boussinesq equations (Spiegel and Weiss, 1982; see also Corfield, 1984). The stability criteria of Gilman (1970), Acheson (1978) and Hughes (1985a) depend crucially on the flow being three-dimensional and are valid for incompressible flows; both Acheson and Hughes assumed incompressibility from the outset and, although Gilman did not, his stability criterion is unaltered by this assumption. For two-dimensional instabilities however, compressibility is essential, and consequently analytic progress can be made only for certain special atmospheres (Parker, 1966; Zweibel and Kulsrud, 1974).

The most common explanation of the instability mechanism for undular modes comes from considering the behaviour of an isolated magnetic flux tube and applying this behaviour, erroneously we believe, to the case of a magnetic layer. As explained by Parker (1955), raising a section of a flux tube will cause a drainage of gas away from the elevated section, thereby increasing the internal magnetic pressure relative to the external gas pressure and consequently enhancing the magnetic buoyancy of the raised portions of the tube. It has been argued that in a magnetic layer an undular perturbation of the field lines would cause the same effect; namely that fluid would run away down the wavy field lines from the "summits" to the "valleys" thereby causing the crests to become lighter and expand further, the troughs to become heavier and be depressed downward. We believe this explanation to be unsatisfactory on two counts. First, we consider it to be a non-linear explanation of a linear instability. The drainage flow argument states that the instability succeeds *as a consequence* of the crests being lightened by the flow and the troughs being made heavier—success therefore depends crucially on the alteration of the mean state, a non-linear effect. Our second, and major, objection though comes from the fact that the argument makes no mention of either the magnetic field gradient or the stratification of the atmosphere, two essential ingredients for determining stability. Since the overall stratification is *not* top-heavy (interchange modes stable) then, *a priori*, there is no good reason for any runaway flow—surely one would naïvely expect displaced gas just to return to its initial position. Indeed, if the presence of any wavy magnetic field is such as to cause a runway flow from crests to troughs, could not any atmosphere be so destabilised?

We believe that the explanation for the instability mechanism is somewhat different. For stratifications of the magnetic field such that the undular modes are unstable, but the interchanges stable, there is gravitational potential energy available for instability due to the field increasing downwards, but the atmosphere is not sufficiently top-heavy that this can be released by a straightforward inversion of the gas. For undular modes the release of this energy can be accomplished by doing work solely against the thermodynamic pressure whereas for interchange modes work must be done against both the thermodynamic *and* the magnetic pressure. In Section 3 we shall elaborate on this mechanism and shall show how instability is achieved for both two- and three-dimensional disturbances.

2. MATHEMATICAL FORMULATION

We shall consider a layer of electrically conducting gas confined in the vertical direction by rigid surfaces at $z=0$ (top) and $z=d$ (bottom), with gravity acting in the positive z-direction. In equilibrium the magnetic field is horizontal, in the y-direction, the field, pressure, density and temperature depending only on z. All diffusivities are taken to be zero. The equilibrium field variables therefore satisfy the well-known equations,

$$0=-dp/dz+\mu_0^{-1}(\nabla \times \mathbf{B}) \times \mathbf{B}\cdot\hat{\mathbf{z}}+\rho g, \qquad p=R\rho T.$$

Elimination of p between these two equations leads to the following expression for ρ,

$$\rho(z)=\exp\left[-\int_0^z \Gamma(z)\,dz\right]\left\{\rho_0-\frac{1}{2\mu_0 R}\int_0^z \frac{1}{T}\frac{dB^2}{dz}\exp\left[\int_0^z \Gamma(z)\,dz\right]dz\right\},$$

where $\rho(z=0)=\rho_0$ and $\Gamma(z)=d(\ln T)/dz-g/RT$.

To obtain numerical solutions the governing equations are made dimensionless by scaling lengths with the layer depth d, density with ρ_0, temperature and magnetic field with some average values, T_0 and B_0 respectively, pressure with the magnetic pressure B_0^2/μ_0 and velocities with a typical Alfvén speed $V_A=(B_0^2/\mu_0\rho_0)^{1/2}$. We shall denote the velocity by $\mathbf{u}=(u, v, w)$, the magnetic field perturbation by $\delta\mathbf{B}=(b_1, b_2, b_3)$ and the perturbations of pressure, density and temperature by δp, $\delta\rho$ and δT respectively. The linearised equations are separable in x, y and t with all variables proportional to e^{st} and with b_1 proportional to $\sin lx \cos my$; b_2, w, δp, $\delta\rho$ and δT proportional to $\cos lx \sin my$; b_3 and v proportional to $\cos lx \cos my$ and u proportional to $\sin lx \sin my$. The dimensionless governing equations may then be expressed as follows.

The perfect gas equation:

$$\delta p/p=\delta\rho/\rho+\delta T/T.$$

The equations of motion:

$$\rho su=l(\delta p+Bb_2)-mBb_1, \qquad \rho sv=-m\delta p+b_3DB,$$

$$\rho sw=-D(\delta p+Bb_2)-mBb_3+\lambda\delta\rho.$$

The continuity equation:

$$s\delta\rho=-l\rho u+m\rho v-D(\rho w).$$

The energy equation:

$$(s\delta T+wDT)+(\gamma-1)T(lu-mv+Dw)=0.$$

The solenoidal constraint on $\mathbf{B}$:

$$lb_1+mb_2+Db_3=0.$$

The magnetic induction equation:

$$sb_1=mBu, \qquad sb_2=-lBu-D(Bw), \qquad sb_3=mBw.$$

In these equations the perturbation variables are now just functions of the vertical coordinate z, the operator D denotes d/dz and $\lambda=gd/V_A^2$.

3. THE INSTABILITY

As stated in the introduction, it is our belief that the undular modes are destabilised not as a consequence of the gas draining from the crests to the troughs of the magnetic field lines, but instead by the release of the gravitational potential energy which is available by virtue of the magnetic field stratification. In this section we explore in some detail, for both two- and three-dimensional motions, just how this energy is released.

3.1 The two-dimensional instability

For these motions l, b_1 and u are all identically zero. Confinement of the motions to the yz-plane presents a severe obstacle to instability and one which can be overcome only by compressible motions. Parker (1966), using a normal-mode analysis, and Zweibel and Kulsrud (1974), using the energy principle of Bernstein *et al.* (1958), have studied such instabilities for the special case of an atmosphere with constant Alfvén speed; these modes however are not captured by the approaches of Acheson (1978) or Hughes (1985a), both of whom assume the flow to be three-dimensional and essentially incompressible. Indeed, in the Boussinesq limit, it can be shown that the only possible modes are purely oscillatory Alfvén waves (see Appendix A).

In order to see how the instability works we shall employ the energy principle of Bernstein *et al.* (1958), which is described in Appendix B. This principle yields the change in potential energy, δW, arising from a small displacement of the gas, the system being stable if $\delta W > 0$ for all possible displacements and unstable if there exists a displacement which makes $\delta W < 0$. From equation (B.6) we know that the formal minimisation of δW for two-dimensional undular motions is given (in dimensional quantities) by

$$\delta W_{min} = \tfrac{1}{2} \int_0^d \{[g\rho' - (\rho^2 g^2/\gamma p)]\xi_z^2 + B^2 \mu_0^{-1} \xi_z'^2\}\, dz, \tag{1}$$

where ξ_z is a displacement in the z-direction and a prime denotes differentiation with respect to z. The corresponding expression for interchanges [equation (B.5)] is

$$\delta W_{min} = \tfrac{1}{2}\int_0^d (\{g\rho' - [\rho^2 g^2/(\gamma p + B^2 \mu_0^{-1})]\}\xi_z^2)\, dz. \tag{2}$$

Obviously instability can arise if $\rho' < 0$, which is the case in a non-magnetic Rayleigh–Taylor instability. However, for the cases we are considering the density increases with depth and hence the negative definite terms in expressions (1) and (2) are solely responsible for the instability. It is therefore crucial to understand the physical nature of these terms.

For the general class of instabilities that work through the release of gravitational potential energy the quantity $w\delta\rho$ must, on the whole, be positive (i.e. light gas moving upwards, heavy gas downwards). In order to induce density fluctuations obviously a certain amount of work must be done against pressure forces. Clearly the most vigorous instability is the one for which the ratio of potential energy released to the work done against pressure forces is largest. The negative definite terms in (1) and (2) are a measure of this ratio, as may be verified by a formal minimisation of equation (B.4).

The reason why undular modes can be more unstable than interchanges is now apparent. In both cases a magnetic field increasing with depth provides extra potential energy but the mechanism by which this is released differs for the two cases. Inspection of equation (1) shows that for the undular modes density fluctuations can be attained by doing work against the thermodynamic pressure alone. Physically this is because compressive motions are along the magnetic field lines and hence do no work against magnetic pressure. By contrast, in the case of interchanges, for which the motions are transverse to the magnetic field, it is impossible to alter the density without also altering the magnetic field. This is reflected by the appearance of the total pressure in the denominator of the second term in the integrand of equation (2).

Since δW must be negative for instability, equation (2) yields a necessary and sufficient condition for instability to interchanges, namely that $\rho' < \rho^2 g/(\gamma p + B^2 \mu_0^{-1})$ somewhere in the gas (Newcomb, 1961). However, due to the presence of the third term, the integrand of equation (1) gives only a necessary condition for instability to undular modes ($\rho' < \rho^2 g/\gamma p$) (Yu, 1965). The appearance of this term, which can be thought of as a result of geometrical constraints, behoves us to look in more detail at the nature of the motions.

For a general flow only the irrotational part induces density fluctuations. However, for a magnetic field in the y-direction, only the part of the flow that is irrotational in the xz-plane does work against magnetic pressure. Although interchange modes cannot escape doing such work since the flow is solely in the xz-plane, in the case of undular modes there are irrotational flows which can in principle be solenoidal in the xz-plane. In two dimensions however such flows are very contrived for, in general, they will have structure in the z-direction owing to the presence of boundaries or vertical inhomogeneities. It is in this sense that the third term of equation (1) results from geometrical constraints since it is not implied by the dynamics. Indeed, as we shall see presently, this term can be completely eliminated for three-dimensional motions.

As noted above, although the energy principle yields a *necessary* condition for instability it is only by solving the full equations that instability to two-dimensional undular modes can be determined for any given situation. To find the exact nature of the solutions we have solved the perturbation equations of Section 2 by a Newton–Raphson–Kantorovich iterative technique and, for simplicity, in our numerical calculations we have assumed the initial equilibrium state to be isothermal and the magnetic field to take the form $\mathbf{B}=B_0(1+\zeta z/d)\hat{\mathbf{y}}$. Figure 1 shows the eigenfunctions of the perturbations

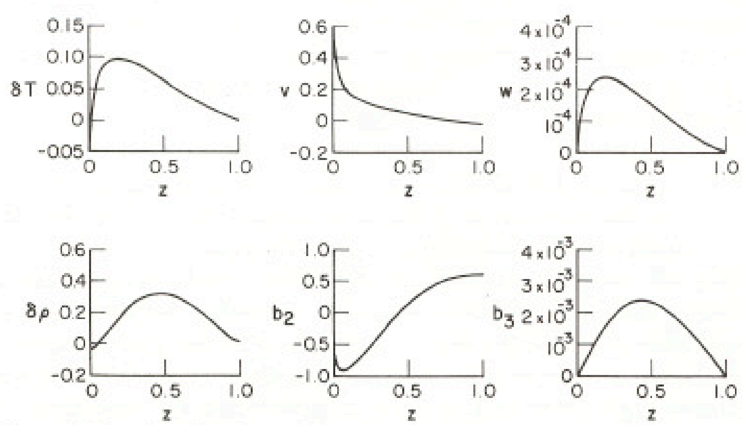

Figure 1 The eigenfunctions of the perturbations for a slightly unstable two-dimensional mode: $\gamma=5/3$, $\lambda=100$, plasma $\beta=10$, $m=10^{-2}$, $\zeta=27.01$, $s=9.4\times10^{-3}$.

for a slightly unstable undular mode and, as expected from the arguments given above, the motion is nearly all a compression (or expansion) in the y-direction, the ratio of w to v being very small indeed. It can also be seen that w is peaked very close to the top of the layer in order to minimise the final term on the right-hand side of equation (1) (i.e. the slope of w is greatest where the field is weakest). In accordance with the requirement that potential energy be released by the motions, $\langle w\delta\rho\rangle$ is indeed positive (angular brackets denoting a volume average), as can be seen from Figure 1. Furthermore, we notice that $\langle wb_2\rangle$ is negative, implying an upward flux of magnetic field and thereby showing that the effect of the flow is to reduce the field gradient responsible for the instability.

3.2 The three-dimensional instability

The three-dimensional instability in a layer of finite depth is much healthier than its two-dimensional counterpart. We mentioned earlier that for undular modes density fluctuations can be induced without doing work against the magnetic pressure. However, whereas in two dimensions *some* work against magnetic pressure is, in general, unavoidable, in three dimensions the extra degree of freedom allows for motions which do *no* work against magnetic pressure whatsoever. In other words, in three dimensions it is always possible to construct irrotational flows which are divergence-free in the xz-plane. In fact (see Appendix B) this criterion can be used to derive an extremum of δW, namely

$$\delta W_{min} = \tfrac{1}{2}\int_0^d [g\rho' - (\rho^2 g^2/\gamma p)]\xi_z^2\, dz. \tag{3}$$

The absence of work done against magnetic pressure implies that once again we obtain a necessary *and* sufficient condition for instability. Three-dimensional motions, being more easily destabilised than those in two dimensions, allow for instability in atmospheres which are always stable to two-dimensional disturbances—in particular, in the Boussinesq limit three-dimensional motions are readily destabilised (see Spiegel and Weiss, 1982; Hughes, 1985a). Figure 2 shows the eigenfunctions of a slightly unstable three-dimensional mode for both a Boussinesq and a more compressible atmosphere.

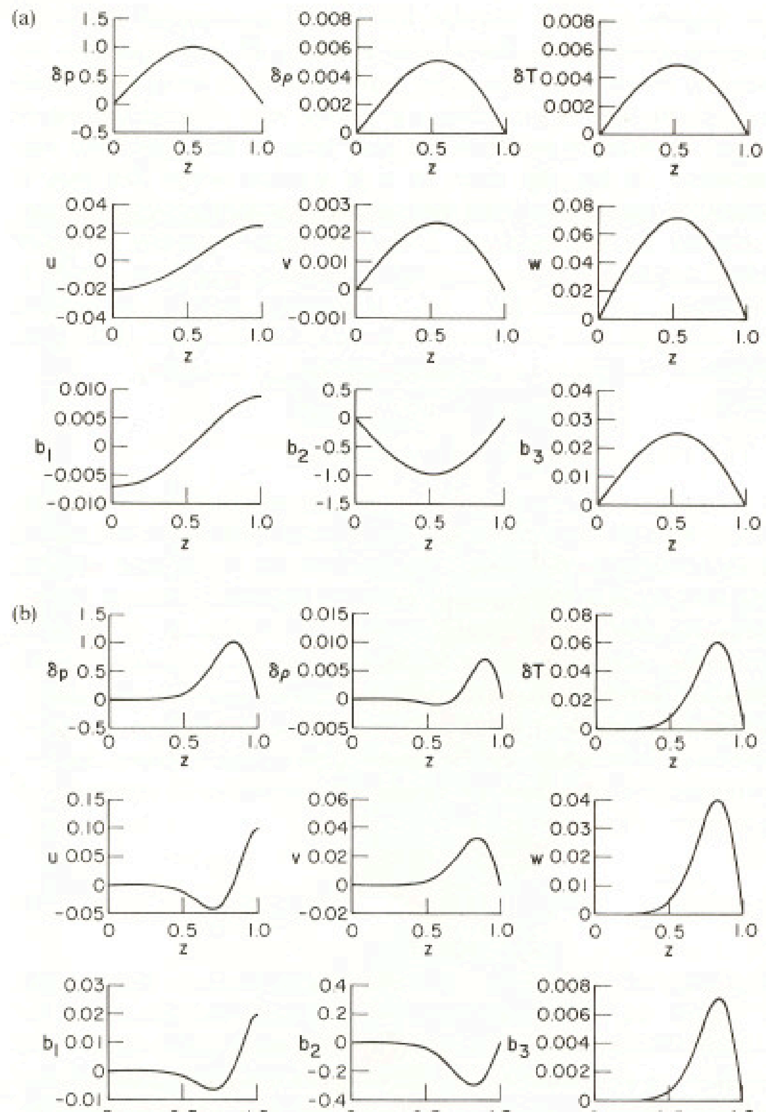

Figure 2 The eigenfunctions of the perturbations for two slightly unstable three-dimensional modes—the atmosphere of Figure 2(a) is Boussinesq, that of Figure 2(b) more compressible. In (a) $\gamma=5/3$, $\lambda=0.05$, $\beta=100$, $m=10^{-3}$, $l=10$, $\zeta=0.04$, $s=2.9\times10^{-3}$. In (b) $\gamma=5/3$, $\lambda=10$, $\beta=10$, $m=10^{-2}$, $l=4$, $\zeta=2.8$, $s=1.9\times10^{-1}$.

For the Boussinesq case (Figure 2a) the square-bracketed term in equation (3) is constant throughout the layer and consequently the eigenfunctions are sines and cosines, as predicted analytically (Spiegel and Weiss, 1982; Hughes, 1985a). When the atmosphere is more compressible (Figure 2b) the square-bracketed term in equation (3) takes a minimum value near the bottom of the layer and therefore, in order to make δW as small as possible, w is peaked in this region. In two dimensions this effect is outweighed by the fact that having w peaked near the *top* of the layer minimises the work done against magnetic pressure—in three dimensions, as noted above, the work done against magnetic pressure can be made arbitrarily small. It can easily be seen, from inspection of Figures 2a and 2b, that, as expected, $\langle wb_2 \rangle < 0$ and $\langle w\delta\rho \rangle > 0$; indeed, in the Boussinesq limit, the stronger condition also holds of wb_2 being negative and $w\delta\rho$ being positive *pointwise*.

4. CONCLUSION

Our main aim in this paper has been to present a new explanation of why a horizontal magnetic field which increases sufficiently rapidly with depth can become unstable to undular modes even though it may be stable to interchanges. The key difference between the two cases is that whereas for interchanges density fluctuations are related to the total (gas + magnetic) pressure, for the undular modes they depend on the gas pressure alone. This explanation does not hinge on a drainage flow from the crests to the troughs of the magnetic field lines, as is sometimes assumed necessary. In fact, we believe that the relationship between the flow and the field is purely geometrical and is not responsible for the instability.

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Appendix A

When the layer under consideration is so thin that variations of thermodynamic quantities are slight, and when the sound speed is large compared to both the flow speed and the Alfvén speed, then the dynamics may be described by the magneto-Boussinesq equations of Spiegel and Weiss (1982). It is the aim of this appendix to examine the nature of the linear, two-dimensional undular modes in this Boussinesq limit. The reader is referred to the papers by Spiegel and Weiss (1982) and Corfield (1984) for a detailed derivation of all the equations.

The lowest order approximation to the continuity equation is just $\nabla \cdot \mathbf{u}=0$, i.e.,

$$\partial v/\partial y+\partial w/\partial z=0, \tag{A.1}$$

using the notation of Section 2.

Similarly, the solenoidal constraint on the magnetic field is

$$\partial b_2/\partial y+\partial b_3/\partial z=0. \tag{A.2}$$

In general, the Boussinesq version of the induction equation is

$$\partial \mathbf{B}/\partial t+(\mathbf{u}\cdot\nabla)\mathbf{B}-(\mathbf{B}\cdot\nabla)\mathbf{u}-wH_\rho^{-1}\mathbf{B}=0, \tag{A.3}$$

where H_ρ is the density scale height. If we consider the y component of this equation then the second and fourth terms are $O(wB_0/H)$ whereas the third term is $O(B_0\,\partial v/\partial y)$. Equation (A.1) tells us that $\partial v/\partial y \sim w/d$ (where d is the layer depth) and consequently, since $d \ll H$, the lowest order version of equation (A.3) is just

$$\partial_t \mathbf{B}=(\mathbf{B}\cdot\nabla)\mathbf{u}.$$

The linearised y component of this equation is

$$\partial_t b_2=B_0\,\partial_y v, \tag{A.4}$$

with the z component giving $\partial_t b_3=B_0\,\partial_y w$, a result consistent with equations (A.1), (A.2) and (A.4).

The linearised equation of motion is

$$\rho_0\,\partial \mathbf{u}/\partial t=g\delta\rho\hat{\mathbf{z}}-\nabla\delta\Pi+\mu_0^{-1}[(\delta\mathbf{B}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\delta\mathbf{B}], \tag{A.5}$$

where $\delta\Pi$ represents the variation in the total pressure and is scaled such that $\delta\Pi/p_0 \sim (d/H)(\delta\rho/\rho_0)$. With this scaling the equation of state becomes

$$\delta\rho/\rho_0 = -\delta T/T_0 - \delta p_m/p_0, \tag{A.6}$$

where δp_m is the variation in the magnetic pressure.

Finally, we have the energy equation which, in its linearised form, is

$$(\partial/\partial t)[\delta T + (\delta p_m/C_p\rho_0)] - w\alpha = 0, \tag{A.7}$$

where α is the subadiabatic temperature gradient.

The y component of equation (A.5) gives

$$\rho_0\, \partial v/\partial t = -\partial(\delta\Pi)/\partial y + \mu_0^{-1}(b_3\, dB/dz + B_0\, \partial b_2/\partial y). \tag{A.8}$$

By virtue of the scaling of $\delta\Pi$, together with equation (A.6), the first term on the right-hand side of equation (A.8) is $O(d/H)$ times the third term. Equation (A.2) shows that the second term is similarly small. Consequently the lowest order approximation to equation (A.8) is

$$\rho_0\, \partial_t v = B_0 \mu_0^{-1}\, \partial_y b_2. \tag{A.9}$$

Combining equations (A.4) and (A.9) gives

$$\partial_t^2 v = (B_0^2/\rho_0\mu_0)\, \partial_y^2 v, \tag{A.10}$$

thus showing the motions to be purely oscillatory Alfvén waves.

In obtaining these results we have not made use of either equation (A.7) or the z component of equation (A.5). To ensure that our result (A.10) is sensible it is necessary to check that it is consistent with these other equations. On making use of equations (A.6) and (A.7) the z component of (A.5) may be expressed as

$$\frac{\partial^2 w}{\partial t^2} = -\frac{\alpha g}{T_0} w - \frac{g}{\gamma p_0}\frac{\partial \delta p_m}{\partial t} - \frac{1}{\rho_0}\frac{\partial^2 \delta\Pi}{\partial t\, \partial z} + \frac{B_0}{\mu_0\rho_0}\frac{\partial^2 b_3}{\partial t\, \partial y}. \tag{A.11}$$

The first, second and third terms on the right-hand side are comparable in magnitude. If $\partial_y \lesssim O[(dH)^{-1/2}]$ then the fourth term is comparable to or smaller than the other terms and equation (A.11) just gives us an equation for $\delta\Pi$. If, on the other hand, $\partial_y \gg O[(dH)^{-1/2}]$ then the fourth term on the right-hand side dominates, giving the balance

$$\partial_t^2 w = (B_0/\mu_0\rho_0)\,\partial_t\,\partial_y b_3.$$

This equation gives no new information but does not contradict any of the earlier results—indeed it can be derived from equations (A.1), (A.2) and (A.9). Thus, whatever the scaling of the wavelength in the y-direction, the only possible two-dimensional undular motions are Alfvén waves. If $\partial_y \lesssim O[(dH)^{-1/2}]$ it is possible to determine the system completely, but if $\partial_y \gg O[(dH)^{-1/2}]$ then it appears that the total pressure perturbation cannot be calculated.

Finally, a brief remark about why the magneto-Boussinesq equations admit unstable solutions in three dimensions but not in two. For three-dimensional motions there are two distinct scalings. In the first, all the terms in the solenoidal constraint on $\mathbf{u}$ (and also on $\mathbf{B}$) are of comparable magnitude and, as for the case of two dimensions described above, the ensuing motions are Alfvén waves. In the second ordering, under which instability *is* possible, u, v and w are assumed to be of comparable magnitude (as are b_1, b_2 and b_3), but the wavelength in the y-direction is $O(H/d)$ times that in the x- or z-directions. Consequently the solenoidal constraints on $\mathbf{u}$ and $\mathbf{B}$ are satisfied principally in the xz-plane, thereby showing the necessity of the third dimension.

Appendix B

This appendix describes how the energy principle of Bernstein *et al.* (1958) may be used to determine the stability of a vertically stratified horizontal magnetic field, following the approach of Newcomb (1961).

By adopting a Lagrangian description of the fluid motion and considering the effect of a small displacement $\boldsymbol{\xi}(\mathbf{r}_0, t)$ of a plasma element initially at $\mathbf{r}_0$ it can be shown (Bernstein *et al.*, 1958) that to lowest order the equation of motion takes the form

$$\rho\ddot{\boldsymbol{\xi}} = \mathbf{F}(\boldsymbol{\xi}), \tag{B.1}$$

where

$$\mathbf{F}(\xi)=\nabla(\gamma p\nabla\cdot\xi+\xi\cdot\nabla p)+\mathbf{j}\times\mathbf{Q}-\mu_0^{-1}\mathbf{B}\times(\nabla\times\mathbf{Q})+\nabla\cdot(\rho\xi)\nabla\phi,$$

$$\mathbf{Q}=\nabla\times(\xi\times\mathbf{B}),$$

a dot denotes differentiation with respect to time, p, ρ and $\mathbf{B}$ assume their equilibrium values and ϕ is the external potential. $\mathbf{F}(\xi)$ is a self-adjoint operator, as may be verified by a vast number of integrations by parts together with repeated use of the initial conditions.

Now the change in potential energy due to the displacement ξ is just

$$\delta W=-\tfrac{1}{2}\int_V \xi\cdot\mathbf{F}(\xi)\,dV. \tag{B.2}$$

It is intuitively clear that if δW is positive for all displacements ξ then the system is stable. Somewhat less obvious is the more powerful result that if δW is negative for *some* displacement (not necessarily one satisfying the equation of motion) then the system is unstable. If the operator $\mathbf{F}(\xi)$ only allows discrete eigenvalues then its eigenfunctions form a complete basis in the space of square-integrable functions and it then follows easily that if there is a square-integrable displacement which makes δW negative then there exists an exponentially growing physical perturbation (Bernstein *et al.*, 1958). In general though, idealised magnetohydrodynamic systems possess a continuous spectrum of eigenvalues and in such cases the proof is more difficult—this was given by Laval *et al.* (1965) who made use only of the self-adjointness of the operator $\mathbf{F}(\xi)$ with no assumptions about the nature of its eigenvalues.

Equation (B.2) may be expressed as

$$\delta W=\tfrac{1}{2}\int_V \{\mu_0^{-1}Q^2-\mathbf{j}\cdot(\mathbf{Q}\times\xi)+\gamma p(\nabla\cdot\xi)^2+(\nabla\cdot\xi)(\xi\cdot\nabla p)$$

$$+(\xi\cdot\mathbf{g})\nabla\cdot(\rho\xi)\}\,dV. \tag{B.3}$$

With the particular equilibrium of Section 2, and with ξ_x assumed to vary as $\sin lx\sin my$, ξ_y as $\cos lx\cos my$ and ξ_z as $\cos lx\sin my$, equation (B.3) becomes

$$\delta W = \tfrac{1}{2}\int_0^d \{\mu_0^{-1}B^2[m^2(\xi_z^2+\xi_x^2)+(\xi_z'+l\xi_x)^2]+\gamma p(\xi_z'-m\xi_y+l\xi_x)^2$$

$$+2\rho g\xi_z(\xi_z'-m\xi_y+l\xi_x)+g\xi_z^2\rho'\}\,dz, \qquad \text{(B.4)}$$

where all the variables are now functions of z only and a prime denotes differentiation with respect to z.

For the interchange modes both m and ξ_y are identically zero. A formal minimisation of the integrand with respect to ξ_x yields

$$\delta W_{min} = \tfrac{1}{2}\int_0^d (\{g\rho' - [\rho^2 g^2/(\gamma p+\mu_0^{-1}B^2)]\}\xi_z^2)\,dz. \qquad \text{(B.5)}$$

For the two-dimensional undular modes l is identically zero. Minimising (B.4) with respect to ξ_y and letting $m\to 0$, thereby eliminating the work done against magnetic tension, gives

$$\delta W_{min} = \tfrac{1}{2}\int_0^d \{[g\rho' - (\rho^2 g^2/\gamma p)]\xi_z^2 + \mu_0^{-1}B^2\xi_z'^2\}\,dz. \qquad \text{(B.6)}$$

For the three-dimensional undular modes minimising (B.4) with respect to ξ_x and ξ_y gives

$$\delta W = \tfrac{1}{2}\int_0^d \left\{\left(m^2\frac{B^2}{\mu_0}+g\rho'-\frac{\rho^2 g^2}{\gamma p}\right)\xi_z^2+\left(\frac{m^2}{m^2+l^2}\right)\frac{B^2}{\mu_0}\xi_z'^2\right\}dz. \qquad \text{(B.7)}$$

Clearly in (B.7) δW is least as $m\to 0$ and $l\to\infty$, giving

$$\delta W_{min} = \tfrac{1}{2}\int_0^d [g\rho' - (\rho^2 g^2/\gamma p)]\xi_z^2\,dz. \qquad \text{(B.8)}$$

Another perhaps more physically intuitive way of obtaining expression (B.8) is to set $(\xi_z'+l\xi_x)$ equal to zero in (B.4). After minimising over ξ_y expression (B.4) becomes

$$\delta W = \tfrac{1}{2}\int_0^d \{[g\rho' - (\rho^2 g^2/\gamma p)]\xi_z^2 + \mu_0^{-1}B^2 m^2[\xi_z^2+(\xi_z'/l)^2]\}\,dz.$$

Expression (B.8) is now recovered by letting $m\to 0$ and $l\to\infty$.