# Taylor Relaxation and its Dynamics

- → Taylor Relaxation
- → Buried bodies on Taylor Hypothesis
  - — Relation to stochastic fields, turbulence
- → RFP
- → Mean Field Theory
- → Selective Decay.

so, ... $K_1 = \phi_1 \phi_2$ → product of fluxes

similarly $K_2 = \phi_2 \phi_1$

∴ $K = 2\phi_1 \phi_2$

if n windings $K = K_1 + K_2 = \pm 2n\phi_1 \phi_2$

⇒ helicity is measure of self-linkage of magnetic configuration.

Topological complexity.

Scale or

Why care → Taylor Conjecture (1974) (J.B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

$\underline{J} \times \underline{B} = 0$

- RFP

~ toroid
~ toroidal current

well fit by
$$B_z = B_0 J_0(\alpha r)$$
$$B_\theta = B_0 J_1(\alpha r)$$

$\underline{J} \times \underline{B} = 0$ ⇔ force free

⇒ why so robust?, especially since RFP so turbulent

Date

see RMP: { J.B. Taylor 1986

→ Taylor Relaxation

[ First example of Self-organization ⇒ Improved confinement.

→ transition to "quiescent period" ⇒ "relaxation" → turbulent resistive

→ magnetic energy minimization ($P_{OH}$ only, and $\beta << 1$)

⇒ what constraint?

→ (a) in ideal plasma, $\int d^3x\, \underline{A}\cdot\underline{B}$ conserved for all $\int d^3x$

i.e. any tube, around line

$$\int_{tube} d^3x\, A\cdot B = \text{const.}$$

line $\alpha, \beta$ s/t $\underline{B} = \underline{\nabla}\alpha \times \underline{\nabla}\beta$

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→ if $\delta \int_{tube} d^3x \left[ \frac{B^2}{8\pi} + \lambda A \cdot B \right] = 0$

$$\boxed{\nabla \times B = \lambda(\alpha, \beta)\, B} \; ; \quad B \cdot \nabla \lambda = 0$$

force free on micro-tube along line

but $\lambda(\alpha, \beta) \neq \lambda(\alpha', \beta')$

i.e → each tube/line defines conserved helicity

→ $\infty$ of invariants, due freezing in.

(b) But relaxation occurs in resistive, turbulent plasma. $\tau_R \sim \frac{L}{V} \sim \tau_A \sqrt{R_m}$

⇒ small tubes are destroyed by reconnection $\tau_R \sim \ell^{3/2}$

⇒ as $t \to \infty$, only very largest tube survives → global helicity is asymptotic survivor

could ~~also~~
view from stoch. lines
→ 1 line

motion → turbulence
resistivity → reconnection



i.e. recall, S-P:

$$V = v_A / \sqrt{R_m} \sim \sqrt{\frac{v_A \eta}{L}}$$

$1/\tau_{R_L} \sim 1/L^{3/2}$ ⇒ smaller scales reconnect faster.

⇒ smaller tubes destroyed first.

∴ 3 arguments for conjecture of global helicity as rugged invariant:

→ enhanced dissipn (above) → largest scales reconnect most slowly

→ stochasticity → if field lines stochastic, then (a la Fermi-MWR) 1 field line → 1 tube of conserved helicity → Global helicity is only inv.

⇒ RFP has only 1 field line.

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→ selective decay → magnetic helicity (inverse cascade) in 3D (MHD)

∴ global (large scale) helicity accumulates. → magnetic energy forward cascades.

N.b. compare: energy

heuristic $\dot{W} \sim -\eta \langle (\nabla B)^2 \rangle$ (if $\nu \to 0$)

$$H = \int d^3x\, A \cdot B \Rightarrow \dot{H} = -2c\eta \langle J \cdot B \rangle$$

$$\dot{W} \sim -2\eta \frac{\langle B^2 \rangle}{L_{eff}^2}$$

$$\dot{H} \sim -\eta \frac{\langle B^2 \rangle}{L_{eff}}$$

if $L_{eff} \sim \Delta \sim L / \sqrt{R_m} \sim \eta^{1/2}$

No.

Date

∴ $W \sim \eta^{0}$ → finite → indep dissipn of η in turb

$K \sim \eta^{1/2} \to 0$

∞ W diss, K ~ cons!

⟹

Routine calc. variation:

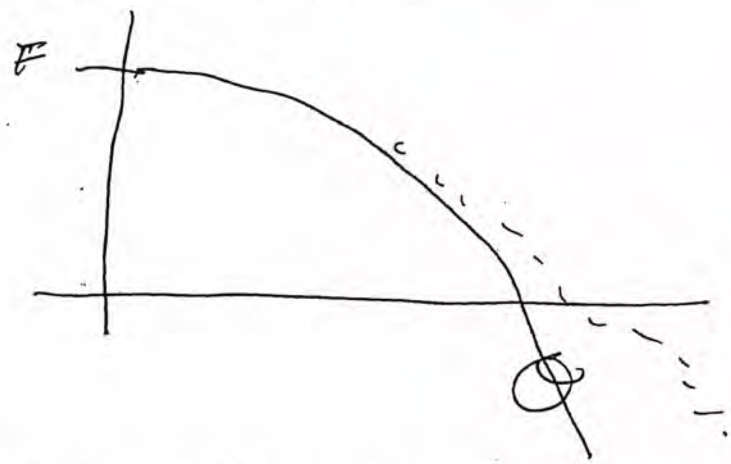
$\nabla \times B = \mu B$

$\nabla \cdot B / B^2 \to \text{const} = \mu$

$J_{\parallel}/B$ homogenized

n.b. $\int d^3x \, A \cdot B$ related to volt-seconds in plasma, via transformer.

Taylor Theory predicts F – Θ curve well




$\Theta = \mu a/2 = 2I/a B_0$

(B₀ created externally)

need $\mu a > 2.4$

$\Theta > 1.2$

$F = B_{z\,wall} / \langle B \rangle$

pretty good . . . —

N.B. An unpleasant reality:

- Relaxation ⟺ stoch/turb.
- stoch/turb → losses.

i.e.

$$\int_{V<n} d^3x\, \eta J^2 = 2\pi r R Q$$

$$Q = \,? \quad \text{i.e.} \quad \sim v_T \tilde{b}^2 \ell_{ec} \;? \quad \sim \frac{\chi_{\parallel}}{L} D_M \;?$$

$\tilde{b}^2$ < relaxation / losses

∴ heat flux driven dynamo ...

Confinement bad.

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point - helicity conserved in flux tubes, to $\eta$
- toroidal plasma $\Rightarrow$ many small tubes

etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite $\eta$, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\int \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow \text{conserved}.$$

$\int$ over plasma volume

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point
- helicity conserved in flux tubes, to $\eta$
- toroidal plasma $\Leftrightarrow$ many small tubes

etc.

$$\tau_R \sim L^{3/2}$$

$$\frac{V}{L} \sim \frac{V_A}{L} / \sqrt{R_M} \sim 1/L^{3/2}$$

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite $\eta$, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\int \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow \text{ⓘ conserved.}$$

($\int$ over plasma volume)

$\therefore$ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x\,\frac{B^2}{8\pi} + \lambda\int_V d^3x\,\underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works ! – indeed amazingly well – for RFPs, spheromaks, etc. Departures only recently being discovered.

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof. !!

Hypothesis: Selective Decay
- energy cascades → small scale
- helicity cascades → large scale (less dissipation)

Dual Cascade

– relevance to driven system ?
i.e. in real RFP, transformer on. $\tau_R \sim L^{3/2}$

→ dynamics? — <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing..

$$\delta\left[\int d^3x \left[\frac{B^2}{8\pi} + \lambda \underline{A}\cdot\underline{B}\right]\right] =$$

$$\frac{\underline{B}}{4\pi}\cdot\delta\underline{B} + \lambda \underline{A}\cdot\delta\underline{B} = 0$$

$$\frac{\nabla\times\underline{A}}{4\pi} + \lambda\underline{A} = 0$$

$\nabla\times$

$$\underline{J} = \mu \underline{B} \qquad \nabla\times\underline{B} = \mu\underline{B}$$

<u>force free</u>

<u>and</u>

$$\frac{\underline{J}\cdot\underline{B}}{B^2} = \mu$$

const

$\nabla_\perp J_\parallel = 0$ → parallel current homogenized

# II.) Dynamics of Taylor Relaxation.

ⓐ → How represent dynamics of relaxation?
How does system evolve to Taylor state?
(general)

ⓑ → How does RFP drive poloidal currents which produce reversed toroidal field (specific)

ⓒ → How relate to more general concepts of relaxation, dynamo? – Self-organized criticality...

ⓐ-ⓒ ⇒ Mean Field Electrodynamics

i.e. how calculate $\langle \tilde{v} \times \tilde{B} \rangle$

→ goal is turbulence driven EMF

→ akin $\langle E\delta f \rangle$ in QLT

→ issues: structure, symmetry
→ origin of irreversibility
conservation properties

→ topic is fundamental to subject of dynamo theory

→ flow counterpart: ~~Zonal flow generation (Monday Lecture)~~

Good resource:

www.igf.edu.pl/KB/HKM

items 28, 46

Keith Moffatt pubs.

(a) Structural / Symmetry Argument

Approach I (Boozer '86)

Write Ohm's Law (Mean Field) in form:

$$\langle \underline{E} \rangle + \langle \underline{v} \rangle \times \langle \underline{B} \rangle = \langle \underline{S} \rangle + \eta \langle \underline{J} \rangle$$

hereafter ignore ($\langle \underline{v} \rangle \times \langle \underline{B} \rangle$)

un-resolved EMF → "something" ($\langle \underline{S} \rangle$)

some unspecified operator.

What is $\langle S \rangle$?

Taylor → i) $\underline{S}$ must not dissipate $H_M$ (conserve $H_M$)

ii) $\underline{S}$ must dissipate $E_M$.

now,

$$\partial_t \int d^3x \, \langle \underline{A} \rangle \cdot \langle \underline{B} \rangle = \partial_t \int d^3x \left[ \langle \underline{A} \rangle \cdot \underline{\nabla} \times \langle \underline{A} \rangle \right]$$

$$= -2c \int d^3x \left[ \left( \langle \underline{E} \rangle + \underline{\nabla} \phi \right) \cdot \langle \underline{B} \rangle \right]$$

$$= -2c \int d^3x \left[ \langle \underline{E} \rangle \cdot \langle \underline{B} \rangle \right] \qquad \int \underline{B} \cdot \underline{\nabla} \phi \neq 0 \text{ to s/t.}$$

now

$$= -2c \int d^3x \left[ \langle S \rangle \cdot \langle \underline{B} \rangle + \eta \langle \underline{J} \rangle \cdot \langle \underline{B} \rangle \right]$$

$$\partial_t \int d^3x \, \langle A \rangle \cdot \langle \underline{B} \rangle = -2c\eta \int d^3x \, \langle \underline{J} \rangle \cdot \langle \underline{B} \rangle$$
$$- 2c \int d^3x \, \langle \underline{B} \rangle \cdot \langle \underline{\mathcal{E}} \rangle$$

Now, to conserve $H_M$, $2^{nd}$ term must integrate to S.T., so:

drop $\langle \ \rangle$

$$\langle \underline{\mathcal{E}} \rangle = \frac{\underline{B}}{B^2} \underline{\nabla} \cdot \underline{\Gamma}_H$$

↳ flux, driving helicity evolution

For form $\underline{\Gamma}_H$, consider energy:

$$\partial_t \int d^3x \, \frac{B^2}{8\pi} = \int d^3x \, \frac{\underline{B}}{4\pi} \cdot \partial_t \underline{B}$$

$$= - \int d^3x \, \frac{\underline{B}}{4\pi} \cdot c \, \underline{\nabla} \times \underline{E}$$

$$= - \int d^3x \, \underline{E} \cdot \underline{J}$$

$$= - \int d^3x \left[ \eta J^2 + \left( \frac{\underline{J} \cdot \underline{B}}{B^2} \right) \underline{\nabla} \cdot \underline{\Gamma}_H \right]$$

$$= - \int d^3x \left[ \eta J^2 - \underline{\Gamma}_H \cdot \underline{\nabla} \left( \frac{\underline{J} \cdot \underline{B}}{B^2} \right) \right]$$

flux   force

i.e. $\frac{dS}{dt} = \alpha \left( - \underline{\nabla} v \cdot \underline{\Gamma}_v \right) = \alpha \, D_v (\underline{\nabla} v)^2$, general form.

(entropy)

apart $\eta$,

$$\partial_t E_M = \int d^3x \, \underline{\Gamma}_H \cdot \nabla (J_\parallel / B)$$

so $\Gamma_H = -\lambda \nabla (J_\parallel / B)$ assures

$$\partial_t E_M = -\int d^3x \, \lambda \left[ \nabla (J_\parallel / B) \right]^2$$

and:

$$\boxed{\langle \underline{E} \rangle = \eta \langle \underline{J} \rangle - \frac{\underline{B}}{B^2} \underline{\nabla} \cdot \left[ + \lambda \underline{\nabla} \left( \frac{\underline{J} \cdot \underline{B}}{B^2} \right) \right]}$$

simplified form:

$$\boxed{\langle E_\parallel \rangle = \eta J_\parallel - \nabla_\perp \cdot \lambda \nabla_\perp J_\parallel}$$

[diffusion of current.]

$\lambda \equiv$ 'hyper-resistivity', 'electron viscosity'

structurally:

chapl. Recomm.

$$\lambda = \frac{c^2}{\omega_{pe}^2} D_J \quad , \quad \text{as} \quad \eta = \frac{c^2}{\omega_{pe}^2} \nu_{ei}$$

($D_J$: diffusivity)

$\lambda \equiv \mu$.

$D_J$ → MHD
→ multi-fluid
→ extended stochastic field argument

→ Exercises :

→ S-P reconnection, with $E_{\parallel} = -\mu \nabla_{\perp}^2 J_{\parallel}$ ?

$$V_R / V_A = 1/(S_{\mu})^{1/4} \qquad S_{\mu} = \frac{V_A L^3}{\mu} \qquad (J!)$$

$$1/S \rightarrow \mu / V_A L^3$$

→ derive structure of $D_J$
for ensemble stochastic fields
(i.e. shifted electron Maxwellian ⇒
$J_{\parallel}(\underline{x})$ ...).

→ Compare $D_J$ to $\chi_e$ for various
turbulence models.

In MHD:

– as seek $\langle E_{\parallel} \rangle$, and concerned with
locally strong field

$$\left( \underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \mu \underline{J} \right) \cdot \underline{B} / |B|$$

Strauss

$$\Rightarrow \boxed{-\frac{1}{c} \partial_t A_{\parallel} - \underline{n} \cdot \nabla \phi - \underline{\nabla A_{\parallel}} \times \hat{n} \cdot \nabla \phi = \mu J_{\parallel}}$$

here $\hat{n} = \underline{B} / |B|$

$B_0 \nabla_{\parallel} \phi$

then for mean field :

$$-\frac{1}{c}\partial_t\langle A\rangle + \partial_r\left[\langle \tilde{v}_r \tilde{\phi}\, \tilde{A}_\parallel\rangle\right] = \eta\langle J_\parallel\rangle$$

↑ flctn. induced EMF

- note naturally in flux form.

- $\langle v_\perp \tilde{\phi}\, \tilde{A}_\parallel\rangle \cong \langle v_\perp \tilde{\phi}\, \delta A_\parallel\rangle + \langle \tilde{A}_\parallel\, \nabla_\perp \delta\phi\rangle$

first term: iterate Ohm's Law ①; second term: iterate vorticity eqn. ②

i.e.

$$\partial_t \delta A_{\parallel_{\underline{k}}} + \Delta\omega_{\underline{k}}\, \delta A_{\parallel_{\underline{k}}} = i k_\parallel\, \delta\phi_{\underline{k}} - \eta k_\perp^2\, \delta A_{\parallel_{\underline{k}}}$$

(turbulent mixing) (bending) (resistive dissp'n.)

①

$$\langle v_\perp \tilde{\phi}\, \delta A_\parallel\rangle = \sum_{\underline{k}} k_\perp k_\parallel \frac{|\tilde{\phi}_{\underline{k}}|^2 (\Delta\omega_{\underline{k}} + \eta k_\perp^2)}{\omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2}$$

→ in pure QLT, irreversibility from resistive diffusion, only. → can be slow unless $k_\perp^2$ large

→ if undid normalizations,

$$\langle v_\perp \phi\, \delta A_\parallel\rangle = \alpha\langle B\rangle \quad \rightarrow \text{alpha effect}$$

$\alpha$ = above formula.

i.e. $k_\perp k_\parallel$ → motion has handed-ness i.e. $\underline{x} \to -\underline{x} \Rightarrow \alpha \to -\alpha$

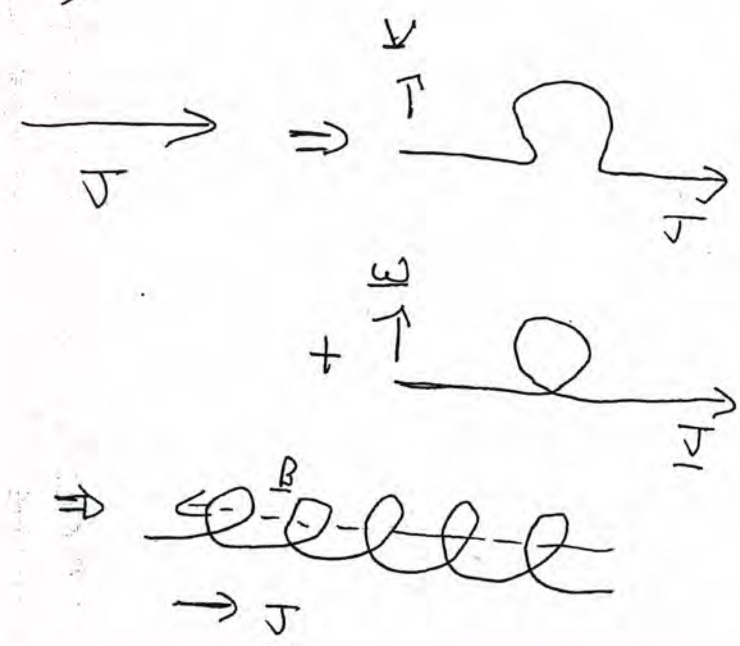
$$k_\perp k_\parallel = \frac{k_\perp^2 x}{L_s}$$ ✓

$$\Rightarrow \frac{\partial \langle A_\parallel \rangle}{\partial t} = \alpha \langle B \rangle$$

$$\frac{\partial}{\partial t} \langle B \rangle = \alpha \langle J \rangle$$

i.e. how generate a field parallel/anti-parallel to a current?

(Parker)




Make $B_J$ from $B_\theta$.

need $\langle \tilde{\underline{v}} \cdot \tilde{\underline{\omega}} \rangle \neq 0 \Rightarrow$ fluctuations have net helicity.

Here $\langle \nabla_\perp \tilde{\phi} \, \nabla_\parallel \tilde{\phi} \rangle$ is magnetized analogue of handedness.

but also ...

$$ⓒ = -\langle \nabla \tilde{A}_{\parallel}\, \delta\phi \rangle$$

vorticity eqn:

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2 \tilde{\phi}$$

$$= \frac{\tilde{B}_r}{B_0} \frac{\partial \langle J_{\parallel} \rangle}{\partial r} + \nabla_{\parallel} \tilde{J}_{\parallel} + \underline{\tilde{B}} \cdot \underline{\nabla} \tilde{J} + \mu \nabla^2 \nabla^2 \phi$$

$$\partial_t \left(-k_\perp^2 \tilde{\phi}_{\underline{k}}\right) + \Delta\omega_{\underline{k}} \left(-k_\perp^2 \tilde{\phi}_{\underline{k}}\right)$$

$$= \frac{\tilde{B}_{r\underline{k}}}{B_0} \frac{\partial \langle J_{\parallel} \rangle}{\partial r} + i k_{\parallel} \hat{A}_{\parallel \underline{k}} (-k_\perp^2) + \mu (k_\perp^2)^2 \tilde{\phi}_{\underline{k}}$$

$$\tilde{\phi}_{\underline{k}} = \frac{-\dfrac{\tilde{B}_{r\underline{k}}}{B_0 k_\perp^2} \dfrac{\partial \langle J_{\parallel} \rangle}{\partial r} + i k_{\parallel} \hat{A}_{\parallel \underline{k}}}{\left(-i\omega + \Delta\omega_{\underline{k}} + \mu k_\perp^2\right)}$$

$$ⓒ = -\sum_{\underline{k}} \frac{k_\perp k_{\parallel} |\tilde{A}_{\underline{k}}|^2 \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2\right)}{\omega^2 + \left(\Delta\omega_{\underline{k}} + \mu k_\perp^2\right)^2}$$

- magnetic $\alpha$ effect
- opposite in [illegible] sign to ①

$$(2)_{(b)} = \sum_{\underline{k}} \frac{|\nabla_\perp \tilde{A}_{\parallel \underline{k}}|^2}{B_0^2 k_\perp^2} \frac{(\Delta\omega_{\underline{k}} + \mu k_\perp^2)}{\omega^2 + (\Delta\omega_{\underline{k}} + \mu k_\perp^2)^2} \frac{\partial \langle J_\parallel \rangle}{\partial r}$$

→ clearly corresponds to hyper-$\eta$.

i.e.

$$-\frac{1}{c} \frac{\partial \langle A_\parallel \rangle}{\partial t} + \partial_r \langle (\nabla_\perp \tilde{\phi}) \tilde{A}_\parallel \rangle = \eta \langle J_\parallel \rangle$$

$$\langle (\nabla_\perp \tilde{\phi}) \tilde{A}_\parallel \rangle = \sum_{\underline{k}} k_\perp k_\parallel \left\{ |\phi_{\underline{k}}|^2 L^{\eta}_{\underline{k}} - |A_{\parallel \underline{k}}|^2 L^{\mu}_{\underline{k}} \right\}$$

($\alpha_K$ over the first term, $\alpha_M$ over the second)

$$\left[ L^{\eta} = (\Delta\omega_{\underline{k}} + \eta k_\perp^2) \Big/ \omega^2 + (\Delta\omega_{\underline{k}} + \eta k_\perp^2)^2 \right]$$

$$+ \sum_{\underline{k}} \left| \frac{\tilde{B}_{r \underline{k}}}{B_0} \right|^2 \frac{L^{\mu}_{\underline{k}}}{k_\perp^2} \frac{\partial \langle J_\parallel \rangle}{\partial r}$$

hyper-resistivity

N.B. – $\alpha$'s both come from bending

– $\alpha_K$, $\alpha_M$ opposite sign.

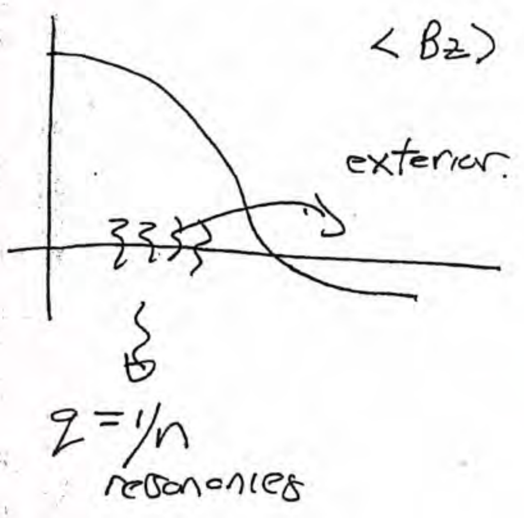
– $\alpha$'s from MHD exterior,

$$\tilde{A}_\parallel \to \frac{k_\parallel \tilde{\phi}_{\underline{k}}}{\omega + i\sigma}$$

- hyper-$\eta$ from (2) resonance
  i.e. where vorticity driven.
  $\Rightarrow$ reconnection process site.

- hyper-$\eta$ tied to basic tearing drive
- $\alpha_M$ + hyper $\eta$ cancel in exterior
  $V_K$ survive in exterior, vanish near Res. surf
- note total EMF encompasses more than hyper-$\eta$ .....

(b) RFP




$q < 1$, $q' < 0$ $\Rightarrow$ K-S unstable

$m = 1$ paradise
(global tearing turbulence)

so to compute induced EMF, seek
$\langle \tilde{V} \times \tilde{B} \rangle \hat{\theta}$ in exterior.

$V = \partial_t \tilde{\xi}$
↳ displacement

$$\tilde{\underline{B}} = \nabla \times \underline{\xi} \times \langle \underline{B} \rangle$$
$$= -\hat{\underline{\xi}} \cdot \nabla \langle B \rangle + \langle B \rangle \cdot \nabla \hat{\underline{\xi}} - \langle B \rangle \nabla \cdot \hat{\underline{\xi}}$$

field advection irrelevant — kink incompressible

i.e. bending is key.

$$\tilde{\underline{B}} \cong \langle B \rangle \cdot \nabla \tilde{\underline{\xi}}$$

$$\langle \tilde{\underline{v}} \times \tilde{\underline{B}} \rangle = \sum_{\underline{k}} \gamma_{\underline{k}} \langle \tilde{\xi}_{-\underline{k}} \times \tilde{B}_{\underline{k}} \rangle$$
$$= \sum_{\underline{k}} \gamma_{\underline{k}} \, \tilde{\xi}_{-\underline{k}} \times i k_{\parallel} \langle B \rangle_{\theta} \, \tilde{\xi}_{\underline{k}}$$

→ field primarily poloidal near $B_z$ reversal region.

$$\nabla \cdot \underline{\xi} = 0 \Rightarrow \frac{\partial_r \tilde{\xi}_r + i k_\theta \tilde{\xi}_\theta}{i k_z} = \tilde{\xi}_z$$

then

$$\langle \underline{\tilde{v}} \times \tilde{\underline{B}} \rangle_{\theta} = \sum_{\underline{k}} \gamma_{\underline{k}} \, i k_{\parallel} \langle B_\theta \rangle \left[ \tilde{\xi}_z^* \tilde{\xi}_x - \xi_x^* \xi_z \right]$$

$$= \sum_{\underline{k}} \frac{\gamma_{\underline{k}} \, i k_{\parallel} \langle B_\theta \rangle}{-i k_z} \left( M \right)$$

$$M = +\left(\partial_r \tilde{\xi}_r^* - i k_\theta \tilde{\xi}_\theta^*\right) \tilde{\xi}_r \quad -$$

$$+ \xi_r^* \left(\partial_r \xi_r + i k_\theta \tilde{\xi}_\theta\right)$$

$$M = + \partial_r |\tilde{\xi}_r|^2 + \left(i k_\theta \left(\tilde{\xi}_\theta^* \xi_r - \xi_r^* \xi_\theta\right)\right)$$

but $\tilde{\xi}_r \Big|_{wall} = 0$

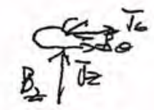
$r_{res} \sim a \Rightarrow \partial_r \gg k_\theta$

$$\boxed{\langle \underline{\tilde{v}} \times \underline{\tilde{B}} \rangle_\theta = +\sum_{\underline{k}} \gamma_{\underline{k}} \frac{k_\parallel}{k_z} \langle B_\theta \rangle \partial_r |\tilde{\xi}_{\underline{k}}|^2}$$

$$\rightarrow k_\parallel / k_z = \left(\frac{m}{r} B_\theta - \frac{n}{R} B_z\right) \Big/ B_\theta$$

$$\cong \frac{m}{r} - \frac{n}{r} q(r)$$

$$= 1/r \left(m - n q(r)\right)$$




so $J_\parallel B_\parallel$ at $r_{res}$

$$k_z = n/R$$

$$k_\parallel / k_z = (R/r)(m/n) - \frac{R}{r} q(r)$$

$$= (R/r)\left(q_{res} - q(r)\right)$$

$$\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle = -\sum_{\underline{n}} |\gamma_{\underline{n}}| \frac{R}{r} \left( q_{res} - q(r) \right) \langle B_\theta \rangle \partial_r |\tilde{\mathcal{E}}_{\underline{n}}|^2$$

$\rightarrow \partial_r |\tilde{\mathcal{E}}_n|^2 < 0$

$\rightarrow \gamma_{\underline{n}} \rightarrow$ irreversibity (!)

$\rightarrow q_{res} - q(r) \Rightarrow$ $< 0$ on axis

$> 0$ at $r_{rev}$

so $\quad T > 0$

$$\langle E \rangle + \langle \tilde{V} \times \tilde{B} \rangle = \eta \langle J_\theta \rangle$$

$$\therefore \langle J_\theta \rangle \cong \frac{1}{\eta} \langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle_\theta$$

$\Rightarrow \langle B_z \rangle < 0 \rightarrow$ kinks drive reversal.

But what about irreversibility and/or locking in ?

S – T – F – R

11 ⌒ ⇒ 1 | [illegible]

$1/n \quad 1/(n+1) \rightarrow \frac{2}{2n+1}$

↘ 0, 1

$1/(n+2),\ 0/1 \rightarrow 1/(n+2)$

m = 0 <u>drives</u> ⇒ reconnection

→ lock in.

ii.) 4/5 Law – See Lecture I.

iii.) Cascades and Relaxation

⇒ Selective Decay

Recall: Taylor Relaxation { 3D; 2D – "Taylor in Flatland" }

Argued: $\int d^3x \, B^2/8\pi$ minimized

subject to constraint of

$\int d^3x \, A \cdot B$ conserved.

$\Rightarrow \quad J_{\parallel} = J \cdot B / B^2 \rightarrow$ const.

(2D $J/A \rightarrow$ const.).

Arguements heuristic. { Power counting (K), stoch. fields, ⋮ }

Now, – dissipation at small scale $\eta, \nu$

– expect energy transfer to small scale

- Inverse {cascade / transfer} of magnetic helicity would set up "Selective Decay" scenario

i.e. magnetic energy scattered to small scales and dissipated
⇒ relaxation

magnetic helicity inverse cascades
⇒ avoids dissipation. Constraint; as survives.

c.f. {Frisch (75), Pouquet, et al. (76) (posted)
see also: Montgomery

- why, where from?

⇒ Primarily: Statistical Mechanics

⇒ c.f.: Frisch 75, though not transparent.

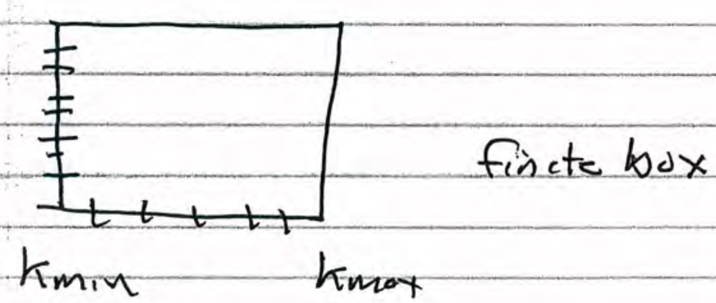
easier → "Taylor in Flatland" Problem.

Recall: Relaxation $\begin{cases} \text{minimizes } \langle B^2 \rangle \\ \text{conserving } \langle A^2 \rangle \end{cases}$

Does this follow from Selective Decay?

⇒ Explore Absolute Equilibrium

i.e.




- remove forcing, dissipation etc.
- input excitation.

For 2D MHD (ignoring cross helicity):

have $A \to X_i$
↳ Modal amplitude

so

$$E_M = \sum_{i=1}^{N} k_i^2 X_i^2$$

$$H = \sum_{i=1}^{N} X_i^2 \qquad - \langle A^2 \rangle$$

$$\phi \rightarrow Y_i$$

$$E_K = \sum_{i=1}^{N} k_i^2 Y_i^2 .$$

Now,

$$H \rightarrow \alpha$$

$$E = E_M + E_K \rightarrow \beta$$

> conserved

conserved, so pdf of this closed system is given by micro-canonical ensemble / distribution:

$$P(X,Y) = \underset{\text{norm}}{c} \exp\left[ -\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2 + \beta k_i^2 Y_i^2 \right]$$

and can integrate out $Y_i$ (KE) part, so:

$$P(X) = c \exp\left[ -\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2 \right]$$

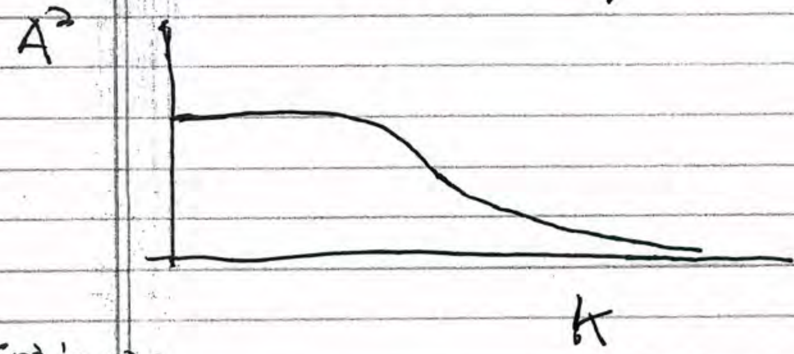
then:

$$\langle A^2(k) \rangle = \int dX_i \, X_i^2 \, P(X_i)$$

$$= 1/(\alpha + \beta k^2)$$

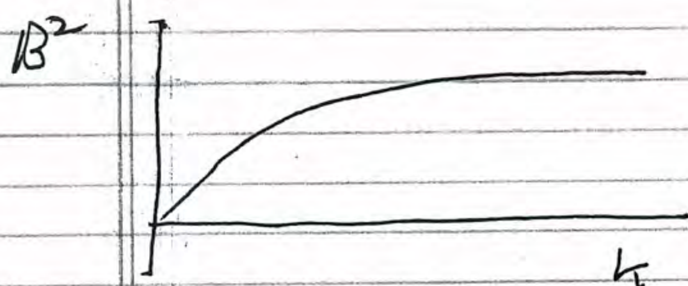
$$\langle B^2(k) \rangle = \left[ k^2 / (\alpha + \beta k^2) \right]$$

observe immediately:




"$A^2$ wants remain at large scale"

$k_{min} \to 0$




"$B^2$ approaches equipartition"

⇒ $A^2$ distribution most populated at larger scales. Decays at small

⇒ $B^2$ distribution most populated at smaller. Approaches equipartition at small scale.

∴ - suggests $A^2$ populates large scales, $B^2$ approaches equipartition.

- suggestive of inverse cascade of $A^2$, along with forward cascade of energy.

- supports Selective Decay Hypothesis as foundation for "Taylor in Flatland".

- similar story for Magnetic Helicity, though more laborious.

N.B. For 2D Fluid:

$$E = \int d^2x \, (\nabla\phi)^2 \quad \text{— energy}$$

$$\Omega = \int d^2y \, (\nabla^2\phi)^2 \quad \text{— enstrophy}$$

$$\Omega_i = k_i^2 E_i$$

$$\underline{v} \rightarrow \underline{X}_i$$

$\therefore$

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2\right]$$

so

$$E(k) = \langle v^2(k) \rangle = 1/(\alpha + \beta k^2)$$

$$\Omega(k) = k^2/(\alpha + \beta k^2)$$

similar suggestion of dual cascade and minimum enstrophy state.

→ Is this story true?

⇒ What does dynamics tell us?
∴ Consider interactions in 2D MHD.

Observe:

- Reduced MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = B_0 \partial_z \phi + \eta \nabla_\perp^2 \psi$$

↕

- 2D MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = \eta \nabla_\perp^2 \psi$$

so; with strong $\underline{B_0}$:

$$\langle A \cdot B \rangle \rightarrow \langle \psi \rangle \underline{B_0}$$

so mean $\langle \psi \rangle$ in 2D ≃ captures magnetic helicity dynamics in strongly magnetized system.

For $\langle A^2 \rangle_k$ transfer, consider closure of $\partial_t \langle A^2 \rangle$ equation, much akin to wave kinetics, though closure required.

See: Diamond, Hughes, Kim (posted).

Can write (see DHK) :

$$\frac{1}{2}\left[\partial_t \langle A^2\rangle_{\underline{k}} + T(k)\right] = -\Gamma_A(k)\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\underline{k}}$$

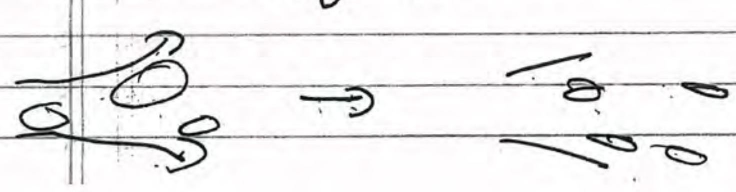
triplet $\langle \nabla\cdot\langle \underline{V}A^2\rangle\rangle_{\underline{k}}$

Flux $\Gamma_A = \left[\tilde{\tau}_0^{\phi}(\underline{k})\langle V^2\rangle_{\underline{k}} - \tilde{\tau}_0^{A}(\underline{k})\langle B^2\rangle_{\underline{k}}\right]$

$$T_{\underline{k}} = \sum_{\underline{k}'}(\underline{k}\cdot\underline{k}'\times\hat{z})^2\,\Theta_{k,k',k''}\left\{\langle\phi^2\rangle_{\underline{k}'}\ ⓵ - \frac{(k'^2-k^2)}{|\underline{k}+\underline{k}'|^2}\langle A^2\rangle_{\underline{k}'}\ ⓶\right\}\{A^2_{\underline{k}}\}$$

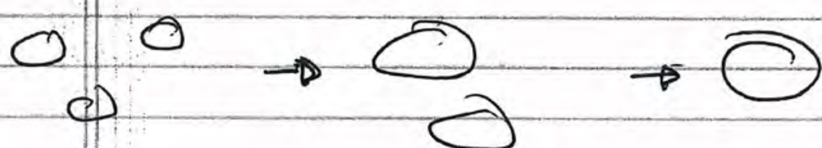
$$⓷\quad - \sum_{\substack{\underline{k}=\underline{p}+\underline{q}\\ \underline{p},\underline{q}}}(\underline{p}\cdot\underline{q}\times\hat{z})^2\,\Theta_{\underline{k},\underline{p},\underline{q}}\langle A^2\rangle_{\underline{p}}\langle\phi^2\rangle_{\underline{q}}$$

①, ③ → coherent damping, incoherent emission
→ akin to scattering of passive scaler, → small scale / chop-up.
→ conserve $\langle\psi^2\rangle$ upon $\sum_k$ together.



(2) → coherent damping/growth - from back reaction ($J \times B$) into Ohm's Law:
→ reshuffle $\langle A^2 \rangle$ to larger scale. sign $k'^2$ vs $k^2$!

→ $\sum_{\underline{k}} \langle A^2 \rangle$ conserves independently.



→ correspondence to condensation and wires (currents) attracting.

(1) + (2) → net effective resistivity sign.
→ see $\Gamma_A$, too. – 'negative resistivity'
– Alfvenized state

⇒ $E_K > E_M$ ⇒ $\langle A^2 \rangle_{\underline{k}}$ shuffled to smaller scale.

$E_M < E_K$ ⇒ $\langle A^2 \rangle_{\underline{k}}$ transfered to larger scale.

and

transfer need not be local!

⇒ In dynamics, $\langle A^2 \rangle$ ; $\langle A \cdot B \rangle$ evolution is complex.

⇒ N.B. Recall flux expulsion:

$\frac{V_{A0}^2}{\tilde{V}^2} R_m < 1 \rightarrow$ A passive, B expelled

$> 1 \rightarrow$ $J \times B$ disrupts vortex, expulsion stops

⇒ $B_0^2 \lessgtr \rho_0 \langle \tilde{v}^2 \rangle / R_m$

but $\langle \tilde{B}^2 \rangle \gg B_0^2$, upon stretching, weak $B_0$ is sufficient!

Zeldovich:

$$\frac{\partial A}{\partial t} + \underline{v} \cdot \underline{\nabla} A = -v_r \frac{\partial \langle A \rangle}{\partial x} + \eta \nabla^2 A$$

$\times A$ and avg. ⇒

$$\eta \langle \tilde{B}^2 \rangle = \langle \tilde{v}_r \tilde{A} \rangle \frac{\partial \langle A \rangle}{\partial x}$$

$$\langle \tilde{B}^2 \rangle = \frac{\eta_T}{\eta} B_0^2$$

$$= \frac{\tilde{v}\, \Delta r_c}{\eta} B_0^2 \cong R_m B_0^2 \quad \checkmark.$$

so, crudely:

$$\langle \tilde{B}^2 \rangle / \cancel{R_m} \lessgtr \rho \langle \tilde{v}^2 \rangle / \cancel{R_m}$$ ✓

⇒ Questions still open!

∴ Taylor conjecture remains a conjecture!

# Outline

i.) Preamble: → From Reconnection to Relaxation and Self-Organization

→ What 'Self-Organization' means

→ Why Principles are important

→ Examples of turbulent self-organization

→ Preview

ii.) Focus I: Relaxation in R.F.P. (J.B. Taylor)

→ RFP relaxation, pre-Taylor

→ Taylor Theory
- Summary
- Physics of helicity constraint + hypothesis
- Outcome and Shortcomings

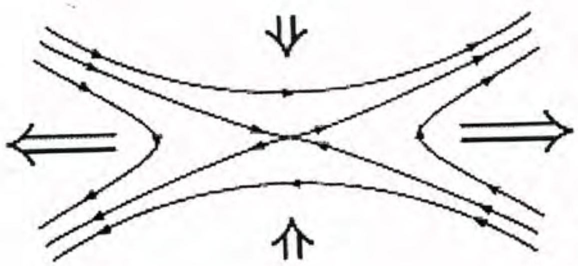
→ Dynamics → Mean Field Theory
- Theoretical Perspective
- Pinch's Perspective
- Some open issues

→ Lessons Learned and Unanswered Questions

## I.) Preamble

→ From Reconnection to Relaxation

- Usually envision as localized event involving irreversibility, dissipation etc. at a singularity



S.-P.

$$V = V_A / Rm^{1/2}$$

- ??? - how describe global dynamics of relaxation and self-organization

→

- multiple, interacting/overlapping reconnection events

→ turbulence, stochastic lines, etc

# Examples of Self-Organization Principles

→ Turbulent Pipe Flow: (Prandtl → She)

$$\sigma = -\nu_T \frac{\partial \langle v_y \rangle}{\partial x} \qquad \nu_T \sim v_* x \qquad \Rightarrow \langle v_y \rangle \sim v_* \ln x$$

Streamwise Momentum undergoes scale invariant mixing

→ Magnetic Relaxation: (Woltjer-Taylor)

(RFP, etc)
(Focus 1) Minimize $E_M$ at conserved global $H_M$ $\Rightarrow$ Force-Free RFP profiles

→ PV Homogenization/Minimum Enstrophy: (Taylor, Prandtl, Batchelor, Bretherton, ...)

(Focus 2)
→ PV tends to mix and homogenize
→ Flow structures emergent from selective decay of potential enstrophy relative energy

→ Shakura-Sunyaev Accretion

→ disk accretion enabled by outward viscous angular momentum flux

## II.) Focus I - Magnetic Relaxation

(Derek C Robinson)

→ Prototype of RFP's: Zeta (UK: late 50's - early 60's )

- toroidal pinch = vessel + gas + transformer
- initial results → violent macro-instability, short life time
- weak $B_T$ → stabilized pinch ↔ sausage instability eliminated
- $I_p > Ip, crit$ ( $\theta > 1+$ ) → access to "Quiescent Period"

→ Properties of Quiescent Period:

- macrostability - reduced fluctuations
- $\tau_E \sim 1\ msec \quad T_e \sim 150eV$
- $B_T(a) < 0$ → reversal

→ Quiescent Period is origin of RFP

# Further Developments

- Fluctuation studies:

turbulence = ↗ $m = 1$ kink-tearing → tend toward force-free state
↘ resistive interchange, ...

- Force-Free Bessel Function Model

$$B_\theta = B_0 J_1(\mu r) \qquad B_z = B_0 J_0(\mu r)$$

$$\mathbf{J} = \alpha \mathbf{B}$$

observed to correlate well with observed B structure

- L. Woltjer (1958) : Force-Free Fields at constant $\alpha$

→ follows from minimized $E_M$ at conserved $\int d^3x \mathbf{A} \cdot \mathbf{B}$

- steady, albeit modest, improvement in RFP performance, operational space

→ Needed: Unifying Principle

# Theory of Turbulent Relaxation (J.B. Taylor, 1974)

→ hypothesize that relaxed state minimizes magnetic energy subject to constant global magnetic helicity

i.e. profiles follow from: $$\delta\left[\int d^3x \frac{B^2}{8\pi} + \lambda \int d^3x \mathbf{A}\cdot\mathbf{B}\right] = 0$$

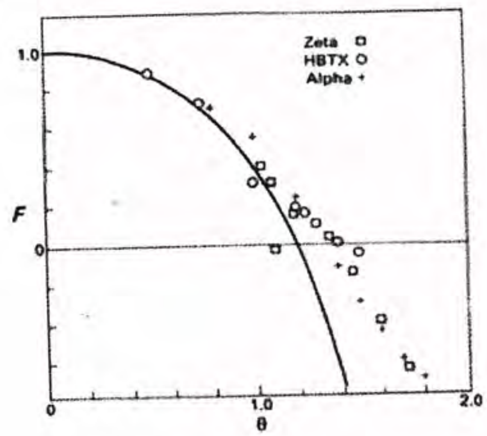
$$\Rightarrow \quad \nabla\times\mathbf{B} = \mu\mathbf{B} \;\; ; \;\; J_{\|}/B = \frac{\mathbf{J}\cdot\mathbf{B}}{B^2} = const$$

Taylor state is:

- force free
- flat/homogenized $J_{\|}/B$
- recovers BFM, with reversal for $\theta = \dfrac{2I_p}{aB_0} > 1.2$

- Works amazingly well

# Result:




$$\theta = \mu a/2 = \frac{2I_p}{aB_0}$$

$$F = B_{z,wall}/\langle B \rangle$$

and numerous other success stories

→ Questions:

- what is magnetic helicity and what does it mean?
- why only global magnetic helicity as constraint?
- Theory predicts end state → what can be said about dynamics?
- What does the pinch say about dynamics?

→ Central Issue: Origin of Irreversibility

# Why Global helicity, Only?

- in ideal plasma, helicity conserved for each line, tube

  i.e. $\mathbf{J} = \mu(\alpha, \beta)\mathbf{B}$ $\mu(\alpha', \beta') \neq \mu(\alpha, \beta)$

- Turbulent mixing eradicates identity of individual flux tubes, lines!

  i.e.

  - if turbulence s/t field lines stochastic, then '1 field line' fills pinch.

    1 line ↔ 1 tube → only global helicity meaningful.

  - in turbulent resistive plasma, reconnection occurs on all scales, but: $\tau_R \sim l^\alpha$ $\alpha > 0$

    ( $\alpha = 3/2$ for S-P reconnection)

    Thus larger tubes persist longer. Global flux tube most robust

  - selective decay: absolute equilibrium stat. mech. suggests possibility of inverse cascade of magnetic helicity (Frisch '75) → large scale helicity most rugged.

# Comments and Caveats

→ Taylor's conjecture that global helicity is most rugged invariant remains a conjecture

→ unproven in any rigorous sense

→ many attempts to expand/supplement the Taylor conjecture have had little lasting impact (apologies to some present....)

→ Most plausible argument for global $H_M$ is stochastization of field lines → forces confinement penalty. No free lunch!

→ Bottom Line:

- Taylor theory, simple and successful

- but, no dynamical insight!

# Dynamics I:

- The question of Dynamics brings us to mean field theory (c.f. Moffat '78 and an infinity of others - see D. Hughes, Thursday Lecture)

- Mean Field Theory → how represent $\langle \tilde{v} \times \tilde{B} \rangle$ ?

  → how relate to relaxation?

- Caveat: - MFT assumes fluctuations are small and quasi-Gaussian. They are often NOT

  - MFT is often very useful, but often fails miserably

- Structural Approach (Boozer): (plasma frame)

$$\langle \mathbf{E} \rangle = \eta \langle \mathbf{J} \rangle + \langle \mathbf{S} \rangle$$

→ something → related to $\langle \tilde{v} \times \tilde{B} \rangle$

$\langle \mathbf{S} \rangle$ conserves $H_M$

$\langle \mathbf{S} \rangle$ dissipates $E_M$

Note this is ad-hoc, forcing $\langle \mathbf{S} \rangle$ to fit the conjecture. Not systematic, in sense of perturbation theory

Now

$$\partial_t H_M = -2c\eta \int d^3x \langle \mathbf{J} \cdot \mathbf{B} \rangle - 2c \int d^3x \langle \mathbf{S} \cdot \mathbf{B} \rangle$$

$$\therefore \langle \mathbf{S} \rangle = \frac{\mathbf{B}}{B^2} \nabla \cdot \mathbf{\Gamma}_H$$

Conservation $H_M$ ➔ $\langle S \rangle \sim \nabla \cdot$ (Helicity flux)

$$\partial_t \int d^3x \frac{B^2}{8\pi} = -\int d^3x \left[ \eta J^2 - \mathbf{\Gamma}_H \cdot \nabla \frac{\langle \mathbf{J} \rangle \cdot \mathbf{B}}{B^2} \right]$$

so

$$\mathbf{\Gamma}_H = -\lambda \nabla (J_{\parallel}/B)$$ , to dissipate $E_M$

→ simplest form consistent with Taylor hypothesis

→ turbulent hyper-resistivity $\lambda = \lambda[\langle \tilde{B}^2 \rangle]$ - can derive from QLT

→ Relaxed state: $\nabla (J_{\parallel}/B) \to 0$ homogenized current → flux vanishes

# Dynamics II: The Pinch's Perspective

- Boozer model not based on fluctuation structure, dynamics

- Aspects of hyper-resistivity do enter, but so do other effects

  → Point: Dominant fluctuations controlling relaxation are m=1 tearing modes resonant in core → global structure

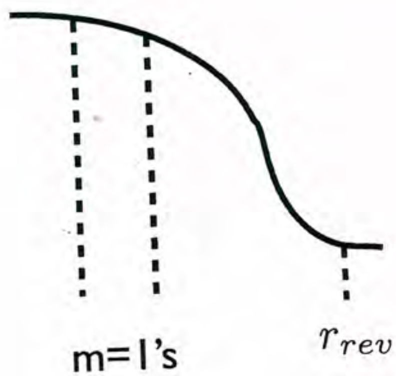
  → Issue: What drives reversal $B_z$ near boundary?

Approach: QL $\langle \tilde{v} \times \tilde{B} \rangle$ in MHD exterior - exercise: derive!

$$\langle \tilde{\mathbf{v}} \times \tilde{\mathbf{B}} \rangle \cong \sum_k |\gamma_k| \frac{R}{r}(q_{res} - q(r))\langle B_\theta \rangle \partial_r (|\tilde{\xi}_r|_k^2)$$

i.e. $\langle J_\theta \rangle$ driven opposite $\langle B_\theta \rangle$ → drives/sustains reversal

→ What of irreversibility - i.e. how is kink-driven reversal 'locked-in'?




→ drive $J_{\|}/B$ flattening, so higher n's destabilized by relaxation front

→ global scattering → propagating reconnection front

m=1, n    m=1, n+1 → m=0, n=1 → driven current sheet, at $r_{rev}$

(difference beat)

sum beat { m=2, 2n+1

but then m=1, n+2 driven → tearing activity, and relaxation region, broadens

→ Bottom Line: How Pinch 'Taylors itself' remains unclear, in detail

# Summary of Magnetic Relaxation

concept: topology

process: stochastization of fields, turbulent reconnection

constraint released: local helicity

players: tearing modes

Mean Field: EMF = $\langle \tilde{v} \times \tilde{B} \rangle$

Global Constraint: $\int d^3x \mathbf{A} \cdot \mathbf{B}$

NL: Helicity Density Flux

Outcome: B-Profile

Shortcoming: Rates, confinement → turbulent transport