Openers:

→ What is $(\nabla\cdot\underline{v} = 0)$ MHD ?

→  disk, rotating      When slow down.

1 fluid model



## Basics of MHD

{N.B.: Read Kulsrud, Chapt. 3, 4}

{1 fluid, Large scale, slow}

→ MHD Equations → Eulerian Fluid

① $$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0}$$ (Continuity)

→ Lorentz, $\underline{E}$ ?

② $$\boxed{\rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla P + \frac{\underline{J}\times\underline{B}}{c} + \underline{f}_{body}}$$ (momentum balance)

[frequently $\underline{f}_{body} = \rho\,\underline{g}$]

③ $$\boxed{\begin{aligned}\frac{d}{dt}S &= \frac{\partial S}{\partial t} + \underline{v}\cdot\nabla S = 0\\ S &= C_v \ln(P/\rho^{\gamma})\end{aligned}}$$

$\uparrow$ entropy

{eqn of state more general} (isentropic fluid)

[frequent form of equation of state]

④ $$\boxed{\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \left(\eta\underline{J},\ \frac{\underline{J}}{\sigma}\right)}$$ (Ohms Law)

[resistivity $\eta$ is usually most significant dissipation]

Ideal MHD

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0$$

and

$$\boxed{\begin{array}{l} (5)\quad \underline{\nabla}\cdot\underline{B} = 0 \\ (6)\quad \underline{\nabla}\times\underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \\ (7)\quad \underline{\nabla}\times\underline{B} = \frac{4\pi}{c}\underline{J} \end{array}}$$

} from Maxwell's EQNS. neglecting displacement current

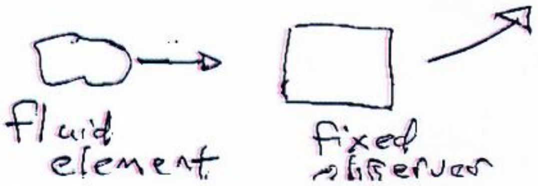
→ Meaning, Restriction, Validity

— MHD is simplest, closed, self-consistent plasma model, and the most heavily exploited for dynamical modelling.

— variants : Reduced MHD → strong $\underline{B}_0$ (tokamaks)
2D MHD
E MHD → stationary ions (ICF)
FLR MHD, Reduced Braginskiy } → MHD + additional effects (MFE, space)
hybrid → { bulk – MHD ; hot species – kinetic (i.e. α's energetics) }
⋮

— MHD – Eulerian

"fluid element" ⇒ "glue"




here "glue" → collisions

applies $L > \lambda_{mfp}$

collisions on $\tau_{phen} < \tau$ fall apart

$(\omega > kv)$

Why MHD often works at low collisionality

→ MHD is:
- 1 fluid - electrons and ions
- strongly collisional
- low frequency
- large scale

i.e. frequencies relevant:

$$\omega \ll \Omega_{e,i};\ \omega_{p_{e,i}};\ \nu_{ee_{e,i}};\ \nu_{ii};\ \omega_{*e,i}$$

etc.

scales relevant:

$$L \gg \lambda_{D_{e,i}},\ \rho_{e,i},\ c/\omega_{p_{e,i}},\ l_{mfp_{e,i}}$$

$$l_{mfp} < L$$

and

collisions isotropise, equilibrate $\underline{\underline{P}}$.

$$\left( \text{i.e.}\quad \underline{\underline{P}} \sim \int d^3v\ \tilde{v}_i \tilde{v}_j\ f(\underline{x}, \underline{v}, t) \right)$$

→ Some Specific Points:

- re: continuity ①;

$$\rho = m_i n_i + m_e n_e$$

i.e. (ions control fluid inertia)

$$\begin{cases} \text{total density} \\ \text{ion dominated} \end{cases}$$

- re: momentum balance ② ;

$\rightarrow \underline{V} = \left( \int d^3v_i \, m_i \underline{v}_i f_i + \int d^3v_e \, m_e \underline{v}_e f_e \right) / \rho$

i.e. ( ions control flow $- \rho \frac{d\underline{V}}{dt}$ )

$\rightarrow$ where has $\underline{E}$ gone? $\rightarrow$ $L \gg \lambda_D$ $\rightarrow$ quasi-neutrality

fluid EQ

is it consistent?

$$\rho_i \frac{d\underline{v}_i}{dt} = n_i q_i \underline{E} + n_i q_i \frac{\underline{v}_i \times \underline{B}}{c} + \cdots$$

$$\rho_e \frac{d\underline{v}_e}{dt} = -n_e q_i \underline{E} - n_e q_i \frac{\underline{v}_e \times \underline{B}}{c} + \cdots$$

if add: $\rightarrow 0$ cancel! $n_i = n_e$ (quasi-neutrality) $\rightarrow$ $\frac{\underline{J} \times \underline{B}}{c}$ (Lorentz force term in momentum balance)

note also: $\rho_i, \rho_e \rightarrow \rho$

$\rightarrow$ re-writing the $\underline{J} \times \underline{B}$ force:

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\nabla \times \underline{B}) \times \underline{B}}{4\pi} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$$

so can write:

$$\rho \frac{d\underline{v}}{dt} = \underbrace{-\underline{\nabla}\left(P + \frac{B^2}{8\pi}\right)}_{\substack{\text{magnetic pressure} \\ \text{(field energy density)}}} + \underbrace{\frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}}_{\text{magnetic tension}}$$

$P_{tot}$ tension

a) What / Why "Magnetic Tension"?

$\underline{B} = B\,\hat{b}$ $\qquad B = |\underline{B}|\,,\ \hat{b} = \underline{B}/B$

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = B\,\hat{b}\cdot\underline{\nabla}\,(B\hat{b})$$

$$= B^2\,\underset{①}{\hat{b}\cdot\underline{\nabla}\hat{b}} + \hat{b}\,\underset{②}{\hat{b}\cdot\underline{\nabla}}\left(\frac{B^2}{2}\right)$$

$\boxed{\hat{b}\cdot\underline{\nabla}\,\hat{b}} \rightarrow$ curvature of $\hat{b}$
(i.e. rate of change of $\hat{b}$ along itself)

$\equiv d\hat{b}/ds$

n.b. in general: curve: $\underline{x}(t)$

tangent: $\underline{T} = d\underline{x}/ds$

$(ds^2 = d\underline{x}\cdot d\underline{x})$ $\qquad s \equiv$ distance along curve curvature $\quad \underline{\kappa} = \frac{d\underline{T}}{ds} = \frac{d\underline{T}/dt}{ds/dt} = \frac{\dot{\underline{T}}}{|\underline{V}|}$

Now: $\underline{\kappa} = \hat{\underline{b}} \cdot \nabla \hat{b}$ → points in direction of turning of $\hat{b}$, orthogonal to $\hat{b}$

i.e.


$\underline{\kappa} = d\underline{T}/ds$

∴ $\underline{\kappa} = -\frac{\hat{n}}{R_c}$ $\quad R_c \equiv$ radius of curvature

as curved field line suggests "tension" → "magnetic tension".

b.) What about ② ?

But $\underline{J} \times \underline{B} \perp \underline{B}$ yet $\nabla \left(\frac{B^2}{8\pi}\right)$ can have component along $\underline{B}$ ??

→ recombining total $\underline{J} \times \underline{B}$ gives:

$$-\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\hat{b}\cdot\nabla\hat{b} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)$$

$$= -\nabla_\perp\left(\frac{B^2}{8\pi}\right) - \hat{b}\hat{b}\cdot\nabla\cancel{\left(\frac{B^2}{8\pi}\right)} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\cancel{\left(\frac{B^2}{8\pi}\right)} + \frac{B^2}{4\pi}\hat{b}\cdot\nabla\hat{b}$$

$$\therefore \boxed{\frac{\underline{J}\times\underline{B}}{c} = -\underline{\nabla}_\perp\left(\frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\hat{b}\cdot\nabla\hat{b}}$$

(3) $dE = đQ - p\,dV$ (Thermo)

$$\begin{cases} đQ = T\,dS \\ dE = C_v\,dT \end{cases}$$
(normalized)

$C_v\,dT = T\,dS - p\,dV$

$V = 1/\rho \qquad dV = -d\rho/\rho^2$

$$C_v\frac{dT}{T} = dS + \frac{d\rho}{\rho}$$

$$\Rightarrow \ln T = \frac{S}{C_v} + \ln \rho^{1/C_v}$$

$$\therefore \boxed{S = C_v \ln\left(T/\rho^{1/C_v}\right)}$$

$p = \rho T$

$$\Rightarrow S = c_v \ln\left(P/\rho^{(c_v+1)/c_v}\right)$$
$$= c_v \ln\left(P/\rho^{\gamma}\right)$$

$\gamma = 5/3$, ideal gas

$\left(c_v = 3/2 \text{ (normalized)}\right)$

$$\frac{dS}{dt} = 0 \Rightarrow \frac{d}{dt}\left(P/\rho^{\gamma}\right) = 0$$

i.e.

$$\boxed{\frac{\partial}{\partial t}\left(P/\rho^{\gamma}\right) + \underline{v}\cdot\underline{\nabla}\left(P/\rho^{\gamma}\right) = 0}$$

eqn. of state

perfect $\begin{cases}\text{homogeneity} \\ \text{stationarity}\end{cases}$ $\left(P/\rho^{\gamma} = \text{const}\right)$

– "adiabatic equation of state"

④ Ohm's Law – most sensitive part of MHD (since controlled by electrons)

MHD variants differ primarily in Ohm's Law
- Hall MHD → Hall term
- EMHD → electron inertia
- Braginsky / drift MHD → $\nabla P$ terms

⋮ etc., etc.

Ohm's Law → subtract moments on electron equations → electrons $\left( \underline{J} = n e (\underline{V}_i - \underline{V}_e) \right)$

Simple resistive MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \frac{\eta}{\varsigma} \underline{J}$$

$\sim \nu_{ei}$ → momentum transfer to ions ...

ideal MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0$$ → field "frozen into" fluid

(5), (6), (7): Only 1 approximation:

$$\underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t}$$

Why drop displacement

$|\partial \underline{E}/\partial t| << |\underline{J}|$ → condition on $\omega$ !?

$$\rightarrow \quad \omega \frac{v B}{c} \overset{?}{<<} \frac{k}{c^{-1}} B$$

$$\Rightarrow |\underline{v}| (\omega/k)/c^2 << 1$$ is condition on $\omega$.

→ Skeptic: "Does it Hang Together"?

i.e. is electric force negligible? consistency

$$\rho \frac{d\underline{v}}{dt} = n q \underline{E} + \ldots\ldots$$

and $q \neq 0$, as
$$n q = \frac{\nabla \cdot \underline{E}}{4\pi}$$
$$\underline{E} = -\frac{\underline{v} \times \underline{B}}{c}$$

so

$$n q \underline{E} = +\frac{(\underline{v} \times \underline{B})}{c} \nabla \cdot \left( \frac{\underline{v} \times \underline{B}}{c} \right) \neq 0 \quad !$$

but
$$\sim \frac{v^2}{c^2} B^2 k$$

$$\sim \frac{v^2}{c^2} (\underline{J} \times \underline{B}) \quad \rightarrow \text{negligible if } v^2/c^2 \ll 1 .$$

Thus, yes indeed it does!

→ Putting it together:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J} \quad , \quad \nabla \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

⇒ the induction equation for $\underline{B}$ evolution ...

$$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}}$$ Induction eqn.

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) — $\underline{V}(\underline{x}, t)$, $\underline{B}(\underline{x}, t)$ evolving simultaneously

↓

- useful and instructive to re-write induction equation

$$\nabla \times \underline{V} \times \underline{B} = -\underline{V} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

so $$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}.$$

This brings us to .......

→ What Does "MHD", as a system, really mean ....?

this is answered most clearly for the case of incompressible MHD ---

$\underline{\nabla} \cdot \underline{V} = 0$ → defines equation of state
→ sets $P_{total}$ field

$\left( \omega/k \ll \underset{\text{sound}}{C_s}, \underset{\text{magnetosonic}}{V_{MS}} \right)$

$$\nabla \cdot \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{V} = -\frac{\underline{\nabla}}{\rho} \left( P + \frac{B^2}{8\pi} \right) + \frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi\rho} \right\}$$

$$\frac{d\rho}{dt} = -\rho \, \underline{\nabla} \cdot \underline{V} = 0$$

so $\rho$ → constant $\rho_0$ (can relax to slow variation)

$$\nabla^2 \left[ \underbrace{\left( P + \frac{\underline{B}^2}{8\pi} \right)}_{\text{total pressure}} \Big/ \rho_0 \right] = \underline{\nabla} \cdot \left( \frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi \rho_0} - \underline{V} \cdot \underline{\nabla} \underline{V} \right)$$

a la' Poisson's equation :

$$\frac{P + B^2}{8\pi} = -\int \frac{d^3x'}{4\pi |x - x'|} \left\{ \underline{\nabla}' \cdot \left( \frac{\underline{B} \cdot \underline{\nabla}' \underline{B}}{4\pi \rho_0} - \underline{V} \cdot \underline{\nabla}' \underline{V} \right) \right\}$$

solves for: $P_{tot}$ field → eliminates eqn. state.

Basic MHD $\nabla\cdot\underline{V}=0$ $\qquad p^* = p_{tot}$

$$\boxed{\begin{aligned} &\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\nabla\left(\frac{p^*}{\rho_0}\right) + \frac{\underline{B}\cdot\nabla\underline{B}}{4\pi\rho_0} \\ &\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\nabla\underline{B} - \eta\nabla^2\underline{B} = \underline{B}\cdot\nabla\underline{V} \end{aligned}}$$

with $\nabla\cdot\underline{V}=0$, constitute equations of incompressible MHD.

→ Rather clearly, this system is one of two, dynamically coupled, evolving vector fields $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$.

→ Compressible MHD is really a problem in 3 fields, two of which are vectors

i.e.
$$\begin{cases} \underline{V}(\underline{x},t) & \to \text{fluid velocity} \\ \underline{B}(\underline{x},t) & \to \text{magnetic field} \\ S(\underline{x},t) & \to \text{entropy} \Rightarrow \text{energy density} \end{cases}$$

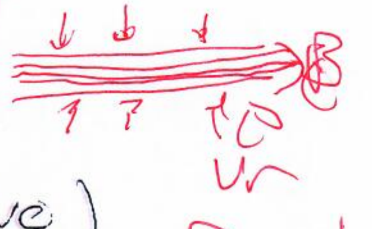
i.e. scalar equation of state provides 3rd field.

→ Key Question: How closely coupled are $\underline{V}$, $\underline{B}$ ?

⇒ the key physics element in MHD ----

⇒ $\boxed{\text{Frozen-in Law, Flux Freezing}}$

2 versions
→ local
→ integrated




① Frozen-in Law

- consider a (for the moment, passive) vector field:
  - frozen into flow $\underline{V}(\underline{x}, t)$
  - consisting of oriented, flexible strands


{ i.e. massless rubber strands in flow }

How does $\underline{\Delta l}$ evolve?

in $dt$, $d(\underline{\Delta l}) = \left(\underline{V}(\underline{l}_0 + \underline{\Delta l}) - \underline{V}(\underline{l}_0)\right) dt$

$$= \underline{\Delta l} \cdot \nabla \underline{V} \; dt$$

$$\therefore \quad \frac{d(\underline{\Delta l})}{dt} = \underline{\Delta l} \cdot \nabla \underline{V}$$

$$\frac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell}\cdot \underline{\nabla V}$$

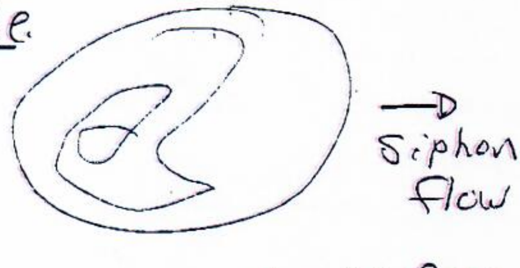
i.e. $$\frac{d}{dt}\underline{\Delta \ell} = \underline{\Delta \ell}\cdot \underline{\underline{S}}$$

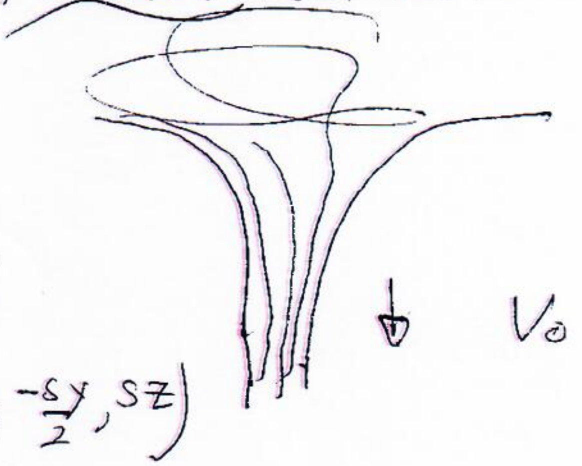
$$\begin{cases} \frac{d}{dt}(\Delta \ell)_i = \Delta \ell_j \cdot S_{ij} \\ S_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \rightarrow \text{strain rate tensor} \end{cases}$$

says that → $\underline{\Delta \ell}$ strands orient along strain

→ strain extends strands ....

i.e.




$$\underline{V} = V_0\left(-\frac{Sx}{2}, -\frac{Sy}{2}, Sz\right)$$

plausible to say that $\underline{\Delta \ell}$ "frozen onto" the flow.

Now, if $\eta \rightarrow 0$, .... in MHD

$$\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V}$$

$$-\underline{\nabla}\cdot\underline{V} = +\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\frac{\partial}{\partial t}\underline{B} + \underline{V}\cdot\underline{\nabla}\,\underline{B} = \frac{d\underline{B}}{dt} = \underline{B}\cdot\underline{\nabla}\,\underline{V} + \frac{\underline{B}}{\rho}\frac{d\rho}{dt}$$

$$\frac{1}{\rho}\frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2}\frac{d\rho}{dt} = \frac{\underline{B}}{\rho}\cdot\nabla V$$

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}\,\underline{V}}$$

→ $\underline{B}/\rho$ obeys same equation as $\underline{\Delta l}$!

→ $\boxed{\underline{B}/\rho \text{ is frozen into flow field } \underline{V}(\underline{x},t)}$

Note: → $\underline{B}/\rho$ is **not** passive → due $\underline{J}\times\underline{B}$ force

→ $\underline{B}$ determines flow, while frozen into it!

→ (essence of coupling problem)

- For $\underline{\nabla}\cdot\underline{V} = 0$, $\underline{B}$ frozen in

→ if $\eta \neq 0$, freezing in is broken ----

i.e. $\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) - \frac{\eta}{\rho}\nabla^2\underline{B} = \frac{\underline{B}}{\rho}\cdot\underline{\nabla}\underline{V}$

form of frozen evolution broken

Vorticity connection

→ Observe: → this motivates attention to resistivity in MHD above other dissipations $\nu$, $\chi$, etc..

→ $\eta$ → $\underline{B}$ diffusion $\sim \eta \nabla^2$

∴ decoupling of $\underline{V}$, $\underline{B}$ occurring on small scales

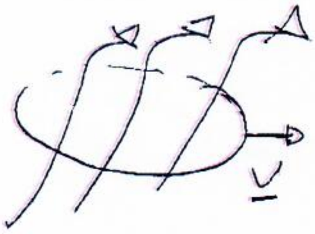
⇒ motivates 'magnetic reconnection' as study of singularity dynamics in MHD.

→ A Word to the Wise: In modelling, describing complex dynamics in MHD (i.e. MHD turbulence, dynamos, etc.) always think carefully about frozen-in law ...

What is frozen in for other systems?

→ Closely Related: Flux Freezing

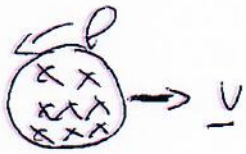
- consider flux thru surface in flow



→ i.e. imaginary loop drawn in flow field...

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

$$\underline{\Phi} = \int \underline{B} \cdot d\underline{S}$$



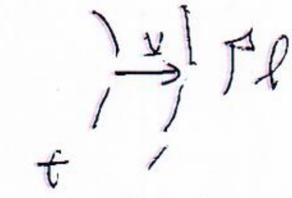
① change in $\underline{B}$ ② change in $d\underline{S}$

$$\frac{d\Phi}{dt} = \int d\underline{S} \cdot \frac{\partial \underline{B}}{\partial t} + \int \frac{d\underline{S}}{dt} \cdot \underline{B}$$

change in $\underline{B}$ motion of loop...

$$① = \int d\underline{S} \cdot \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$= \oint d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$



$$dS = \underline{V} \Delta t \times d\underline{\ell}$$

For ②

so $d\underline{S} = \underline{V} dt \times d\underline{\ell}$

→ change in $\underline{S}$ in $dt$.

$dt\,(d\Phi②/dt)$

$$② dt = \int (\underline{V} dt \times d\underline{\ell}) \cdot \underline{B} = d\Phi$$

$$\left.\frac{d\Phi}{dt}\right)_② = \int (\underline{V} \times d\underline{\ell}) \cdot B = -\int d\underline{\ell} \cdot (\underline{V} \times \underline{B})$$

so $$\boxed{\frac{d\Phi}{dt} = ① + ② = 0}$$

so $\Rightarrow$ magnetic flux invariant $\Leftrightarrow$ cancellation

$\Rightarrow$ in absence of resistivity, flux thru surface in flow is invariant, or frozen in

$\Rightarrow$ no surprise: $\underline{B}$ frozen in $\Rightarrow$ $\Phi$ frozen in

$\Rightarrow$ analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v} \cdot d\underline{l} = \int d\underline{a} \cdot \underline{\omega} \qquad \omega = \underline{\nabla} \times \underline{v}$$

In inviscid hydro, $(\nu \rightarrow 0)$ circulation $\Gamma_c$ is conserved.

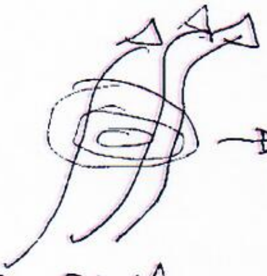
Exercise : Prove this!
Note relation between $\underline{\omega}$ equation and $\underline{B}$ eqn. Assume $\rho = \text{const}$, $\underline{g} = 0$

Extra Credit: ① Discuss the extension to the case where $\rho \neq \text{const}$.
② what is 'frozen in' for Vlasov plasma?

→ What Does "Freezing" Mean?

⇒ can relate field evolution in a flow to density evolution, since $\underline{B}/\rho$ is "frozen in".

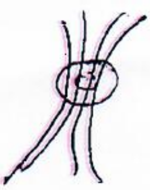
Application:



→ proto-stellar cloud

Star formation

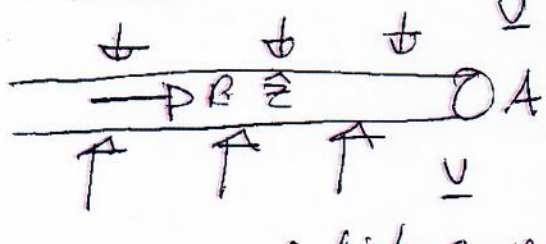
$\underline{B}$-field →



(?)

Simple Cases → How does $\underline{B}$ change in a flow?

(i)




radial compression

Cylinder of cross-sectional area A squeezed in radial flow. $\underline{B} \equiv B\hat{z}$

2 ways:

$$\frac{d}{dt}(B\hat{z}/\rho) = \frac{B\hat{z}\cdot\nabla\underline{V}}{\rho}$$

$$= 0$$

$\underline{V} = v\hat{r}$

$\Rightarrow \underline{V} \perp \underline{B}$

so $\underline{B}/\rho$ = const

Now: $\rho A L$ = const so $\underline{B \sim A^{-1}}$
$\rho \sim A^{-1}$
(L const.)

or

Flux Frozen: $BA = \Phi$ = const.
$\rho AL$ = const $= M$
$L$ const.

$BA \sim \Phi_0$, $B \sim A^{-1}$
$\rho A \sim M_0$, $\rho \sim A^{-1}$
so $B \sim \rho^{(1)} \Rightarrow B/\rho \sim$ const!

ii) $V = V(z)\hat{z}$ — compressible!
$\longrightarrow B\hat{z}$ i.e. stretch, 1D

here $\frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V} \neq 0$, but easier to work with $\underline{B}$ than $\underline{B}/\rho$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{v} - \underline{B}\, \underline{\nabla} \cdot \underline{v}$$

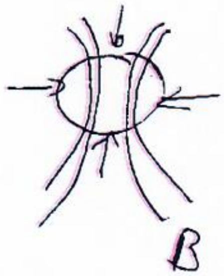
$$= B \frac{\partial}{\partial z} V(z) - B \frac{\partial V(z)}{\partial z}$$

$$= 0 \ !$$

For $\rho$, $\quad \dfrac{d\rho}{dt} = -\rho\, \underline{\nabla}\cdot\underline{V} = -\rho \dfrac{\partial V_z}{\partial z}$

here $\underline{B}$ invariant, $\rho$ changes $\qquad \dfrac{d}{dt}\dfrac{\underline{B}}{\rho} = \dfrac{\underline{B}}{\rho}\cdot\nabla\underline{V}$

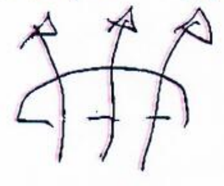
i.e. $\quad B \sim \rho^{(0)}$ $\qquad$ Freezing in $\not\Rightarrow$ const

iii.) 


collapsing sphere: $\underline{V} = V\hat{r}$

(i.e. $\Phi = 0$, total sphere)

consider hemispherical surface (i.e. mushroom cap)



$$\Phi \sim BR^2 \sim \text{const}$$

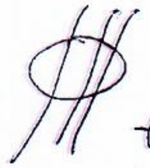
$$M \sim \rho R^3 \sim \text{const.}$$

$\Rightarrow \quad B \sim r^{-2}$
$\quad\;\; \rho \sim r^{-3}$ $\quad \Rightarrow \quad \underline{B/\rho^{2/3} \sim \text{const.}}$

why the scaling ? $\Leftrightarrow$ why of interest ?

$\rightarrow \left\{ \begin{array}{l} \text{"implosion"} \\ \text{gravitational collapse} \\ \text{equation of state} \end{array} \right\}$ problems sensitive to equation of state of material collapsing

<u>If</u>: $p \rightarrow p_{tot} = p + \frac{B^2}{8\pi}$



collapse threaded by magnetic field

$$p = p_0 (\rho/\rho_0)^{\gamma}$$

then natural to ask: Can one write $B^2 = B^2(\rho)$ and thus extend equation of state to encompass magnetic pressure contribution?

$\Rightarrow$ proceed via flux-freezing!

$$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$$

so $p_{B^2}$ has "$\gamma_{eff} = 4/3$". This resembles equation of state for degenerate gas (see Handouts I).

$\Rightarrow$ More on this in discussion of flux freezing and <u>Virial</u> <u>theorems</u> ...

$\rightarrow$ Pragmatic Question: <u>Is</u> flux 'frozen' during star formation? $\Leftrightarrow$ Does resistivity matter?

$\eta \sim \frac{4 \times 10^6}{T_{eV}^{3/2}} \ cm^2/sec.$ (Spitzer)

start $\rightarrow$ collapse $\rightarrow$ protostar

$n \sim 1 \text{atom}/cm^3 \longrightarrow$ $\rho \sim 1 g/cm^3$

$\Rightarrow n \sim 10^{24}_{atm}/cm^3$ (related $N_A$)

but

$B/\rho^{2/3} \sim const$

$\Rightarrow B/B_0 \sim (10^{24})^{2/3} \sim 10^{16}$ ! huge amplification

so $B_0 \sim 10^{-6}$ G, characteristic of ISM

$\Rightarrow B \sim 10^{10}$ G in protostar

$\therefore P_{B^2} \sim 10^{19} erg/cm^3$ $(P_{B^2} \sim B^2/8\pi)$

but $P_{Th}$ for normal star $\sim 10^{14} erg/cm^3$

$P_{B^2} \gg P_{Th}$ ?? $\Rightarrow$ clearly flux-freezing is bad assumption.

→ In terms of time scales:

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{v} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

$$\underset{①}{\frac{1}{T_{collapse}}} \sim \underset{②}{\frac{1}{T_{dynamic}}} + \underset{③}{\frac{\eta}{L^2}}$$

$\frac{\eta}{L^2}$ ≡ $1/T_{diffn.}$

3 scales, 2 balance
i.e. ① & ② ③ negligible
① & ③ ② negligible.

if $T_{collapse} \ll T_{diffn} \rightarrow \{$ flux frozen, OK

$T_{collapse} \gg T_{diffn} \rightarrow \{$ must consider diffusion, freezing invalid

N.B.: In star formation, $T_{coll.} \ll T_{diffn}$

but ISM has large neutral component.

Plasma-neutral drag sets dissipation
→ Ambipolar diffusion.