

# MHD Turbulence I

see:  
Galtier  
(Kulsrud)  
Frisch

- ca) Key Features of Hydro. Turbulence
- cc.) Basis Facts re: MHD Turbulence
- cc.c.) Phenomenology aka' Kraichnan / Iroshnikov
- cu.) Critical Balance and Phenomenology aka' Goldreich-Sridhar
- cv.) 4/5 Law Analogue + Further.

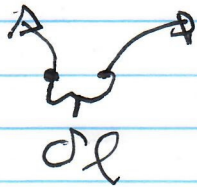
ci.) Key Features of Hydro. Turbulence (3D)  
 (Heavily based on experiment -  $Ca$  vs  $Re$ )  
 - chaotic state at high  $Re$   
 characterized by broad self-similar range and nonlinear transfer (flux rule).

- dissipates energy; dissipation rate indep.  $Re$ , as  $Re \rightarrow \infty$  (but  $\nu \neq 0$ ).

i.e.  $\langle \vec{F} \cdot \vec{v} \rangle = \epsilon = \nu \langle (\nabla v)^2 \rangle = \nu \langle \omega^2 \rangle$   
 $\Rightarrow$  singularity formation  $l_v \sim \nu^{1/4}$

-  $l_d < l < l_0$   
 $\epsilon = \frac{\nu \langle \epsilon \rangle}{T(\epsilon)} = \frac{\nu \langle \epsilon \rangle^3}{\rho}$ , etc.  $\frac{1}{T(\epsilon)} \rightarrow$  scale similarity constrained by Galilean invariance.  
 $l_d = \nu^{3/4} \epsilon^{1/4}$   
 $\Rightarrow E(k) \sim \epsilon^{2/3} k^{-5/3}$  "turbulent flow is rough"

- alternatively



$$\frac{d \overline{dl}}{dt} = v(dl)$$

$$dl^2 \sim \epsilon^2 t^3$$

→ Super-diffusive separation

- Some Things you can Trust:

a.) Karman - Howarth Egn. (Molubto uses)

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \overline{\partial_p} \left\langle \overbrace{(\partial u \cdot \partial u)}^{u(x+l) - u(x)} \delta u \right\rangle$$

$$+ 2\nu \partial_{l_i, l_i}^2 \left\langle \frac{u_i u_i'}{2} \right\rangle$$

$$u_c = \underline{u}(x)$$

$$u'_c = \underline{u}'(x')$$

$$x' = x + l$$

with external forcing:

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \overline{\partial_p} \left\langle (\partial u_i)^2 \delta u \right\rangle$$

$$+ 2\nu \partial_{l_i, l_i}^2 \left\langle \frac{u_i u_i'}{2} \right\rangle + \overline{F}$$

⇒ scale energetics balance

b.) 4/5 Law

Balancing external stirring with energy dissipation (must for steady state):

$$\partial_t \left\langle \frac{u_i u_i'}{2} \right\rangle = \frac{1}{4} \nabla_p \cdot \langle \partial u_i \rangle^2 \partial y \rangle + 2 \nu \partial_{ln}^2 \partial_{ln} \left\langle \frac{u_i u_i'}{2} \right\rangle + \epsilon$$

cr.

in inertial range:

$$\langle \partial u_i \rangle^2 \rangle = -\frac{4}{5} \epsilon l \quad \rightarrow \text{decay}$$

i.e. energy transfer at scale  $l$  (arbitrary in inertial range)

$$\boxed{\langle \frac{\partial u_i \partial u_i'}{l} \rangle = -\frac{4}{5} \epsilon}$$

proportional (with 4/5 factor) to dissipation rate.

b.) MHD (3D, Incompressible)

often  
 ① - wave strong, slowly varying external magnetic field

$$\rho (\partial_t \underline{v} + \underline{v} \cdot \nabla \underline{v}) = -\nabla p^* + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi} + \gamma \nabla^2 \underline{v} + \underline{f}$$

$$\partial_t \underline{B} + \underline{v} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{v} + \eta \nabla^2 \underline{B}$$

→ general  
 → need not B<sub>0</sub>

$$\nabla \cdot \underline{B} = 0$$

Now here can (for B<sub>0</sub>) use reduced MHD:

$$\rho (\partial_t \nabla_{\perp}^2 \phi + \underline{v}_{\perp} \cdot \nabla_{\perp} \nabla_{\perp}^2 \phi) = B_0 \partial_z \nabla^2 A + \nabla_{\perp} A \times z \cdot \nabla_{\perp} \nabla^2 A + \underline{f}_{\perp} + \gamma \nabla^2 \nabla^2 \phi$$

⊥ → emphasize

$$\partial_t A + \underline{v}_{\perp} \cdot \nabla_{\perp} A = B_0 \partial_z \phi + \eta \nabla^2 A$$

→ anisotropic turbulence if B<sub>0</sub> strong. (2D + SAW in z)

→ wave component i.e. MHD turbulence is part of waves (Alfvenic)

⇒ expect use methodology of wave turbulence.

B<sub>0</sub> strong → wave turbulence.

- ⊛  $\rightarrow$  nonlinear transfer in  $\perp$ , i.e.  $k_{\perp}$  (3D turbulence has coupling w/ 2D)
- $\rightarrow$  what controls triad coherence?  $\Rightarrow$  Alfvénic transfer (over scale)  $\sim \omega \rightarrow \nu$

From HW:

- ⊛ Nonlinear interaction exclusively via counter-propagating Elsasser populations

i.e.  $\underline{Z}^{\pm} = \underline{v} \pm \underline{b}$   $\underline{b} \equiv \text{part. } \tilde{B}$   
 2 MHD eqns  $\Rightarrow$

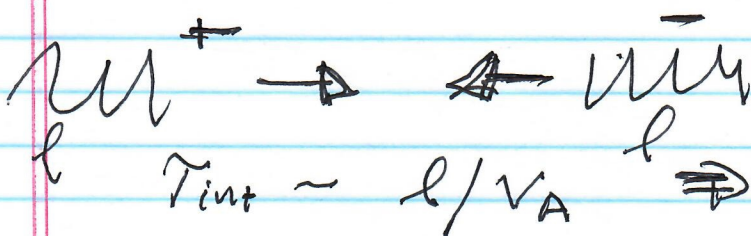
$$\partial_t \underline{Z}^{\pm} \mp \underline{b}_0 \cdot \nabla \underline{Z}^{\pm} + \underline{Z}^{\mp} \cdot \nabla \underline{Z}^{\pm} = -\nabla \rho^*$$

$$+ \underline{v}_{\pm} \cdot \nabla^2 \underline{Z}^{\pm} + \underline{\nu} \cdot \nabla^2 \underline{Z}^{\pm}$$

$$\underline{v}_{\pm} = \frac{v_{\pm 1}}{2}$$

so NL coupling exclusively via  $\underline{Z}^{\mp} \cdot \nabla \underline{Z}^{\pm}$

so NL transfer:



$\nwarrow$   
 interaction  
 diff. populations

i.e. counter-propagating packets' interaction time limited to  $l/v_A$ .

i.e. saves us from renormalization in absence of dispersion.

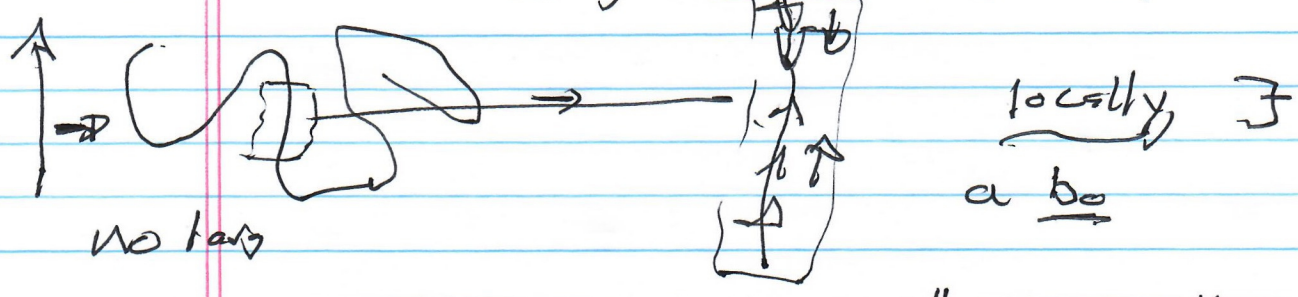
classic

so, some phenomenology:

(Iroshnikov, Kraichnan)  
(64, 65)

→ nonlinear transfer - isotropic turbulence

For weak mean  $B_0$ ,  $b_0 \sim b_{rms} > |B_0|$



→ so, nonlinear scattering from "collisions" of counter-propagating packets



so (modulo 4/5!) - T.B.D.

universally waves, = counter E/S. populations locality?

→  $\epsilon \sim \frac{Z^2}{l^3}$   $l \equiv$  scale

→ Alfvén wave cascade

→ const. energy throughput

Now Alfvénic transit time on scale  $l$ , in  $b_0$  is  $\frac{1}{2}$

$T_A = l/b_0$

Far transfer:

- transfer by waves scattering
- define  $\tau_{Tr}$  by:

$$\langle \delta Z^2(\tau_{Tr}) \rangle \sim \langle Z^2 \rangle$$

$\tau_{Tr}$  sufficient # kicks to make

i.e. randomization of scattering in amplitude  
 $\delta Z_{rms} \sim Z$

SB

$$Z_p(t + \tau_A) = Z_p(t) + \tau_A \delta Z_p$$

kick in 1 A after transit time

$$\delta Z_p \approx \tau_A \frac{Z_p^2}{p} \rightarrow \text{kick in } \tau_A$$

so  $\frac{Z_p^2}{p} \rightarrow$  kick.

$$\Rightarrow \langle \delta Z_p^2 \rangle = \left( \tau_A \frac{Z_p^2}{p} \right)^2 \frac{t}{\tau_A} \rightarrow \text{accumulated kicks}$$

thus:

$$\langle \delta Z_p^2 \rangle \sim \langle Z_p^2 \rangle$$

$$\Rightarrow Z_p^2 \approx \tau_A^2 \frac{(Z_p^2)^2}{p^2} \frac{\tau_{Tr}}{\tau_A}$$

$$\frac{1}{\tau_{Tr}} \approx \frac{Z_p^2}{p^2} \tau_A$$

determiner transfer rate.

$$\underline{\text{so}} \quad \frac{1}{\gamma_{Tr}} \approx \frac{Z_e^2}{l^2} \frac{l}{b_0}$$

$$\approx \frac{Z_e^2}{l b_0}$$

Thus:

$$\epsilon \approx \frac{Z_e^2}{\gamma_{Tr}} \approx \frac{Z_e^2 Z_e^2}{l b_0}$$

const. rate  $k = k_{inc}$

$$Z_e^3 \approx \sqrt{b_0 \epsilon} l^{1/2}$$

$$Z_e \approx (b_0 \epsilon)^{1/4} l^{1/4}$$

$$E(k) \approx \sqrt{\epsilon b_0} k^{-3/2}$$

Kraichnan spectrum

N.B. — as work with  $Z$   
 $E_M \sim E_k$

(equal Elsasser populations)

$$- \frac{1}{\gamma_{Tr}} \approx \frac{Z_e^2}{l^2} \frac{l}{b_0} \approx \frac{\gamma_A}{\gamma_{Tedy}} \frac{1}{\gamma_{Tedy}}$$



complete K41:  $1/T_{Tr} \sim \frac{\epsilon}{T_{eddy}}$

Transfer rate reduced by factor  $T_A/T_{eddy}$ !

Also, recall for weak wave turbulence:

$CNS = \sum_{k'} |V|^2 N_{k'} N_k \tilde{T}_{TC}$  → general structure of collision integral

(tried) coherence time → interaction

so

$\frac{\epsilon}{T_{Tr}} \approx \sum_{k'} |V|^2 N_{k'} \tilde{T}_{TC}$

but  $N_k \rightarrow Z \epsilon^2$  (intensity) (assumes local transfer)

$|V|^2 \sim |\nabla^2| \sim \frac{1}{l^2} \rightarrow \frac{\epsilon}{l^2}$

$\tilde{T}_{TC} \sim T_A \rightarrow$  tried coherence  
Alfvenic packet transit time

$1/T_{Tr} \sim \frac{Z \epsilon^2}{l^2} T_A \Leftrightarrow$  W.T.T.

fundamentally wave scattering process!

- 'triad'  $\rightarrow$  2 Alfs + cell.  
 $\rightarrow$  akin NLLD.

$\underline{\omega} = \underline{k} + \underline{k}'$   
(conservation)  
 $\omega_{\underline{z}} = \omega_{\underline{n}} + \omega_{\underline{n}'}$   
 $0 = k_{\parallel} v_A - k'_{\parallel} v_A$

- Alfsen waves non-dispersive, but coherence time controlled by packet propagation

$\Rightarrow$  prevents negative spectra, etc.  
 (Coherence time not so large)

- Strong  $B_0$ . (explicit)

Some simple observations:

- turbulence clearly anisotropic
- nonlinear transfer in  $k_{\parallel}$ .

So, consider weak wave turbulence!

Consider:  $k_{\perp} \rightarrow 1/l_{\perp}$   
 $k_{\parallel}$

$k_{\perp} > k_{\parallel}$  so

so 
$$E \sim \frac{(Z(l_{\perp}))^2 (Z(l_{\parallel}))^2}{l_{\perp}^2 |k_{\parallel} v_A|}$$

$\Delta k_{\parallel} v_A \rightarrow$  coh. time  $\tau$

ie. tacetly  $|k_{\parallel} v_A| \sim |k_{\parallel} v_A|$

$$Z(l_{\perp})^2 \approx (\epsilon |k_{\parallel} V_A|)^{1/2} l_{\perp}$$

$$\Rightarrow E(k_{\perp}) \approx (\epsilon |k_{\parallel} V_A|)^{1/2} / k_{\perp}^2 \quad \text{sharp}$$

and exciting for DOS in  $k_{\parallel}$ :

dist. restricted

$$E(k_{\perp}, k_{\parallel}) \sim [\epsilon V_A]^{1/2} / k_{\parallel}^{1/2} k_{\perp}^2$$

- strongly anisotropic

Weak wave turbulence of electron

-  $k_{\parallel}$  frozen.

$$\text{Now, } Z(l_{\perp}) \sim \delta B(l_{\perp}) \sim (\epsilon |k_{\parallel} V_A|)^{1/4} l_{\perp}^{1/2}$$

$$W \sim k_{\parallel} V_A$$

recall:  $v_{\parallel} = \partial_z + \frac{\partial B_{\perp} \cdot \nabla_{\perp}}{B_0}$

Kudo #

$$k_{\parallel} \sim \frac{\partial B_{\perp} \cdot \nabla_{\perp}}{B_0} / \partial_z \sim \frac{\rho_0 \omega \partial B_{\perp} / B_0}{\Delta_{\perp}}$$

(see 235)

# kicks in coherence length

$$\text{Now, } \frac{\partial B_{\perp}}{l_{\perp}} \sim (\epsilon |k_{\parallel} V_A|)^{1/4} / l_{\perp}^{1/2}$$

⇒ For W.T.T., expect:

$k_{\perp} < 1$  (diffusive picture)

but,

$k_{\perp}$  rises as  $l_{\perp}$  drops

i.e.  $k_{\perp} \uparrow$  as  $\text{pays}$  thru  $l_{\perp}$   
cascade  $\downarrow$  —  $\uparrow \uparrow$  what happens

⇒ Begs the question:

How high can  $k_{\perp}$  go and still retain physics of Alfvén wave cascade?

enter the:

Critical Balance Conjecture

- Goldreich - Sridhar (1995)  
(cf also Kadomtsev - Pogutse, 1978)

⇒ MHD inertial range in strong field will sit at  $k_{\perp} \sim 1$

i.e.  $\delta B_{\perp} \cdot \nu_{\perp} \sim B_0 \nu_{\perp}$

$Z(l_{\perp}) \sim k_{\perp} V_A$   
 $\frac{1}{l_{\perp}}$   
⋮

i.e. transit time sets bound on crit. strength  
 $\tau_A \rightarrow \tau$ ,  $\tau_{\text{tr}} \rightarrow \tau$   
Teddy Teddy

but: 
$$E \sim \frac{(z(l_{\perp}))^2 (z(l_{\perp}))^2}{l_{\perp}^2} \frac{1}{k_{\perp} v_A}$$

$$\begin{aligned} &\rightarrow \frac{(z(l_{\perp}))^2 (z(l_{\perp}))^2}{l_{\perp}^2 z(l_{\perp})} \\ &= \frac{z(l_{\perp})^3}{l_{\perp}} \end{aligned}$$

$$\Rightarrow z(l_{\perp}) \sim (E l_{\perp})^{1/3}$$

$$E(k_{\parallel}, k_{\perp}) \sim E^{2/3} k_{\perp}^{-5/3}$$

G-S spectrum  
 $\rightarrow$  back to  $k_{\perp}^4$   
 but different physics  
 $\rightarrow$  fits data

and anisotropy:

$$k_{\parallel} \sim z, \quad B_0 \cdot \nabla \sim \bar{B}_{\perp} \cdot \nabla_{\perp}$$

$$B_0 k_{\parallel} \sim E^{1/3} \frac{l_{\perp}^{1/3}}{l_{\perp}}$$

$k_{\parallel}$  vs  
 $k_{\perp}$   
 relation

$$k_{\parallel} \sim \frac{E^{1/3}}{B_0} k_{\perp}^{+2/3}$$

specifics "G-S cone"  
 in  $k$  space

i.e.  $k_{\parallel} \ll k_{\perp}$

$\rightarrow$  cascade develops preferentially in perp.

Why Believe

→ analogue of 4/5 Law (August, Politano)

$$-\frac{4}{3} \epsilon^{\pm} l = \langle (\delta u \cdot \delta u + \delta b \cdot \delta b) \delta u_{\parallel} \rangle$$

↳ main term → term  $\bar{E}_k$

$$-2 \langle (\delta u \cdot \delta b) b_{\parallel} \rangle$$

total energy  
conservation

induction → term  $\bar{E}_M$

~ reflects flip-flop in energy between channels

or

$$-\frac{4}{3} \epsilon^{\pm} l = \langle (\delta z^{\pm} \cdot \delta z^{\pm}) \delta z_e^{\mp} \rangle$$

no dissipation for 1 stream only.

→ why? → relate induced E-field.