

5.) 'Least Action' and the Energy Principle in MHD

→ Introduction

- we now arrive at the MHD Energy Principle, which is a highlight of MHD, plasma physics and classical physics, in general.

- Energy Principle → stability

i.e. till now $\left\{ \begin{array}{l} 218B - \text{waves, etc.} \\ 218A - \text{trivial instabilities (i.e. 2-stream, bump-on-tail, J-driven current-acoustic)} \end{array} \right.$

realistic plasmas $\left\{ \begin{array}{l} \text{lab} \\ \text{or} \\ \text{astro} \end{array} \right\}$ → free energy $\left(\begin{array}{l} \nabla P \\ \nabla J \text{ etc.} \end{array} \right)$

(+) complex geometry, b.c.'s, etc.

→ instabilities with complex dynamics ...

i.e. Rayleigh-Benard → ∇S
Interchanges → $R, \nabla P$ (includes Rayleigh-Taylor)
kinks, tearing → $\nabla J, \nabla(\cdot)$

Relaxation, turbulence, shocks ...
limits on performance (lab)
restrictions on morphology (lab and astro)

- brute force, frontal assault on instabilities often leads to heavy casualties

∴
- need a simple criterion, i.e. a necessary/sufficient criterion to identify and characterize instabilities

⇒ Energy Principle !

- Energy Principle is very much in spirit of R-R Variational principle → no surprise, as both based on self-adjointness of linear operator

- Proceed via:

- sketch of Principle of Least Action for Ideal MHD
⇒ Lagrangian formulation (Kulsrud 4.7)

N.B. This underlies formulation in terms of displacement . . .

- MHD eigenmode equation (generalized simple wave studies so far), second order W

⇒

- energy principle.

(Kulsrud 7.1, 7.2)
(Kadomtsev Article)

- applications (various)

i.) Principle of Least Action for MHD

- For ideal MHD, can immediately write

$$L = \int d^3x \left[\frac{\rho v^2}{2} \right] - W \quad (\text{Lagrangian})$$

$$W = \int d^3x \left(\frac{\rho}{\delta-1} + \frac{\beta^2}{8\pi} + \rho\phi \right)$$

$$\delta' = \int dt L$$

↳ action

$$\omega \quad \mathcal{L} = \frac{\rho v^2}{2} - \left(\frac{\rho}{\delta-1} + \frac{\beta^2}{8\pi} + \rho\phi \right)$$

and can derive MHD equations by $\delta' L = 0$
i.e. Principle of Least Action

- key point: how parametrize trajectory variations??

i.e. for string: (easy)

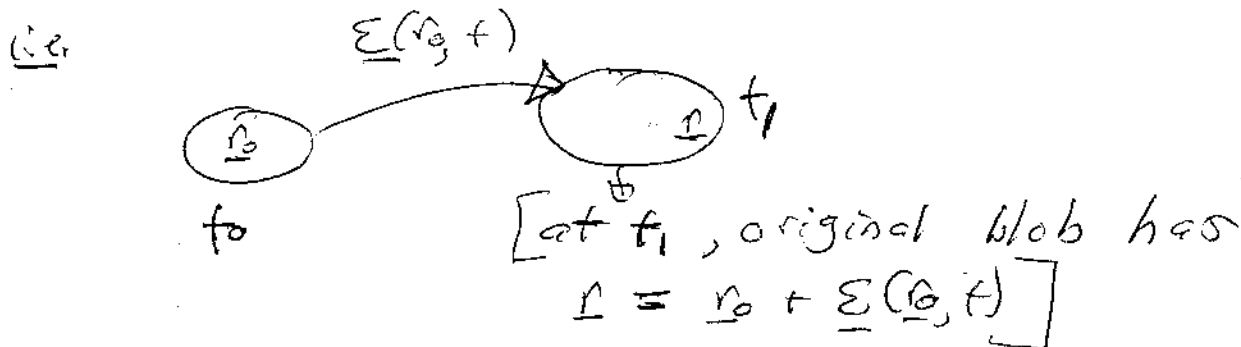
$$S = \int dt \int_0^{L_0} dx \left[\frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \sqrt{\left(1 + \left(\frac{\partial y}{\partial x} \right)^2 \right)^{1/2} - 1} \right]$$

$$\delta' L = \delta L / \delta y \Rightarrow \frac{\partial L}{\partial y_t} \delta y_t + \frac{\partial L}{\partial y_x} \delta y_x \quad \text{etc. ...}$$

∴ analogy with string suggests displacement
is:

→ natural way to formulate Least Action
for ideal MHD

→ natural link of MHD dynamics to particle
dynamics



- how relate $\underline{\xi}(r_0, t)$ to Eulerian velocity?

c.e. during dt , fluid element moves

$$\text{from } r = r_0 + \underline{\xi}(r_0, t) \quad \longrightarrow \quad \text{to } r_0 + \underline{\xi}(r_0, t) + \left(\frac{\partial \underline{\xi}}{\partial t}\right) dt$$

$$\therefore \underline{v}(r_0 + \underline{\xi}(r_0, t), t) = \frac{\partial \underline{\xi}(r_0, t)}{\partial t}$$

→ 3 components of $\underline{\xi}$ satisfy 3 nonlinear
odes with $\underline{\xi}(r_0, t_0) = 0$ as c.c.

→ theory of ode's assures solution exists.

Now, as in wave theory, can write all changes in MHD quantities in terms of displacements, ξ.

$$\delta \rho = -\underline{\nabla} \cdot [\rho(\underline{r}, t) \delta \underline{\xi}(\underline{r}, t)]$$

$$\delta p = -\gamma \rho(\underline{r}, t) \underline{\nabla} \cdot \delta \underline{\xi}(\underline{r}, t) - \delta \underline{\xi}(\underline{r}, t) \cdot \underline{\nabla} p(\underline{r}, t)$$

$$\delta \underline{B} = \underline{\nabla} \times (\delta \underline{\xi}(\underline{r}, t) \times \underline{B}(\underline{r}, t))$$

and

$$\delta V(\underline{r}, t) = \underline{V}(\underline{r}, t) \cdot \underline{\nabla} \delta \underline{\xi}(\underline{r}, t) - \delta \underline{\xi}(\underline{r}, t) \cdot \underline{\nabla} V(\underline{r}, t) + \partial \delta \underline{\xi}(\underline{r}, t) / \partial t$$

so now, can consider δS

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \delta \mathcal{L}$$

$$= \int_{t_1}^{t_2} dt \int d^3x \left(\delta \rho \frac{V^2}{2} + \rho \underline{V} \cdot \delta \underline{V} - \frac{\delta p}{\gamma - 1} - \frac{\underline{B} \cdot \delta \underline{B}}{4\pi} - \delta \rho \phi \right)$$

plugging in δ quantities \Rightarrow

$\delta K E$

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \underbrace{\underline{\nabla} \cdot (-\rho \underline{\delta E}) \frac{V^2}{2} + \rho \underline{V} \cdot (\underline{V} \cdot \underline{\nabla} \delta E - \delta E \cdot \underline{\nabla} V)}_{\delta T K E} + \frac{\partial \delta E}{\partial t} \right\} + \int_{t_1}^{t_2} dt \int d^3x \left(\frac{\gamma \rho \underline{\nabla} \cdot \delta E + \delta E \cdot \underline{\nabla} \rho}{\gamma - 1} \right) - \int_{t_1}^{t_2} dt \int d^3x \frac{\underline{B} \cdot \underline{\nabla} \times (\rho \delta E \times \underline{B})}{4\pi} + \int_{t_1}^{t_2} dt \int d^3x \underline{\nabla} \cdot (\rho \delta E) \phi$$

Now $\delta \underline{\Sigma} \Big|_{t_1, t_2} = 0$, $\delta \underline{\Sigma} \Big|_{\text{bdry}} = 0$
 bdry \rightarrow const restriction

so drop a lot \Rightarrow (with b.c.'s)

$$\delta S' = \int_{t_1}^{t_2} dt \int d^3x \left\{ \delta E \cdot \left[\underbrace{\rho \underline{\nabla} V^2}_{\frac{\partial \rho}{\partial t}} - \underline{\nabla} \cdot (\rho \underline{V} \underline{V}) - \underbrace{\rho \underline{\nabla} V^2}_{\rho \nabla \phi} - \frac{\partial (\rho V)}{\partial t} \right] - \frac{\underline{J} \times \underline{B} \cdot \underline{\nabla} \delta E + \delta E \cdot \underline{\nabla} \rho}{\gamma - 1} - \delta E \cdot \rho \underline{\nabla} \phi + \delta E \cdot \frac{(\underline{\nabla} \times \underline{B}) \times \underline{B}}{4\pi} \right\}$$

So

$$\delta S = - \int_{t_1}^{t_2} \int d^3x \delta \underline{\varepsilon} \cdot \left[\frac{\partial (\rho \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v} \underline{v}) + \underline{\nabla} p - \underline{j} \times \underline{B} + \rho \underline{\nabla} \phi \right]$$

So

$$\delta S = 0 \quad \text{and} \quad \delta \underline{\varepsilon} \neq 0 \Rightarrow$$

$$\frac{\partial (\rho \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v} \underline{v}) = -\underline{\nabla} p + \underline{j} \times \underline{B} - \rho \underline{\nabla} \phi$$

and $\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{v}) \Rightarrow$

$$\rho \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right] = -\underline{\nabla} p + \underline{j} \times \underline{B} - \rho \underline{\nabla} \phi$$

\Rightarrow equation of motion of ideal MHD emerges as "Lagrange's Equation".

Note: for case of $\underline{v} = 0 \rightarrow$ equilibrium solution then $\delta S = 0$ gives:

$$\underline{\nabla} p = \underline{j} \times \underline{B} - \rho \underline{\nabla} \phi$$

Moral of this story:

→ can derive MHD equations from
Principle of Least Action

→ displacement is a useful way to formulate
ideal MHD dynamics

Now this brings us to:

ii.) Energy Principle - Simple Form

Consider inhomogeneous static equilibrium/critical
state with:

$\rho = \text{const}$ ($C \rightarrow \infty$)

equilibrium conditions

$$\begin{cases} \nabla p_0 = \underline{J}_0 \times \underline{B}_0 - \rho \underline{g} \\ \nabla \times \underline{B}_0 = \frac{4\pi}{c} \underline{J}_0 \\ \nabla \cdot \underline{B}_0 = 0 \end{cases}$$

[n.b. gravity
will be dropped
in places]

$$\text{a) } \begin{cases} p_0 = p_0(r_0) \\ \text{time independent} \\ \text{but inhomogeneous} \end{cases}$$

and no flow or $\begin{cases} \text{self-} \\ \text{gravity} \end{cases}$...

Further assume → rigid wall bounds
system $(\uparrow \downarrow)$

$$\begin{aligned} \rightarrow \underline{v} \cdot \hat{n} \Big|_{\text{wall}} &= 0 \\ \underline{B} \cdot \hat{n} \Big|_{\text{wall}} &= 0 \end{aligned}$$

and now ... \rightarrow perturb system from eqbm
by $\underline{\underline{\epsilon}}$

$$\rightarrow \text{so, at } t=0 : \\ \underline{\underline{\Sigma}}(r) = \underline{\underline{\Sigma}}_0(r)$$

$$\frac{\partial \underline{\underline{\Sigma}}(r)}{\partial t} = \frac{\partial \hat{\underline{\underline{\Sigma}}}_0(r)}{\partial t}$$

\rightarrow keep only linear terms in $\underline{\underline{\epsilon}} \Rightarrow$
 $\underline{\underline{r}} = \underline{\underline{r}}_0 + \epsilon(\underline{\underline{r}}_0, t)$

and $\underline{\underline{r}}_0 \rightarrow \underline{\underline{r}}$ in argument of perturbed quantities.

so

$$\rightarrow \rho(t, r) = \rho_0 - \nabla \cdot (\rho_0 \underline{\underline{\epsilon}})$$

$$\rho(t, \underline{\underline{r}}) = \rho_0 - \gamma \rho_0 \nabla \cdot \underline{\underline{\epsilon}} - \underline{\underline{\epsilon}} \cdot \nabla \rho_0$$

$$\underline{\underline{B}}(t, \underline{\underline{r}}) = \underline{\underline{B}}_0 + \nabla \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0)$$

$$\underline{\underline{H}}(t, \underline{\underline{r}}) = \underline{\underline{j}}_0 + \nabla \times [\nabla \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0)]$$

\rightarrow putting it into equation of motion
(linearized) \Rightarrow

$$\rho_0 \frac{\partial^2 \underline{\Sigma}}{\partial t^2} = \underline{F}(\underline{\Sigma})$$

where:

$$\begin{aligned} \underline{F}(\underline{\Sigma}) = & \frac{1}{4\pi} \left[\nabla \times \left[\nabla \times (\underline{\Sigma} \times \underline{B}_0) \right] \right] \times \underline{B}_0 \\ & + \underline{J}_0 \times \left[\nabla \times (\underline{\Sigma} \times \underline{B}_0) \right] - g \nabla \cdot (\rho_0 \underline{\Sigma}) \\ & + \nabla \left[\underline{\Sigma} \cdot \nabla \rho_0 + \delta \rho_0 (\nabla \cdot \underline{\Sigma}) \right] \\ & - \underline{\nabla} \rho \end{aligned}$$

with b.c. $\begin{cases} \underline{\Sigma} \cdot \underline{n} = 0 & \text{on surface} \\ \underline{B} \cdot \underline{n} = 0 & \text{on surface} \end{cases}$

Key Point:

$$\rightarrow \underline{F}(\underline{\Sigma}) \text{ is self-adjoint} !!$$

i.e.

$$\int d^3x \underline{\eta} \cdot \underline{F}(\underline{\Sigma}) = \int d^3x \underline{\Sigma} \cdot \underline{F}(\underline{\eta})$$

→ to prove: see Kulshrud, pblm. 6
(coming on pblm Set III)

or consider the following (an indirect proof) ...
Legendre's involved...

→ can write total energy, to
second order (in displacement) as:

$$\text{c.e. } E = \int d^3x \frac{\rho_0(r)}{2} \left(\frac{\partial \underline{\xi}}{\partial t} \right)^2 + W(\underline{\xi}, \underline{\xi})$$

2nd order bit of:

$$\int \left(\frac{\rho}{\gamma-1} + \frac{B^2}{8\pi} + \rho\phi \right) d^3x$$

Now:

$$\rightarrow W = W_0 + \underbrace{W_1(\underline{\xi})}_{\text{first order}} + \underbrace{W_2(\underline{\xi}, \underline{\xi})}_{\text{second order}}$$

→ total energy is conserved, for any $\underline{\xi}$

with initial conditions $\underline{\xi}_0, \dot{\underline{\xi}}_0$,

provided $\underline{\xi} \cdot \hat{n} = \dot{\underline{\xi}} \cdot \hat{n} = 0$ (b.c.)

Now, $dE/dt = 0 \Rightarrow$

$$\frac{dE}{dt} = \int d^3x \rho_0 \left\{ \frac{\partial \underline{\underline{\xi}}}{\partial t} \cdot \frac{\partial^2 \underline{\underline{\xi}}}{\partial t^2} \right\} + W_1 \left(\frac{\partial \underline{\underline{\xi}}}{\partial t} \right) \\ + W_2 \left(\frac{\partial \underline{\underline{\xi}}}{\partial t}, \underline{\underline{\xi}} \right) + W_2 \left(\underline{\underline{\xi}}, \frac{\partial \underline{\underline{\xi}}}{\partial t} \right) = 0$$

and $\rho_0 \frac{\partial^2 \underline{\underline{\xi}}}{\partial t^2} = \underline{\underline{F}}(\underline{\underline{\xi}}) \Rightarrow$

$$\frac{dE}{dt} = \int d^3x \left[\frac{\partial \underline{\underline{\xi}}}{\partial t} \cdot \underline{\underline{F}}(\underline{\underline{\xi}}) \right] + W_1 \left(\frac{\partial \underline{\underline{\xi}}}{\partial t} \right) \\ + W_2 \left(\frac{\partial \underline{\underline{\xi}}}{\partial t}, \underline{\underline{\xi}} \right) + W_2 \left(\underline{\underline{\xi}}, \frac{\partial \underline{\underline{\xi}}}{\partial t} \right)$$

but since $dE/dt = 0$ is always true, it is true at $t=0$, a particular time

setting $\underline{\underline{\xi}}_0 \equiv \underline{\underline{\eta}} \Rightarrow$
 \hookrightarrow a particular displ. ...

$$\int d^3x \underline{\underline{\eta}} \cdot \underline{\underline{F}}(\underline{\underline{\xi}}) + W_1(\underline{\underline{\eta}}) + W_2(\underline{\underline{\eta}}, \underline{\underline{\xi}}) \\ + W_2(\underline{\underline{\xi}}, \underline{\underline{\eta}}) = 0$$

now, $W_1(\underline{\eta}) = 0$ so (no velocity dependence)
on i.c.

$$\int d^3x \left(\underline{\eta} \cdot F(\underline{\varepsilon}) \right) + \left[W_2(\underline{\eta}, \underline{\varepsilon}) + W_2(\underline{\varepsilon}, \underline{\eta}) \right] = 0$$

or more clearly \Rightarrow

$$\int d^3x \underline{\eta} \cdot F(\underline{\varepsilon}) = - \left[W_2(\underline{\eta}, \underline{\varepsilon}) + W_2(\underline{\varepsilon}, \underline{\eta}) \right]$$

so RHS symmetric under $\underline{\eta} \leftrightarrow \underline{\varepsilon}$
interchange:

so so is LHS \downarrow i.e.

$$\int d^3x \underline{\eta} \cdot F(\underline{\varepsilon}) = \int d^3x \underline{\varepsilon} \cdot F(\underline{\eta})$$

and have proved self-adjointness \downarrow

\rightarrow finally, useful to note that if now
 $\underline{\eta} = \underline{\varepsilon}$

$$W_2(\underline{\varepsilon}, \underline{\varepsilon}) = -\frac{1}{2} \int d^3x \left[\underline{\varepsilon} \cdot F(\underline{\varepsilon}) \right]$$

— a handy expression for W_2 in terms F \downarrow

so now, have shown that:

→ $\underline{F}(\underline{\xi})$ self-adjoint

→ $W_2(\underline{\xi})$, the potential energy of displacement $\underline{\xi}$, can be expressed as:

$$W_2(\underline{\xi}) = -\frac{1}{2} \int d^3x [\underline{\xi} \cdot \underline{F}(\underline{\xi})]$$

From these, we show several important results:

- reality of ω^2 and "exchange of stabilities"
↔ due to structure of instability in ideal MHD
- orthogonality of eigenfunctions
- variational structure

1) reality of ω^2 , "exchange of stabilities"

$$\underline{\xi} = \tilde{\sum} \alpha_j e^{-i\omega t}$$

$$-\rho \omega^2 \underline{\xi} = \underline{F}(\underline{\xi}) \quad (1)$$

$$\rho \omega^{2*} \underline{\xi}^* = \underline{F}(\underline{\xi}^*) \quad (2)$$

sub. (1) into (2)
explicitly real

$$\underline{\Sigma}^* * (1) - \underline{\Sigma} * (2) \Rightarrow$$

$$-\rho_0 (\omega^2 - \omega^{2*}) \underline{\Sigma}^* \cdot \underline{\Sigma} = \underline{\Sigma}^* \cdot \underline{F}(\underline{\Sigma}) - \underline{\Sigma} \cdot \underline{F}(\underline{\Sigma}^*)$$

and integrating \Rightarrow

$$-\rho_0 (\omega^2 - \omega^{2*}) \int d^3x (\underline{\Sigma}^* \cdot \underline{\Sigma}) = \int d^3x [\underline{\Sigma}^* \cdot \underline{F}(\underline{\Sigma}) - \underline{\Sigma} \cdot \underline{F}(\underline{\Sigma}^*)] = 0, \text{ by self-adjoint property}$$

$$\Rightarrow \underline{\Sigma}^* \cdot \underline{\Sigma} \text{ real} \Rightarrow (\omega^2)^* = \omega^2$$

$\Rightarrow \omega^2$ is real

$\omega^2 > 0 \rightarrow$ stability

$\omega^2 < 0 \rightarrow$ instability, but purely growing
 \leadsto no oscillation

Contrast to instabilities with which you should be familiar:

\rightarrow bump-on-tail

$$\omega = \omega_k^0 + i\gamma_k$$

wave + inverse dissipation
 carrier



→ two stream $\epsilon = 1 - \frac{\omega_{p0}^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kv_0)^2}$

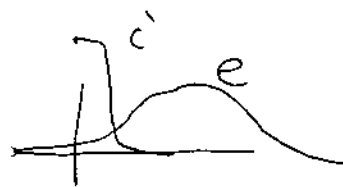
~ coupling of $\left\{ \begin{array}{l} \text{positive energy wave in plasma} \\ \text{negative energy wave in beam} \end{array} \right.$

~ "reactive" counter-part of bump on tail \Rightarrow
can have ω^2 real

→ beam + dissipation \Rightarrow negative energy wave
 \oplus dissipation \Rightarrow growth

$$\omega = \omega_r + i\gamma$$

- current-driven ion-acoustic

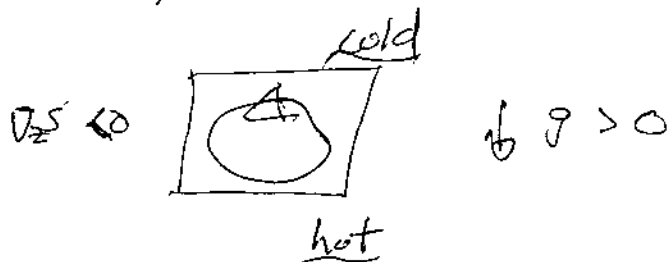


$$\omega = \omega_r + i\gamma \quad \gamma = (c) \frac{\partial f_e^{(c)}}{\partial v} - (e) \frac{\partial f_e^{(e)}}{\partial v}$$

wave + competition of dissipation and dissipation

→ ideal Rayleigh-Benard Convection

$$\omega^2 = - \frac{k_H^2}{k_H^2 + k_V^2} g \frac{\partial \rho'}{\partial z}$$



of these, ideal MHD instabilities similar in structure to convection and ω^2 real cases of 2-stream, and different in structure from the others

4. In ideal MHD, instability defines structure of eigenfunction, i.e. $\tilde{\Sigma} = \tilde{\Sigma}(n, \gamma)$.

N.B. In ideal MHD, only scale in problem is system size \leftrightarrow boundaries. Contrast Sweet-Parker reconnection ($\Delta/L \ll 1$), a case of resistive MHD.

proceeding \Rightarrow

Since ω^2 real, ω^2 must pass thru $\omega^2 = 0$ as the system evolves from stable to unstable.

this evolution is called "exchange of stabilities"

\Rightarrow marginal displacement solves $F(\underline{\Sigma}) = 0$

N.B. \Rightarrow solution of $\underline{F}(\underline{\xi}) = 0$ gives linear stability boundary, in parameter space

ii.) orthogonality

consider two solutions to $-\rho_0 \omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$,

$$-\rho_0 \omega_1^2 \underline{\xi}_1 = \underline{F}(\underline{\xi}_1) \quad \times \underline{\xi}_2$$

$$-\rho_0 \omega_2^2 \underline{\xi}_2 = \underline{F}(\underline{\xi}_2) \quad \times \underline{\xi}_1$$

$$-(\omega_1^2 - \omega_2^2) \int d^3x \rho_0 \underline{\xi}_1 \cdot \underline{\xi}_2 = \int d^3x \left[\underline{\xi}_2 \cdot \underline{F}(\underline{\xi}_1) - \underline{\xi}_1 \cdot \underline{F}(\underline{\xi}_2) \right] \\ = 0, \text{ by self-adjointness}$$

$$\omega_1^2 \neq \omega_2^2 \Rightarrow \int d^3x \rho_0 \underline{\xi}_1 \cdot \underline{\xi}_2 = 0$$

\Rightarrow orthonormality, with weighting function ρ_0 .

The point of all this is that now we can set up a variational quadratic form, aka' beloved Sturm-Liouville theory.

$$-\nabla^2 \omega^2 \underline{\underline{\epsilon}} = \underline{\underline{F}}(\underline{\underline{\epsilon}})$$

and $\otimes \frac{\underline{\underline{\epsilon}}}{2} \Rightarrow$

$$\omega^2 = \frac{-\int d^3x \underline{\underline{\epsilon}} \cdot \underline{\underline{F}}(\underline{\underline{\epsilon}})/2}{\int \rho_0 \underline{\underline{\epsilon}}^2/2}$$

$$= W_2(\underline{\underline{\epsilon}}) / \int \rho \underline{\underline{\epsilon}}^2/2$$

\Rightarrow with $k(\underline{\underline{\epsilon}}) \equiv \int d^3x \rho \underline{\underline{\epsilon}}^2/2$, have

$$\left\{ \omega^2 = W_2(\underline{\underline{\epsilon}}) / k(\underline{\underline{\epsilon}}) \right\} \rightarrow \left\{ \begin{array}{l} \text{variational, quadratic} \\ \text{form} \end{array} \right.$$

and we know that, since all requirements satisfied, that

\rightarrow any trial $\underline{\underline{\epsilon}}$ plugged into $W_2(\underline{\underline{\epsilon}})/k(\underline{\underline{\epsilon}})$ yields $\omega^2(\underline{\underline{\epsilon}}) > \omega_T^2$

\hookrightarrow the true eigenvalue

the variational result is always upper bound.

→ so, we know that

- if can find a trial $\underline{\underline{\varepsilon}}$ such that

$$W_2(\underline{\underline{\varepsilon}}) < 0$$

- then, configuration is surely unstable

∴ this yields the desired necessary and sufficient condition for instability namely that it be possible to find a $\underline{\underline{\varepsilon}}$ such that

$$\underline{W_2(\underline{\underline{\varepsilon}}) < 0.}$$

hereafter, we write $W_2(\underline{\underline{\varepsilon}}) = \delta W(\underline{\underline{\varepsilon}})$,

so the MHD Energy Principle is just:

instability iff \exists well behaved $\underline{\underline{\varepsilon}}$ s/t

$$\delta W(\underline{\underline{\varepsilon}}) < 0$$

N.B.

- in physical terms, E.P. \Rightarrow instability if can find a displacement which lowers the energy. Note linear instability \leftrightarrow δW to $O(\underline{\underline{\epsilon}}^2)$ considered

- know $\delta W(\underline{\underline{\epsilon}}) = -\frac{1}{2} \int d^3x \underline{\underline{\epsilon}} \cdot \underline{\underline{F}}(\underline{\underline{\epsilon}})$

so, now must manipulate δW into physically useful form, a.e. recall

$$\begin{aligned}
 \underline{\underline{F}}(\underline{\underline{\epsilon}}) &= \frac{1}{4\pi} \left\{ \begin{array}{l} \underline{\underline{J}} \times \underline{\underline{B}}_0 \quad -\textcircled{1} \\ \nabla \times [\nabla \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0)] \end{array} \right\} \times \underline{\underline{B}}_0 \\
 &+ \underline{\underline{J}}_0 \times \left[\begin{array}{l} \underline{\underline{J}}_0 \times \delta \underline{\underline{B}} \quad -\textcircled{2} \\ \nabla \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) \end{array} \right] \\
 &+ \underline{\underline{D}} \left[\begin{array}{l} \nabla \cdot \underline{\underline{\epsilon}} \quad -\textcircled{3} \\ \underline{\underline{\epsilon}} \cdot \nabla \rho_0 \end{array} \right] + \nabla \cdot (\rho_0 \underline{\underline{\epsilon}}) \underline{\underline{D}} \phi \quad -\textcircled{4} \\
 &= \underline{\underline{F}}_1 + \underline{\underline{F}}_2 + \underline{\underline{F}}_3 + \underline{\underline{F}}_4
 \end{aligned}$$

Remember here, all $\underline{\underline{B}}_0$, $\underline{\underline{\rho}}_0$, $\underline{\underline{p}}_0$ etc. inhomogeneous, and $\underline{\underline{\epsilon}} \cdot \underline{\underline{n}}$ and $\underline{\underline{B}} \cdot \underline{\underline{n}}$ on boundary.

- remains to manipulate $-\int [\underline{\underline{\epsilon}} \cdot \underline{\underline{F}}(\underline{\underline{\epsilon}})/2] d^3x$
into "illumination" form

- key is sign of dW , so seek to extract quadratic terms, as unambiguous.

\Rightarrow let the crank begin!

$$\textcircled{1} dW_0 = -\frac{1}{2} \int \underline{\underline{\epsilon}} \cdot \underline{\underline{F}}_0(\underline{\underline{\epsilon}}) d^3x$$

$$= -\frac{1}{2} \int d^3x \underline{\underline{\epsilon}} \cdot \left\{ \frac{1}{4\pi} \left[\underline{\underline{\nabla}} \times [\underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0)] \right] \times \underline{\underline{B}}_0 \right\}$$

$$= \frac{1}{8\pi} \int d^3x \left(\underline{\underline{\nabla}} \times [\underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0)] \right) \cdot \underline{\underline{\epsilon}} \times \underline{\underline{B}}_0$$

$$= \frac{1}{8\pi} \int d^3x \underline{\underline{\nabla}} \cdot \left[\underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) \right]$$

$$+ \frac{1}{8\pi} \int d^3x \left(\underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) \right) \cdot \left(\underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) \right)$$

if $\underline{\underline{Q}} \equiv \underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) = d\underline{\underline{B}}$, from induction

$$dW_0 = \int d^3x \frac{\underline{\underline{Q}}^2}{8\pi} + \frac{1}{8\pi} \int d\underline{\underline{S}} \cdot \left(\underline{\underline{\nabla}} \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0) \right) \times (\underline{\underline{\epsilon}} \times \underline{\underline{B}}_0)$$

$$\Rightarrow \oint_{\text{Surface}} \delta W_0 = -\frac{1}{8\pi} \int dS \left[\hat{n} \cdot \underline{B}_0 \underline{\epsilon} \cdot \underline{Q} - (\hat{n} \cdot \underline{\epsilon}) \underline{B}_0 \cdot \underline{Q} \right]$$

$$\delta W_0 = \int d^3x \frac{Q^2}{8\pi}$$

$$\delta W_0 = -\frac{1}{2} \int d^3x \underline{\epsilon} \cdot \underline{J}_0 \times [\nabla \times (\underline{\epsilon} \times \underline{B}_0)]$$

$$= -\frac{1}{2} \int d^3x \underline{\epsilon} \cdot (\underline{J}_0 \times \underline{Q})$$

$$= +\frac{1}{2} \int d^3x \underline{J}_0 \cdot (\underline{\epsilon} \times \underline{Q})$$

$$\delta W_0 = -\frac{1}{2} \int d^3x \underline{\epsilon} \cdot \underline{\nabla} [\rho_0 \underline{\nabla} \cdot \underline{\epsilon} + \underline{\epsilon} \cdot \underline{\nabla} \rho_0]$$

c.b.p $\underline{\epsilon} \cdot \hat{n} = 0$ on boundary

$$\Rightarrow \delta W_0 = \int d^3x \frac{1}{2} \left[\rho_0 (\underline{\nabla} \cdot \underline{\epsilon})^2 + (\underline{\nabla} \cdot \underline{\epsilon}) \underline{\epsilon} \cdot \underline{\nabla} \rho_0 \right]$$

• & last but not least...

$$\delta W_{\oplus} = - \int \frac{d^3x}{2} \underline{\Sigma} \cdot \nabla \cdot (\rho_0 \underline{\Sigma}) \nabla \phi$$

$$= - \frac{1}{2} \int d^3x (\underline{\Sigma} \cdot \nabla \phi) \nabla \cdot (\rho_0 \underline{\Sigma})$$

so, putting the whole mess together

$$\delta W = \frac{1}{2} \int d^3x \left\{ \begin{array}{l} \textcircled{1} \quad \textcircled{2} \\ \frac{\underline{Q}^2}{4\pi} + \underline{J}_0(\underline{x}) \cdot (\underline{\Sigma} \times \underline{Q}) \\ \textcircled{3} \quad \textcircled{4} \quad \textcircled{5} \\ + \gamma \rho_0(\underline{x}) (\nabla \cdot \underline{\Sigma})^2 + (\underline{\Sigma} \cdot \nabla \rho_0(\underline{x})) \nabla \cdot \underline{\Sigma} - (\underline{\Sigma} \cdot \nabla \phi) \nabla \cdot (\rho_0 \underline{\Sigma}) \end{array} \right\}$$

$\underline{Q} = \nabla \times (\underline{\Sigma} \times \underline{A}_0)$

note: general characteristics

- $\textcircled{1} \rightarrow > 0 \rightarrow$ field line bending
 - $\textcircled{3} \rightarrow > 0 \rightarrow$ compression
- } \rightarrow always stabilizing
 $\delta W > 0$

Free energy sources:

$\nabla \rho_0(\underline{x})$ in $\textcircled{5} \rightarrow$ density gradient

$\underline{J}_0(\underline{x})$ in $\textcircled{2} \rightarrow$ current profile

$\nabla \rho_0(\underline{x})$ in $\textcircled{4} \rightarrow$ pressure gradient
gravity and ρ_0 in $\textcircled{5}$

\Rightarrow can make $\delta W < 0$, for certain profiles
and $\underline{\Sigma} \Rightarrow$ free energy sources for instability.

Note:

→ dW is imprecise

→ dW does not reveal much about growth rates

but

→ very useful for simple quality assessment of stability

→ can elucidate

- complex problem
- problem in which infer re: equilibrium not precise.

∴ further developments in theory remain, but better to consider some examples

⇒

iii) Convection and Interchange Instabilities
→ A Simple Application of the Energy Principle

consider 4 related examples:

- Convection and the Schwarzschild Criterion
- Rayleigh-Taylor Instability
- Interchange Instability
- Interchange without Gravity

i) Schwarzschild Criterion and Convection

i.e. stellar atmosphere

$$\begin{array}{c} \text{---} \\ \circ \quad \circ \quad \circ \\ \circ \quad \circ \quad \circ \\ \text{---} \end{array} \quad z \uparrow \quad \left(\rho g = \frac{d\rho}{dz} \right)$$

$$\frac{d\rho}{dz} < 0, \quad \frac{d\rho}{dz} < 0$$

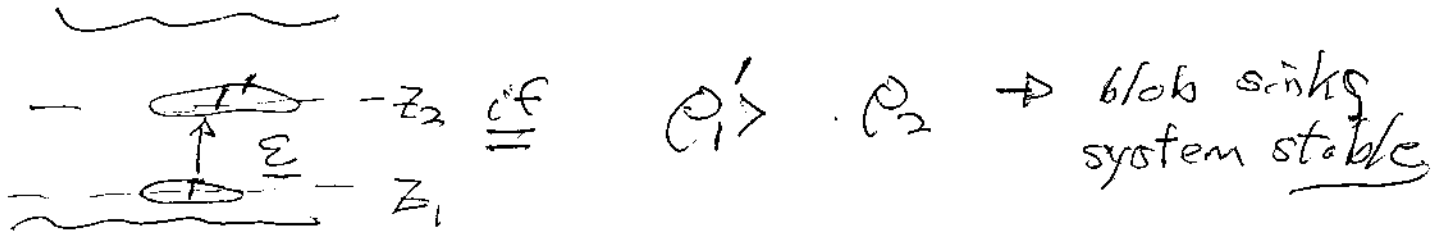
and

$$\rho \rho^{-\gamma} = \text{const.}$$

(basic means of heat transport)

is basic idea of convection, consider a virtual displacement of a slug/blob of gas upward

⇒ physical argument



$\rho_1' < \rho_2 \rightarrow$ blob rises, system unstable

For infinitesimal displacement, $\epsilon \sim \Delta z \Rightarrow$

$$\rho_2 = \rho_1 + \frac{d\rho_1}{dz} \Delta z$$

For ρ_1' , \rightarrow system is isentropic \Rightarrow
 $P \rho^{-\gamma} = \text{const.}$ applies

\rightarrow displaced blob (i.e. ρ_1') comes to rapid pressure equilibration with surroundings

i.e. $\frac{\Delta z}{c_s} \ll T_{\text{rise}}$ $\Leftrightarrow \gamma < k c_s$
 \sim (nearly incompressible)

$$\rho_1' = \rho_1 + \Delta z \frac{d\rho_1}{dz} = \rho_2$$

$$P_1 \rho_1^{-\gamma} = P_1' \rho_1'^{-\gamma}$$

$$\Rightarrow \rho_1 \rho_1^{-\gamma} = \left(\rho_1 + \Delta z \frac{d\rho_1}{dz} \right) \rho_1^{1-\gamma}$$

$$\Rightarrow \left(\frac{\rho_1'}{\rho_1} \right)^\gamma = 1 + \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

$$\frac{\rho_1'}{\rho_1} = \left(1 + \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz} \right)^{1/\gamma} \approx 1 + \frac{\Delta z}{\gamma \rho_1} \frac{d\rho_1}{dz}$$

$$\frac{\rho_1'}{\rho_1} = 1 + \frac{1}{\gamma} \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

\Rightarrow buoyant blob if:

$$\frac{\rho_1'}{\rho_1} < \frac{\rho_2}{\rho_1} \Rightarrow \frac{\Delta z}{\gamma \rho_1} \frac{d\rho_1}{dz} < \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

$$\Rightarrow \frac{1}{\gamma} \frac{1}{\rho_1} \frac{d\rho_1}{dz} < \frac{1}{\rho_1} \frac{d\rho_1}{dz}$$

or, as both gradients negative

$$\frac{1}{\gamma} \left| \frac{1}{\rho_1} \frac{d\rho_1}{dz} \right| > \frac{1}{\rho_1} \left| \frac{d\rho_1}{dz} \right|$$

Schwarzschild
criterion for
Convective Instability

and as $S \equiv \ln(\rho \rho^{-\gamma})$

$$\frac{dS}{dz} = \frac{1}{\rho} \frac{d\rho}{dz} - \frac{\gamma}{\rho} \frac{d\rho}{dz}$$

\Rightarrow blob buoyant if $\frac{dS}{dz} < 0 \rightarrow$ "superadiabotically stratified"

sinks/restored if $\frac{dS}{dz} > 0 \rightarrow$ "subadiabotically stratified"

Marginal $dS/dz = 0 \rightarrow$ adiabotically stratified

Note: \rightarrow Schwarzschild instability criterion \Leftrightarrow answers "is free energy available, locally" \Leftrightarrow ideal

\rightarrow Rayleigh # criterion $\Rightarrow Ra > Ra_{crit}$
 \Rightarrow does free energy overcome dissipation?

Now, what does dW say?

$$\text{Recall: } dW = \frac{1}{2} \int d^3x \left[\frac{\underline{Q}^2}{4\pi} + \gamma \rho (\underline{v} \cdot \underline{E})^2 + \underline{j}_0 \cdot (\underline{E} \times \underline{Q}) \right. \\ \left. + (\underline{E} \cdot \underline{v} \rho_0) (\underline{v} \cdot \underline{E}) - (\underline{E} \cdot \underline{v} \phi) \underline{v} \cdot (\rho_0 \underline{E}) \right]$$

pure hydro $\rightarrow \underline{Q} = 0, \underline{j}_0 = 0$

$$\frac{d\rho}{dz} = \rho g \rightarrow \text{hydrostatic equilibrium} \\ \underline{v} \rho = \underline{v} \times \underline{B} + \rho \underline{g}$$

$$\underline{g} = \nabla\phi \quad , \quad \underline{g} \text{ downward}$$

$$2dW = \int d^3x \left[\gamma\rho (\underline{\nabla}\cdot\underline{\varepsilon})^2 + (\underline{\varepsilon}\cdot\underline{\nabla}\rho)(\underline{\nabla}\cdot\underline{\varepsilon}) \right. \\ \left. + (\underline{\varepsilon}\cdot\underline{g}) (\underline{\varepsilon}\cdot\underline{\nabla}\rho_0 + \rho_0 \underline{\nabla}\cdot\underline{\varepsilon}) \right] \\ = \int d^3x \left[\gamma\rho (\underline{\nabla}\cdot\underline{\varepsilon})^2 + (\underline{\nabla}\cdot\underline{\varepsilon}) (\underline{\varepsilon}\cdot(\underline{\nabla}\rho + \underline{g}\rho_0)) \right. \\ \left. + (\underline{\varepsilon}\cdot\underline{g}) (\underline{\varepsilon}\cdot\underline{\nabla}\rho_0) \right]$$

$$\text{but } \underline{\nabla}\rho = \rho \underline{g} \quad (\text{eibm condition}) \Rightarrow$$

$$2dW = \int d^3x \left[\gamma\rho (\underline{\nabla}\cdot\underline{\varepsilon})^2 + 2 \frac{(\underline{\nabla}\cdot\underline{\varepsilon})(\underline{\varepsilon}\cdot\underline{\nabla}\rho)}{\gamma\rho} + \left(\frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho} \right)^2 \right. \\ \left. - \gamma\rho \left(\frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho} \right)^2 + (\underline{\varepsilon}\cdot\underline{g}) (\underline{\varepsilon}\cdot\underline{\nabla}\rho_0) \right] \\ = \int d^3x \left[\gamma\rho (\underline{\nabla}\cdot\underline{\varepsilon} + \frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho})^2 - \left(\frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho} \right)^2 + (\underline{\varepsilon}\cdot\underline{g}) (\underline{\varepsilon}\cdot\underline{\nabla}\rho_0) \right] \\ = \int d^3x \left[\gamma\rho (\underline{\nabla}\cdot\underline{\varepsilon} + \frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho})^2 - \frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho} \left(\frac{\underline{\varepsilon}\cdot\underline{\nabla}\rho}{\gamma\rho} - \underline{\varepsilon}\cdot\underline{\nabla}\rho_0 \right) \right]$$

where used equilibrium condition again, so
 \Rightarrow

$$2dW = \int d^3x \left[\delta P \left(\underline{\nabla} \cdot \underline{\varepsilon} + \underline{\varepsilon} \cdot \underline{\nabla} P \right)^2 - \underline{\varepsilon} \cdot \underline{\nabla} P \frac{\underline{\varepsilon} \cdot \underline{\nabla}}{\gamma} \ln(\rho_0^{-\gamma}) \right]$$

Now, object is to

- \rightarrow explore possible displacements to see if $dW < 0$ possible
- \rightarrow uncover any general condition

Now, expect $\underline{\varepsilon}$ to have form:

$$\underline{\varepsilon} = \text{re} \left[\underline{\underline{\varepsilon}}(\underline{z}) e^{ikx} \right] \quad (\text{must be real!})$$

so can choose $\underline{\nabla} \cdot \underline{\varepsilon} = -\underline{\varepsilon} \cdot \underline{\nabla} P$

\rightarrow equivalent to setting a relation between $\varepsilon_x, \varepsilon_z$.

$$\rightarrow \underline{\nabla} \cdot \underline{\varepsilon} \sim \frac{\underline{\varepsilon}}{\gamma P} \frac{dP}{dz} \sim \frac{\underline{\varepsilon}}{\gamma L_p}$$

\hookrightarrow pressure scale height

so $\frac{|\underline{\nabla} \cdot \underline{\varepsilon}|}{|\underline{\varepsilon}|} \sim 1/L_p \rightarrow$ "weakly compressible",
 in accord with physical argument

contrast $\frac{|\nabla \cdot \underline{\epsilon}|}{|\underline{\epsilon}|} \sim |k| \rightarrow$ "strongly compressible" limit

$$\text{So } 2dW = - \int d^3x \left[\frac{\underline{\Sigma} \cdot \nabla \rho}{\gamma} \underline{\Sigma} \cdot \nabla \ln(\rho \rho^{-\gamma}) \right]$$

$$\frac{d\rho}{dz} \neq \nabla \rho \quad \text{and} \quad \frac{d\rho}{dz} < 0 \Rightarrow$$

if have any range of z over which

$$\frac{d \ln(\rho \rho^{-\gamma})}{dz} < 0$$

\Rightarrow have $\underline{\Sigma} \neq 0$ there, and $dW < 0$

\Rightarrow instability, with criterion/condition that

$$\boxed{\frac{d \ln(\rho \rho^{-\gamma})}{dz} < 0}$$

\rightarrow Schwarzschild condition recovered

Now, can go further, and ask what is effect of magnetic field?

i.e. - consider $\underline{B} = B_0 \hat{x}$

then

$$\delta W = \delta W_0 + \int d^3x \frac{Q^2}{8\pi}$$

↑
what we have

$$\underline{Q} = \underline{\nabla} \times (\underline{\epsilon} \times \underline{B}_0) \quad \text{(homogeneous)}$$

$$\underline{Q} = \underline{B}_0 \cdot \underline{\nabla} \underline{\epsilon} - \underline{\epsilon} \cdot \underline{\nabla} \underline{B}_0 - \underline{B}_0 \underline{\nabla} \cdot \underline{\epsilon}$$

Now, to minimize δW ,

$$\therefore Q = -B_0 \underline{\nabla} \cdot \underline{\epsilon}$$

$$\underline{B}_0 \cdot \underline{\nabla} \underline{\epsilon} = 0$$

→ flute displacement

$$k_{\parallel} = 0$$

→ no bending energy expended

$$\delta W = \delta W_0 + \int d^3x \frac{B_0^2}{8\pi} (\underline{\nabla} \cdot \underline{\epsilon})^2$$

but from before have, $\underline{\nabla} \cdot \underline{\epsilon} = -\frac{\underline{\epsilon} \cdot \underline{\nabla} \rho}{\gamma \rho}$

$$\delta W = \int d^3x \left[\frac{B_0^2}{8\pi} \left(\frac{\underline{\epsilon} \cdot \underline{\nabla} \rho}{\gamma \rho} \right)^2 - \left(\frac{\underline{\epsilon} \cdot \underline{\nabla} \rho}{\gamma} \right) \frac{\underline{\epsilon} \cdot \underline{\nabla} \ln(\rho \rho^{-\gamma})}{2} \right]$$

$$\Delta W \sim \int d^3x \left[\rho_{\text{mag}} \frac{E^2}{\gamma^2 L_p^2} - \frac{\rho_{\text{th}}}{\gamma L_p} E^2 \left| \frac{dS}{dz} \right| \right]$$

$$\Delta W < 0 \quad \text{if} \quad \left| \frac{dS}{dz} \right| > \frac{\rho_{\text{mag}}}{\rho_{\text{th}} \gamma L_p}$$

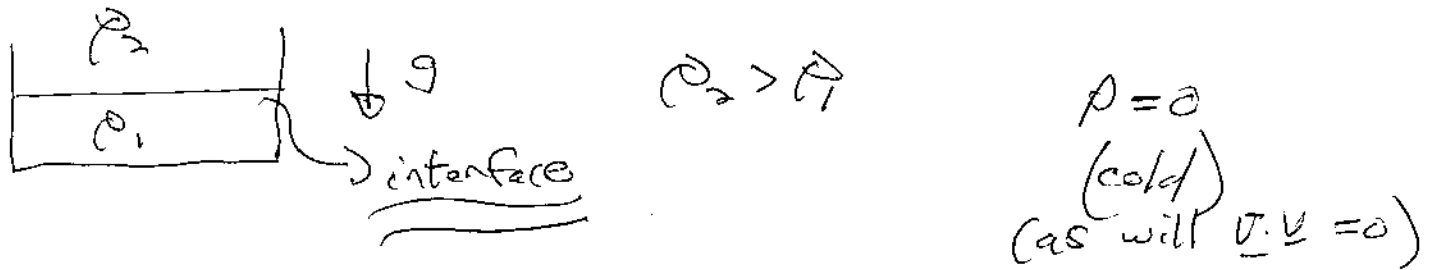
$$\Rightarrow \frac{dS}{dz} < \frac{1}{\gamma \beta} \left(\frac{dP}{dz} \right)$$

$\gamma \beta$ indicates \rightarrow magnetic field stabilizing
 \rightarrow need critical entropy gradient $\sim \frac{1}{\beta L_p}$ for instability.


Moral of the story:

- \rightarrow energy principle recovers essential physical criterion (Schwarzschild)
- \rightarrow enables simple, quick, albeit imprecise insights into more complicated stability problems.

2.) Rayleigh-Taylor Instability \rightarrow critical to implosions (ICF)



\rightarrow while nominally at equilibrium, configuration is unstable (heavy "falls" into light)

\rightarrow  \rightarrow ripples, "spike and bubble"

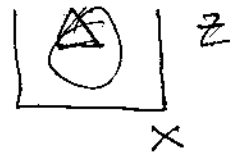
$$\gamma^2 = |kg| \frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}$$

\rightarrow here $\underline{v} \cdot \underline{v} = 0$

\rightarrow if continuous profile $\downarrow g$ $\rho_0 \uparrow$

$$\frac{\partial \underline{\tilde{v}}}{\partial t} = - \frac{\nabla \tilde{p}}{\rho_0} - g \frac{\tilde{\rho}}{\rho_0} \underline{\hat{z}} \quad g > 0$$

$$\frac{\partial}{\partial t} (\nabla \times \underline{\tilde{v}}) \cdot \underline{\hat{y}} = 0 - g \nabla_x \left(\frac{\tilde{\rho}}{\rho_0} \underline{\hat{z}} \right)$$



$$\underline{v} = -\partial_z \phi \underline{\hat{x}} + \partial_x \phi \underline{\hat{z}}$$

$$\nabla \cdot \underline{v} = 0$$

$$-\frac{\partial}{\partial t} \nabla^2 \phi = g \partial_x \left(\frac{\rho}{\rho_0} \right)$$



$$\frac{\partial \rho}{\partial t} = -\partial_x \tilde{\phi} \frac{d\rho_0}{dz} \Rightarrow \omega^2 = -\frac{k_x^2 g}{k^2 L_p}$$

$$\gamma^2 = \frac{k_x^2}{k^2} g/L_p$$

$$g > 0$$

$$1/L_p > 0$$

interchange structure

Now, what would δW say?

$$\delta W = \frac{1}{2} \int d^3x \left[\frac{\mathcal{Q}^2}{4\pi} + \gamma \rho (\nabla \cdot \underline{\mathcal{E}})^2 + \underline{\mathcal{D}} \cdot (\underline{\mathcal{E}} \times \mathcal{Q}) \right.$$

$$\left. + (\underline{\mathcal{E}} \cdot \nabla \rho_0) (\nabla \cdot \underline{\mathcal{E}}) - (\underline{\mathcal{E}} \cdot \nabla \phi) (\nabla \cdot \rho_0 \underline{\mathcal{E}}) \right]$$

$$\underline{\mathcal{Q}} = 0, \underline{j} = 0, \underline{P} = 0, \nabla \cdot \underline{\mathcal{E}} = 0$$

$$\delta W = \int d^3x \left[-(\underline{\mathcal{E}} \cdot \nabla \phi) (\rho_0 \nabla \cdot \underline{\mathcal{E}} + \underline{\mathcal{E}} \cdot \nabla \rho_0) \right]$$

$$= \int d^3x \left[+(\underline{\mathcal{E}} \cdot \underline{g}) (\underline{\mathcal{E}} \cdot \nabla \rho_0) \right]$$

$$\delta W = \int \frac{d^3x}{2} \left[(\underline{\mathcal{E}} \cdot \underline{g}) (\underline{\mathcal{E}} \cdot \nabla \rho_0) \right]$$

$$\delta W = \int \frac{d^3x}{2} [(\underline{\xi} \cdot \underline{g})(\underline{\xi} \cdot \nabla \rho_0)]$$

$g < 0$ so if $\rho_0 > 0$ ($d\rho_0/dz > 0$) anywhere

$\Rightarrow \delta W < 0 \rightarrow$ instability

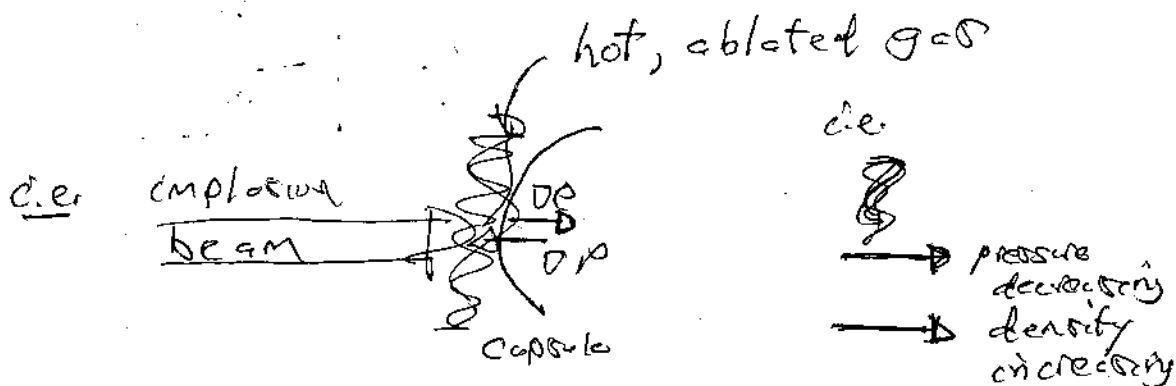
Now, if equilibrium hydrostatic:

$$\nabla p = \rho \underline{g} \quad \Rightarrow$$

$$\delta W = \int \frac{d^3x}{2} [(\underline{\xi} \cdot \nabla p) \left(\frac{\underline{\xi} \cdot \nabla \rho_0}{\rho_0} \right)]$$

\Rightarrow Rayleigh Taylor instability will result whenever $(\nabla p)(\nabla \rho) < 0$

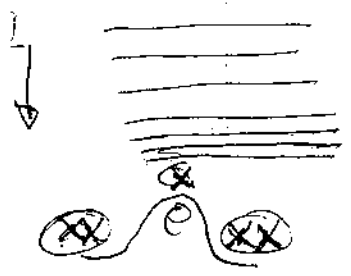
\rightarrow pressure density gradients opposite
 i.e. heavy supported by light (i.e. pressure highest at bottom)



iii) Interchange Instability

(basic confinement consideration)

→ consider plasma confined by magnetic pressure gradient



$$\nabla p = \underline{J} \times \underline{B} + \rho \underline{g}$$

$$\frac{dp}{dz} = -\nabla \left(\frac{B^2}{8\pi} \right) + \frac{\underline{B} \cdot \nabla B}{4\pi} + \rho \underline{g}$$

strat. lines

$\rho \ll 1$

$$-\nabla \left(\frac{B^2}{8\pi} \right) = \rho \underline{g}$$

$\underline{g} = -g \underline{z}$
 $\rho \rightarrow 0$

equilibrium

→

$$\Delta W = \int d^3x \left[\frac{Q^2}{8\pi} + (\underline{Q} \cdot \underline{\epsilon})^2 \gamma_0 + \underline{j}_0 \cdot (\underline{\epsilon} \times \underline{Q}) + (\underline{\epsilon} \cdot \nabla \rho_0)(\underline{Q} \cdot \underline{\epsilon}) - (\underline{\epsilon} \cdot \nabla \phi)(\nabla \cdot (\underline{Q} \cdot \underline{\epsilon})) \right]$$

$\underline{j}_0 = 0$
 $\rho_0 = 0$

$$\Delta W = \int d^3x \left[\frac{Q^2}{8\pi} + (\underline{\epsilon} \cdot \underline{g}) (\underline{\epsilon} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \underline{\epsilon}) \right]$$

Here, must address \mathcal{Q} ,

$$\underline{\mathcal{Q}} = \underline{B}_0 \cdot \underline{\nabla} \underline{\Sigma} - \underline{\Sigma} \cdot \underline{\nabla} \underline{B} - \underline{B}_0 \underline{\nabla} \cdot \underline{\Sigma}$$

Now, can have $\mathcal{Q} = 0$ if:

$$\rightarrow \underline{B}_0 \cdot \underline{\nabla} \underline{\Sigma} = 0 \quad \text{c.e.} \quad \underline{\Sigma} \text{ constant along } \underline{B}_0$$

$$\Rightarrow \kappa_{||} = 0$$

and

$$\rightarrow \underline{\nabla} \cdot \underline{\Sigma} = - \frac{\underline{\Sigma} \cdot \underline{\nabla} \underline{B}_0}{\underline{B}_0}$$

\therefore

$$\begin{aligned} dW &= \int d^3x \left[(\underline{\Sigma} \cdot \underline{g}) \rho_0 \left(\frac{\underline{\Sigma} \cdot \underline{\nabla} \rho_0}{\rho_0} - \frac{\underline{\Sigma} \cdot \underline{\nabla} \underline{B}_0}{\underline{B}_0} \right) \right] \\ &= \int d^3x \left[(\underline{\Sigma} \cdot \underline{g} \underline{B}_0) \underline{\Sigma} \cdot \underline{\nabla} \ln(\rho/B) \right] \end{aligned}$$

$g < 0 \Rightarrow$ if $\nabla \ln(\rho/B) > 0$ anywhere

\therefore instability there \downarrow

Now:

→ obvious parallel to Rayleigh-Taylor is

$$\nabla \rho > 0 \iff \nabla \ln(\rho/B) > 0$$

→ as $k_{||} = 0$, field lines not bent

so can think of instability motion as interchange of flux tubes



Key question: Does interchange lower/raise potential energy?

interchange conserves magnetic flux

$$\Phi_2 = \int B_2 da = B_2 A_2$$

$$\Phi_1 = \int B_1 da = B_1 A_1$$

$$M_2 = \left(\frac{\rho}{B}\right)_2 \Phi_2$$

$$M_1 = \left(\frac{\rho}{B}\right)_1 \Phi_1$$

$M \Rightarrow m/\text{length}$

but $\phi_1 = \phi_2 \Rightarrow$

$$M_2 = \left(\frac{\rho}{B}\right)_2 \Phi$$

so $\Delta M > 0 \Rightarrow \Delta \left(\frac{\rho}{B}\right) > 0$
 $< \quad \quad \quad <$

\Rightarrow if ρ/B increases interchange will liberate gravitational potential energy, d.e.

instability, aka $R-T_i$

\Rightarrow Why care?


- (interchange) instability severely degrades plasma confinement

- curing interchange stability is key element in device design \rightarrow "minimum-B"

(v.) Interchange without Gravity

- in the context of magnetic confinement, "g" is a crutch, to represent

curved field lines

- d.e. 

$$\underline{a} = \frac{v^2}{R_0} \rightarrow \underline{F}_{\text{eff}}$$

[Centrifugal acceleration
on particle]

- Natural to investigate interchanges without
"g" \Rightarrow pressure gradient drive
(expansion free energy)

- Now

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + \gamma \rho (\nabla \cdot \underline{\epsilon})^2 + \underline{\epsilon} \cdot \nabla P (\nabla \cdot \underline{\epsilon}) + \underline{j} \cdot \underline{\epsilon} \times \underline{Q} \right]$$

Now, $\underline{Q} = 0 \rightarrow$ avoid banding, etc.

$$\nabla \times (\underline{\epsilon} \times \underline{B}_0) = 0$$

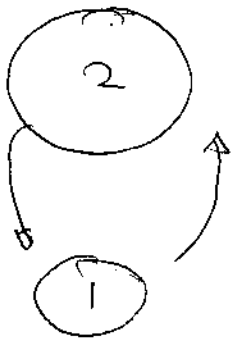
$$\Rightarrow \underline{\epsilon} \times \underline{B}_0 = \nabla \phi$$

\hookrightarrow some scalar potential

and $\underline{B} \cdot \nabla \phi = 0 \Rightarrow \phi$ constant along lines
of force ...

and can formulate dW in terms ϕ , or....

\Rightarrow consider interchange, with flux conservation



$$\bar{\Phi}_1 = \bar{\Phi}_2$$

Does interchange raise or lower energy?

$$\Delta E = [\text{final energy of } \textcircled{1}] - [\text{initial energy of } \textcircled{1}] \\ + [\text{final energy of } \textcircled{2}] - [\text{initial energy of } \textcircled{2}]$$

where interchange

- a) "puts" $\textcircled{1}$ into $\textcircled{2}$ slot
 "puts" $\textcircled{2}$ into $\textcircled{1}$ slot

b) keeps $\rho \rho^{-\gamma} = \rho v^{\gamma} = \text{const.}$

$V \equiv$ volume of flux tube

$$\Rightarrow \text{final energy of } \textcircled{1} \rightarrow (\text{new } \rho) V_2 / (\gamma - 1)$$

$$\text{final energy of } \textcircled{2} \rightarrow (\text{new } \rho) V_1 / \gamma - 1$$

so ...

$$\Delta E = \Delta W = \frac{1}{(\gamma-1)} \left[(\rho'_1 V_2 - \rho_1 V_1) + (\rho'_2 V_1 - \rho_2 V_2) \right]$$

$$\text{and } \left. \begin{aligned} \rho'_1 V_2 \delta &= \rho_1 V_1 \delta \\ \rho'_2 V_1 \delta &= \rho_2 V_2 \delta \end{aligned} \right\}$$

from eqn. state

 $\rho' \equiv$ pressures of displaced flux tubes

(argument akin to Schwarzschild)

 \Rightarrow

$$(\gamma-1) \Delta W = \left\{ \rho_1 \left[\left(\frac{V_1}{V_2} \right)^\gamma V_2 - V_1 \right] + \rho_2 \left[\left(\frac{V_2}{V_1} \right)^\gamma V_1 - V_2 \right] \right\}$$

$$V_2 = V_1 + \delta V$$

$$\rho_2 = \rho_1 + \delta \rho$$

$$(\Delta W)(\gamma-1) = \left\{ \rho_1 \left[\left(\frac{V_1}{V_1 + \delta V} \right)^\gamma (V_1 + \delta V) - V_1 \right] + (\rho_1 + \delta \rho) \left[\left(\frac{V_1 + \delta V}{V_1} \right)^\gamma V_1 - (V_1 + \delta V) \right] \right\}$$

$$\begin{aligned}
(\gamma-1) \Delta W &= \left\{ P_1 V_1 \left[\left(1 + \frac{\partial V}{V}\right)^{-(\gamma-1)} - 1 \right] \right. \\
&\quad \left. + P_1 V_1 \left(1 + \frac{\partial P}{P}\right) \left[\left(1 + \frac{\partial V}{V}\right)^\gamma - \left(1 + \frac{\partial V}{V}\right) \right] \right\} \\
&= P_1 V_1 \left\{ \left[\cancel{1} - (\gamma-1) \frac{\partial V}{V} + \frac{(\gamma-1)(\gamma)}{2} \left(\frac{\partial V}{V}\right)^2 \cancel{-1} \right] \right. \\
&\quad \left. + \left(1 + \frac{\partial P}{P}\right) \left[\cancel{1} + \gamma \frac{\partial V}{V} + \gamma \frac{(\gamma-1)}{2} \left(\frac{\partial V}{V}\right)^2 \cancel{-1} - \frac{\partial V}{V} \right] \right\} \\
&= P_1 V_1 \left\{ \cancel{-(\gamma-1) \frac{\partial V}{V}} + \frac{(\gamma-1)(\gamma)}{2} \left(\frac{\partial V}{V}\right)^2 \right. \\
&\quad \left. + \cancel{\gamma \frac{\partial V}{V}} - \cancel{\frac{\partial V}{V}} + \frac{\partial P}{P} (\gamma-1) \frac{\partial V}{V} + \gamma \frac{(\gamma-1)}{2} \left(\frac{\partial V}{V}\right)^2 \right\}
\end{aligned}$$

$$\boxed{\frac{\Delta W}{P_1 V_1} = \gamma \left(\frac{\partial V}{V}\right)^2 + \frac{\partial P}{P} \frac{\partial V}{V}}$$

> 0

> 0 or < 0

\Rightarrow generic expression for interchange dW

clearly,

$$\frac{\Delta V}{V} \sim (\underline{v} \cdot \underline{\epsilon}) \quad , \quad \frac{\Delta p}{p} \sim \underline{\epsilon} \cdot \underline{v} \underline{p}$$

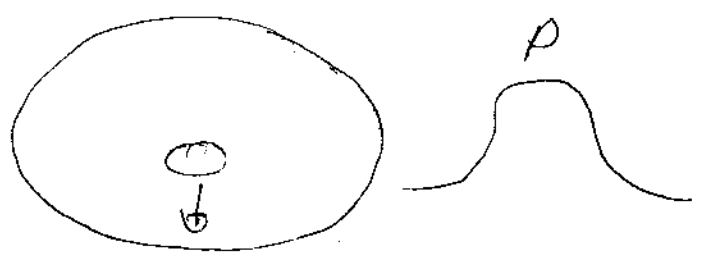
and

→ expansion free energy relaxation ⇒

$$\underline{\Delta P < 0}$$

→ c.e.

pressure higher in center, so occurs



$\Delta p < 0 \Rightarrow$ relaxation

∴ key is sign $\frac{\Delta V}{V}$

- $> 0 \rightarrow$ instability
- $< 0 \rightarrow$ stability

→ Now, for flute perturbation ($k_{||} = 0$)

$$V = \int S dl$$

$S \equiv$ cross-sectional area of tube.

$$\text{but } \Phi = B(l) S(l) = \text{const}$$

⇒

$$V = \Phi \int \frac{dl}{B} \quad \Rightarrow \quad \frac{\delta V}{V} < 0$$

⇒

$$\delta \int \frac{dl}{B} < 0$$

→ condition for interchange stability

$$\delta p > 0 \\ \delta \rho < 0$$

→ content of criterion is that configuration should have a minimum in B in the core, to confine pressure

c.e.



then stable if:

$$\delta \int \frac{dl}{B} < 0$$

⇒ "minimum B " criterion for stability.

→ if define $\psi \rightarrow$ label of surface enclosing const flux Φ



∴ $V(\psi) \equiv$ volume enclosed by flux surface

$p(\psi) \equiv$ pressure enclosed

$$dp/d\psi < 0 \quad \Rightarrow \quad \text{need} \quad \frac{d^2V}{d\psi^2} > 0$$

\leftrightarrow minimum-B

→ can re-write instability criterion

$$\begin{aligned} \delta W &= p_1 \delta V \left(\gamma \frac{\delta V}{V_1} + \frac{\delta p}{p_1} \right) \\ &= p_1 \delta V \left[\delta \ln(pV^\gamma) \right] \end{aligned}$$

so $\delta(pV^\gamma) < 0 \rightarrow$ inst. (akin Schwarzschild)

Also, if tube \odot has flux ψ , then

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$$v = u \psi$$

\Rightarrow

$$\frac{dW}{\psi} = \rho \delta u \frac{d(\rho u^{\delta})}{\rho u^{\delta}} < 0$$

→ What does it Mean?

$$V = \int d\ell A = \oint \frac{d\ell}{B}$$

⏟
volume

now $\nabla P \Rightarrow$ "expansion free energy"

$$dV > 0 \Rightarrow d \int \frac{d\ell}{B} > 0 \rightarrow \text{fluid element expands}$$

\Rightarrow tends reduce W_p

$$dV < 0 \Rightarrow d \int \frac{d\ell}{B} < 0 \rightarrow \text{fluid element compresses}$$

\Rightarrow tends increase W_p

$$dV > 0 \rightarrow \text{'maximum } B \text{'}$$

$$dV < 0 \rightarrow \text{'minimum } B \text{'}$$

Can then define:

$$E_p = -p U, \quad U = -\oint \frac{d\ell}{B}$$

⏟
potential energy of tube

(i.e. for sign convention)

∴ can argue tube tends to move in direction of lower U .

→ equilibrium for $p = p(U)$

then, not surprisingly, can develop parallel between convection and interchange

i.e.

Convection	Interchange
gravitational potential energy	$E_p \rightarrow$ expansion energy
blob	flux tube
displace blob	displace tube
$\rho' < \rho_{\text{ambient}}$ \Rightarrow buoyant rise	$\frac{dV}{V} > 0$ \Rightarrow expansion continues (with $\frac{d\rho}{d\mu} < 0$) (squeezed out)
adiabatic profile $\frac{d\rho}{\rho} = \gamma \frac{d\rho}{\rho}$	adiabatic displacement $-\gamma \rho \frac{d\mu}{\mu} = \frac{d\rho}{d\mu} d\mu$
Schwarzschild Criterion $\frac{d\rho}{\rho} < \gamma \frac{d\rho}{\rho}$ \Rightarrow instability	Interchange Criterion $\frac{d\rho}{d\mu} > -\gamma \rho \frac{d\mu}{\mu}$ \Rightarrow $\left \frac{d\rho}{d\mu} \right > \frac{-\gamma \rho}{\mu}$ $\frac{d\rho}{d\mu}$ sufficiently steep \rightarrow overcome comp.

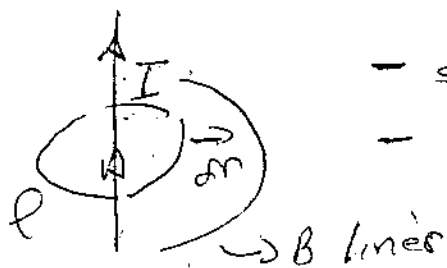
∴ for instability: $\left| \frac{dP}{du} \right| > \frac{-\gamma P}{u}$
 \int change from relaxation \rightarrow adiabatic pressure change

for stability, need:

$$\left| \frac{dP}{du} \right| < \frac{\gamma P}{|u|}$$

\rightarrow Consider some configurations (magnetic)

a) single wire



- stability to displacement δr
 - $\delta P / \rho$ limit \int

now $\oint \frac{dl}{B}$

$$dl = 2\pi r$$

$$B = 2I/r$$

$$dl/B \sim \frac{\pi r^2}{I}$$

\rightarrow wire is not "minimum" $= B$
 i.e. actually maximal
 \rightarrow will have a OPcrit.

for op limit: $\frac{dP}{dU} < \frac{\gamma P}{|U|}$

$$U \equiv -\int \frac{dP}{B} \sim -r^2$$

$$\frac{dP}{dU} = \frac{dP}{dr} \frac{dr}{dU}$$

U scalar \Rightarrow
I cancels

$$= \left| \frac{dP}{dr} \right| \left(\frac{1}{2r} \right)$$

$$\Rightarrow \left| \frac{1}{P} \frac{dP}{dr} \right| < \frac{\gamma(2r)}{r^2}$$

$$\therefore \left| \frac{1}{P} \frac{dP}{dr} \right| < \frac{2\gamma}{r}$$

$$\Rightarrow \left| \frac{d \ln P}{d \ln r} \right| < 2\gamma$$

\rightarrow imposes limit on pressure gradient for interchange stability. \Rightarrow "B limits"

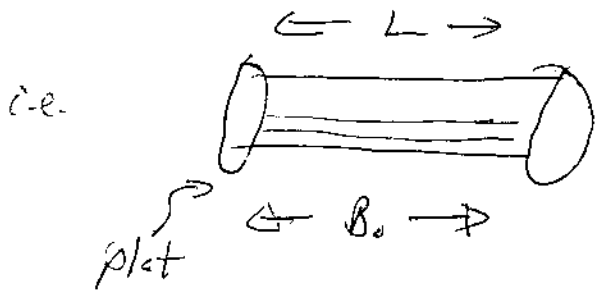
b) can approach point dipole similarly \rightarrow der earth.
i.e. $B \sim 1/r^3$
 $dL \sim r$ $\Rightarrow U \sim -r^4$

similar reasoning $\Rightarrow -\frac{d \ln P}{d \ln r} < 4\gamma$

→ Line Tying and Conducting End Plates

- Till now, have ignored boundary

→ consider plasma between two conducting end plates

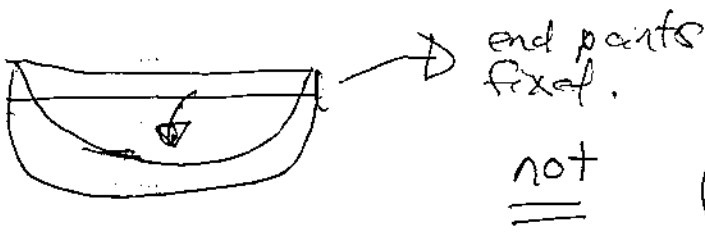


$$\underline{E}_f = 0 \text{ on plate}$$

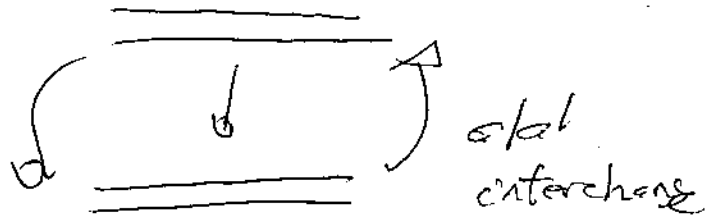
$$\Rightarrow \Sigma \Big|_{\text{plate}} = 0$$

lines are fixed

i.e. displacement has form:



not



⇒ field lines bent ↓

$$\omega, \underline{U}_0 = 0 \Rightarrow$$

$$\delta W = \int d^3x \left[\frac{Q^2}{8\pi} + \gamma \rho (\nabla \cdot \underline{\underline{\epsilon}})^2 + (\underline{\underline{\epsilon}} \cdot \nabla \rho_0) (\nabla \cdot \underline{\underline{\epsilon}}) \right]$$

$$\underline{\underline{Q}} = \nabla \times \underline{\underline{\epsilon}} \times \underline{\underline{B}}$$

$$= \underline{\underline{B}}_0 \cdot \nabla \underline{\underline{\epsilon}} - \underline{\underline{\epsilon}} \cdot \nabla \underline{\underline{B}}_0 - \underline{\underline{B}}_0 \nabla \cdot \underline{\underline{\epsilon}}$$

$$\nabla \cdot \underline{\underline{\epsilon}} \neq 0 \quad \text{new stabilizing effect}$$

$$\delta W = \int d^3x \left[\frac{(\underline{\underline{B}}_0 \cdot \nabla \underline{\underline{\epsilon}} - \underline{\underline{B}}_0 \nabla \cdot \underline{\underline{\epsilon}})^2}{8\pi} + \gamma \rho (\nabla \cdot \underline{\underline{\epsilon}})^2 + (\underline{\underline{\epsilon}} \cdot \nabla \rho_0) \nabla \cdot \underline{\underline{\epsilon}} \right]$$

i.e. cut take $\underline{\underline{B}}_0 \cdot \nabla \underline{\underline{\epsilon}} = 0$, any more

so $Q \sim B_0 \frac{\partial \epsilon_n}{\partial z}$
new

i.e. can make $(\nabla \cdot \underline{\underline{\epsilon}}) B_0$
smaller -- $\frac{\partial \cdot \underline{\underline{v}}}{\partial z} \approx 0$

$$\delta W \sim \int \left[\frac{B_0^2}{8\pi} \left(\frac{\partial \epsilon_n}{\partial z} \right)^2 + \underbrace{\gamma \rho \left(\frac{\partial u}{\partial z} \right)^2 + \rho \frac{\partial u}{\partial z}}_{\text{old}} \right]$$

i.e. schematic ...

$$\frac{\partial \epsilon_n}{\partial z} \sim \frac{\epsilon_n}{L}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial z} \epsilon_n$$

$$\delta p = \nabla p \epsilon_n$$

$$\Rightarrow \delta W \sim V \left(\frac{B_0^2}{8\pi L^2} + \delta \rho \left(\frac{\nabla \psi}{u} \right)^2 + \frac{\nabla \rho \nabla \psi}{u} \right) \epsilon^2$$

$\therefore \delta W < 0 \rightarrow$ instability \Rightarrow

$$\text{instability if } -\frac{\nabla \rho \nabla \psi}{u} < \delta \rho \left(\frac{\nabla \psi}{u} \right)^2 + \frac{B^2}{8\pi L^2}$$

\Rightarrow line tying raises
critical pressure gradient

\int
additional
stabilizing
effect

\Rightarrow clearly stabilizing \Rightarrow β limit!

Physics \rightarrow fixing end points forces
bending of field lines

\rightarrow loss : interchange structure

\rightarrow energy expended coupling to
plucking magnetic field lines.