B.) 'Least Action' and the Energy Principle in MHD

→ Introduction

- we now arrive at the MHD Energy Principle, which is a highlight of MHD, plasma physics and classical physics, in general.

- Energy Principle → stability

i.e. till now:
- 218B – waves, etc.
- 218A – trivial instabilities (i.e. 2-stream, bump-on-tail, J-driven ion-acoustic)

realistic plasmas {lab or astro} → free energy (∇P, ∇J etc.) + complex geometry, b.c.'s, etc.

→ instabilities with complex dynamics ...

i.e.
- Rayleigh-Benard → ∇S
- Interchanges → κ, ∇P (includes Rayleigh-Taylor)
- kinks, tearing → ∇J, $q(r)$

- relaxation, turbulence, shocks ...
- limits on performance (lab)
- restrictions on morphology (lab and astro)

- brute force, frontal assault on instabilities often leads to heavy casualties ....

∴

- need a simple criterion, i.e. a necessary/sufficient criterion to identify and characterize instabilities

⇒ Energy Principle !

- Energy Principle is very much in spirit of R-R Variational principle ⟷ no surprise as both based on self-adjointness of linear operator

- Proceed via:

- sketch of Principle of Least Action for Ideal MHD ⇒ Lagrangian formulation (Kulsrud 4.7)

N.B. This underlies formulation in terms of displacement...

- MHD eigenmode equation (generalizes simple wave studies so far), second order W

⇒

- energy principle. (Kulsrud 7.1, 7.2) (Kadomtsev Article)

— applications (various)

(i.) Principle of Least Action for MHD

— For ideal MHD, can immediately write

$$L = \int d^3x \left[\frac{\rho v^2}{2} \right] - W \qquad \text{(Lagrangian)}$$

$$S = \int dt\, L$$
↳ action

$$W = \int d^3x \left(\frac{P}{\gamma-1} + \frac{B^2}{8\pi} + \rho\phi \right)$$

$$\therefore \quad \mathcal{L} = \frac{\rho v^2}{2} - \left(\frac{P}{\gamma-1} + \frac{B^2}{8\pi} + \rho\phi \right)$$

and can derive MHD equations by $\delta L = 0$
i.e. Principle of Least Action

— key point: how parametrize trajectory variations ??

i.e. for string: (easy)

$$S = \int dt \int_0^{L_0} dx \left[\frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2 - T\left[\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)^{1/2} - 1\right]\right]$$

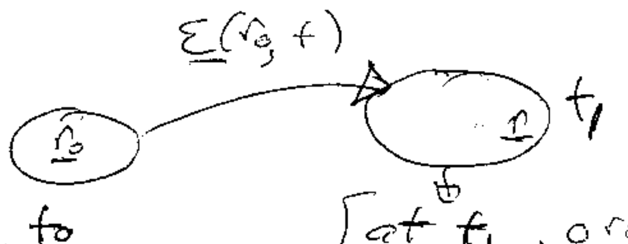
$$\delta L = \delta L/\delta y \Rightarrow \frac{\partial L}{\partial y_t}\delta y_t + \frac{\partial L}{\partial y_x}\delta y_x \quad \text{etc...}$$

$\therefore$ analogy with string suggests displacement is!

$\Rightarrow$ natural way to formulate Least Action for ideal MHD

$\Rightarrow$ natural link of MHD dynamics to particle dynamics

i.e.



[at t_1, original blob has $\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t)$]

– how relate $\underline{\xi}(\underline{r}_0, t)$ to Eulerian velocity?

i.e. during δt, fluid element moves

from $\quad\quad$ to

$$\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t) \longrightarrow \underline{r}_0 + \underline{\xi}(\underline{r}_0, t) + \left(\partial \underline{\xi} / \partial t\right) \delta t$$

$$\therefore \quad \underline{V}(\underline{r}_0 + \underline{\xi}(\underline{r}_0, t), t) = \frac{\partial \underline{\xi}}{\partial t}(\underline{r}_0, t)$$

$\Rightarrow$ 3 components of $\underline{\xi}$ satisfy 3 nonlinear odes with $\underline{\xi}(\underline{r}_0, t_0) = 0$ as i.c.

→ theory of ode's assures solution exists.

Now, as in wave theory, can write all changes in MHD quantities in terms of displacements, i.e.

$$\delta\rho = -\underline{\nabla}\cdot\left[\rho(\underline{r},t)\,\delta\underline{\xi}(\underline{r},t)\right]$$

$$\delta P = -\gamma P(\underline{r},t)\,\underline{\nabla}\cdot\delta\underline{\xi}(\underline{r},t) - \delta\underline{\xi}(\underline{r},t)\cdot\underline{\nabla}P(\underline{r},t)$$

$$\delta\underline{B} = \underline{\nabla}\times\left(\delta\underline{\xi}(\underline{r},t)\times\underline{B}(\underline{r},t)\right)$$

and

$$\delta V(\underline{r},t) = \underline{V}(\underline{r},t)\cdot\nabla\,\delta\underline{\xi}(\underline{r},t) - \delta\underline{\xi}(\underline{r},t)\cdot\underline{\nabla}\,\underline{V}(\underline{r},t) + \partial\,\delta\underline{\xi}(\underline{r},t)/\partial t$$

so now, can consider δS

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x\, \delta\mathcal{L}$$

$$= \int_{t_1}^{t_2} dt \int d^3x \left(\delta\rho\,\frac{\underline{V}^2}{2} + \rho\underline{V}\cdot\delta\underline{V} - \frac{\delta P}{\gamma-1} - \frac{\underline{B}\cdot\delta\underline{B}}{4\pi} - \delta\rho\,\phi \right)$$

plugging in δ quantities $\Rightarrow$

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \underbrace{\nabla\cdot(-\rho\,\delta\underline{\xi})\frac{V^2}{2} + \rho\underline{V}\cdot\left(\underline{V}\cdot\nabla\delta\underline{\xi} - \delta\underline{\xi}\cdot\nabla\underline{V} + \frac{\partial\delta\underline{\xi}}{\partial t}\right)}_{\delta KE} \right\} + \int_{t_1}^{t_2} dt\, d^3x \underbrace{\left(\frac{\gamma p\,\nabla\cdot\delta\underline{\xi} + \delta\underline{\xi}\cdot\nabla p}{\gamma-1}\right)}_{\delta ThE}$$

$$- \int_{t_1}^{t_2} dt \int d^3x \underbrace{\frac{\underline{B}\cdot\nabla\times(\delta\underline{\xi}\times\underline{B})}{4\pi}}_{\delta E_B} + \int_{t_1}^{t_2} dt \int d^3x\, \nabla\cdot(\rho\,\delta\underline{\xi})\,\phi$$

Now $\delta\underline{\xi}\Big|_{t_1, t_2} = 0$, $\delta\underline{\xi}\Big|_{\text{bndry}} = 0$ $\rightarrow$ cmpt restriction

so ibp a lot $\Rightarrow$ (with b.c.'s)

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left\{ \delta\underline{\xi}\cdot\left[\rho\nabla\frac{V^2}{2} - \nabla\cdot(\rho\underline{V}\,\underline{V}) - \rho\nabla\frac{V^2}{2} - \frac{\partial}{\partial t}(\rho\underline{V})\right] - \frac{\delta\underline{\xi}\cdot\gamma\nabla p + \delta\underline{\xi}\cdot\nabla p}{(\gamma-1)} - \delta\underline{\xi}\cdot\rho\nabla\phi + \delta\underline{\xi}\cdot\frac{(\nabla\times\underline{B})\times\underline{B}}{4\pi} \right\}$$

∇p $\quad$ $\rho\nabla\phi$ $\quad$ $\underline{J}\times\underline{B}$

So

$$\delta S = -\int_{t_1}^{t_2} dt \int d^3x \; \delta\underline{\xi} \cdot \left[\frac{\partial}{\partial t}(\rho \underline{V}) + \nabla \cdot (\rho \underline{V}\,\underline{V}) + \underline{\nabla} p - \underline{J} \times \underline{B} + \rho \underline{\nabla} \phi \right]$$

So

$$\delta S = 0 \quad \text{and} \quad \delta\underline{\xi} \neq 0 \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho \underline{V}) + \underline{\nabla} \cdot (\rho \underline{V}\,\underline{V}) = -\underline{\nabla} p + \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi$$

and $\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{V}) \Rightarrow$

$$\boxed{\rho \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{V} \right] = -\underline{\nabla} p + \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi}$$

$\Rightarrow$ equation of motion of ideal MHD emerges as "Lagrange's Equation".

Note: for case of $\underline{V} = 0 \rightarrow$ equilibrium solution then $\delta S = 0$ gives:

$$\underline{\nabla} p = \underline{J} \times \underline{B} - \rho \underline{\nabla} \phi$$

Moral of this story:

→ can derive MHD equations from Principle of Least Action

→ displacement is a useful way to formulate ideal MHD dynamics

Now this brings us to:

ii.) Energy Principle – Simple Form

Consider inhomogeneous static equilibrium/initial state with:

$$\left.\begin{aligned} \underline{\nabla} P_0 &= \underline{J_0} \times \underline{B_0} - \underline{\rho} \, \underline{g} \quad \overset{\rho\text{ const}}{\underset{g}{}} \quad (c \to 1) \\ \underline{\nabla} \times \underline{B_0} &= \frac{4\pi}{c} \underline{J_0} \\ \underline{\nabla} \cdot \underline{B_0} &= 0 \end{aligned}\right\} \text{eqbrm conditions}$$

n.b. gravity will be dropped in places

all $\begin{cases} P_0 = P_0(r_0) \\ \text{time independent} \\ \text{but inhomogeneous!} \end{cases}$

and no flow or $\begin{cases} \text{self-} \\ \text{gravity} \end{cases}$ ....

Further assume → rigid wall bounds system (!!)

→ $\underline{V} \cdot \hat{n} \Big|_{\text{wall}} = 0$

$\underline{B} \cdot \hat{n} \Big|_{\text{wall}} = 0$

and now ... $\Rightarrow$ perturb system from eqbrm by $\underline{\xi}$

$\Rightarrow$ so, at $t=0$:

$$\underline{\xi}(r) = \underline{\xi}_0(\underline{r})$$

$$\frac{\partial \underline{\xi}(r)}{\partial t} = \frac{\partial \hat{\underline{\xi}}_0(r)}{\partial t}$$

$\Rightarrow$ keep only linear terms in $\underline{\xi}$ $\Rightarrow$

$\underline{r} = \underline{r}_0 + \xi(\underline{r}_0, t)$

and $\underline{r}_0 \to \underline{r}$ in argument of perturbed quantities.

So

$$\Rightarrow \quad \rho(t, r) = \rho_0 - \nabla\cdot(\rho_0\,\underline{\xi})$$

$$P(t, \underline{r}) = P_0 - \gamma P_0\,\underline{\nabla}\cdot\underline{\xi} - \underline{\xi}\cdot\underline{\nabla} P_0$$

$$\underline{B}(t, \underline{r}) = \underline{B}_0 + \underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)$$

$$4\pi J(\underline{r}, t) = \underline{j}_0 + \underline{\nabla}\times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right]$$

, putting it into equation of motion (linearized) $\Rightarrow$

$$\boxed{\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi})}$$

where:

$$F(\xi) = \frac{1}{4\pi}\left[\underbrace{\nabla\times\left[\nabla\times(\underline{\xi}\times\underline{B}_0)\right]\Big)\times B_0}_{\delta \underline{J}\times B_0}\right.$$

$$+ \underbrace{\underline{J}_0 \times \left[\nabla\times(\xi\times\underline{B}_0)\right]}_{\underline{J}\times\delta\underline{B}} \qquad \underbrace{-\,\underline{g}\,\underline{\nabla}\cdot(\rho_0\underline{\xi})}_{-g\,\delta\rho}$$

$$+ \underbrace{\underline{\nabla}\left[\underline{\xi}\cdot\underline{\nabla}P_0 + \gamma P_0(\underline{\nabla}\cdot\underline{\xi})\right]}_{-\underline{\nabla}\tilde{P}}$$

with b.c. $\begin{cases} \underline{\xi}\cdot\vec{n} = 0 & \text{on surface} \\ \underline{B}\cdot\vec{n} = 0 & \text{on surface} \end{cases}$

Key Point:

$\rightarrow$ $\boxed{\underline{F}(\underline{\xi}) \text{ is self-adjoint}}$!!

i.e.

$$\int d^3x\; \underline{\eta}\cdot\underline{F}(\xi) = \int d^3x\; \underline{\xi}\cdot F(\underline{\eta})$$

→ to prove: see Kulsrud, Pblm. 6
(coming on Pblm Set III)

or consider the following (an indirect proof) ....
legerdemain involved...

→ can write total energy, to second order (in displacement) as:

i.e. $E = \int d^3x \, \frac{\rho_0(r)}{2} \left(\frac{\partial \xi}{\partial t}\right)^2 + W(\xi, \xi)$

2nd order bit of:

$\int \left(\frac{p}{\gamma - 1} + \frac{B^2}{8\pi} + \rho\phi \right) d^3x$

Now:

→ $W = W_0 + \underbrace{W_1(\xi)}_{\text{first order}} + \underbrace{W_2(\xi, \xi)}_{\text{second order}}$

→ total energy is conserved, for any ξ

with initial conditions ξ_0, $\dot{\xi}_0$,

provided $\xi \cdot \hat{n} = \dot{\xi} \cdot \hat{n} = 0$ (b.c.)

Now, $dE/dt = 0 \Rightarrow$

$$\frac{dE}{dt} = \int_V d^3x\, \rho_0 \left\{ \frac{\partial \underline{\xi}}{\partial t} \cdot \frac{\partial^2 \underline{\xi}}{\partial t^2} \right\} + W_1\left(\frac{\partial \underline{\xi}}{\partial t}\right)$$

$$+ W_2\left(\frac{\partial \underline{\xi}}{\partial t}, \underline{\xi}\right) + W_2\left(\underline{\xi}, \frac{\partial \underline{\xi}}{\partial t}\right) = 0$$

and $\rho_0 \dfrac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\xi) \Rightarrow$

$$\frac{dE}{dt} = \int d^3x \left[\frac{\partial \underline{\xi}}{\partial t} \cdot \underline{F}(\underline{\xi})\right] + W_1\left(\frac{\partial \underline{\xi}}{\partial t}\right)$$

$$+ W_2\left(\frac{\partial \underline{\xi}}{\partial t}, \underline{\xi}\right) + W_2\left(\underline{\xi}, \frac{\partial \underline{\xi}}{\partial t}\right)$$

but since $dE/dt = 0$ is always true, it is true at $t=0$, a particular time

setting $\dot{\underline{\xi}}_0 \equiv \underline{\eta} \Rightarrow$

$\hookrightarrow$ a particular displ. ...

$$\int d^3x\, \underline{\eta} \cdot \underline{F}(\underline{\xi}) + W_1(\underline{\eta}) + W_2(\underline{\eta}, \underline{\xi})$$

$$+ W_2(\underline{\xi}, \underline{\eta}) = 0$$

now, $W_1(\underline{\eta}) = 0$ so (no velocity dependence on i.c.)

$$\int d^3x \left(\underline{\eta}\cdot F(\underline{\xi})\right) + \left[W_2(\underline{\eta},\underline{\xi}) + W_2(\underline{\xi},\underline{\eta})\right] = 0$$

or more clearly $\Rightarrow$

$$\int d^3x \; \underline{\eta}\cdot F(\underline{\xi}) = -\left[W_2(\underline{\eta},\underline{\xi}) + W_2(\underline{\xi},\underline{\eta})\right]$$

so RHS symmetric under $\underline{\eta} \leftrightarrow \underline{\xi}$ interchange

so so is LHS ! i.e.

$$\int d^3x \; \underline{\eta}\cdot F(\underline{\xi}) = \int d^3x \; \underline{\xi}\cdot F(\underline{\eta})$$

and have proved self-adjointness !

→ finally, useful to note that if now $\underline{\eta} = \underline{\xi}$

$$\boxed{W_2(\underline{\xi},\underline{\xi}) = -\frac{1}{2}\int d^3x \left[\underline{\xi}\cdot F(\underline{\xi})\right]}$$

— a handy expression for W_2 in terms F !

so now, have shown that:

→ $\underline{F}(\underline{\xi})$ self-adjoint

→ $W_2(\underline{\xi})$, the potential energy of displacement $\underline{\xi}$, can be expressed as:

$$W_2(\underline{\xi}) = -\frac{1}{2}\int d^3x \left[\underline{\xi}\cdot\underline{F}(\underline{\xi})\right]$$

From these, we show several important results:

- reality of ω^2 and "exchange of stabilities"
  ↔ clue to structure of instability in ideal MHD
- orthogonality of eigenfunctions
- variational structure

i) reality of ω^2, "exchange of stabilities"

$$\underline{\xi} = \tilde{\xi}(x)\, e^{-i\omega t}$$

$$-\rho_0\omega^2\,\underline{\xi} = \underline{F}(\underline{\xi}) \qquad (1)$$

$$-\rho_0\omega^{2*}\,\underline{\xi}^* = \underline{F}(\underline{\xi}^*) \qquad (2)$$

N.B. F is explicitly real

$$\underline{\xi}^* \cdot (1) - \underline{\xi} \cdot (2) \Rightarrow$$

$$-\rho_0 (\omega^2 - \omega^{2*})\, \underline{\xi}^* \cdot \underline{\xi} = \underline{\xi}^* \cdot \underline{F}(\underline{\xi}) - \underline{\xi} \cdot \underline{F}(\underline{\xi}^*)$$

and integrating $\Rightarrow$

$$-\rho_0 (\omega^2 - \omega^{2*}) \int d^3x \, (\underline{\xi}^* \cdot \underline{\xi}) = \int d^3x \left[\underline{\xi}^* \cdot \underline{F}(\underline{\xi}) - \underline{\xi} \cdot \underline{F}(\underline{\xi}^*) \right]$$

$= 0$, by self-adjoint property

so $\underline{\xi}^* \cdot \underline{\xi}$ real $\Rightarrow$ $(\omega^2)^* = \omega^2$

$\Rightarrow$ ω^2 is real

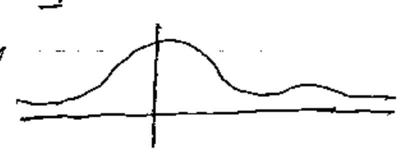
$\therefore$ $\omega^2 > 0 \rightarrow$ stability

$\omega^2 < 0 \rightarrow$ instability, but purely growing
$\rightsquigarrow$ no oscillation

Contrast to instabilities with which you should be familiar:

$\Rightarrow$ bump-on-tail $\qquad \omega = \omega_k^0 + i\gamma_k$

$\gamma \sim \partial f_0 / \partial v$



Wave + inverse dissipation
$\downarrow$
carrier

→ two stream $\epsilon = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - k v_0)^2}$

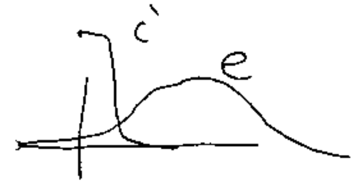
~ coupling of $\begin{cases} \text{positive energy wave in plasma} \\ \text{negative energy wave in beam} \end{cases}$

~ "reactive" counter-part of bump on tail ⇔ can have ω^2 real

→ beam + dissipation ⇒ negative energy wave ⊕ dissipation ⇒ growth

$\omega = \omega_r + i\gamma$

→ current-driven ion-acoustic

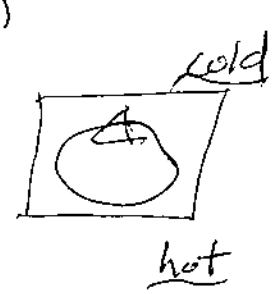


$\omega = \omega_r + i\gamma \qquad \gamma = (\,)\frac{\partial f_e^{(0)}}{\partial v} - (\,)\frac{\partial f_i^{(0)}}{\partial v}$

wave + competition of dissipation and dissipation

→ ideal Rayleigh-Benard Convection

$\omega^2 = -\frac{k_H^2}{k_H^2 + k_V^2}\, g \frac{\partial S'}{\partial z} \qquad \nabla_z S < 0$



$\downarrow g > 0$

of these, ideal MHD instabilities similar in structure to convection and ω^2 real cases of 2-stream, and <u>different</u> in structure from the others

4. In ideal MHD, instability defines structure of eigenfunction, i.e. $\underline{\tilde{\xi}} = \underline{\tilde{\xi}}(\underline{r}, \gamma)$.

N.B. In ideal MHD, only scale in problem is system size $\leftrightarrow$ boundaries. Contrast Sweet-Parker reconnection ($\Delta/L < 1$), a case of resistive MHD.

proceeding $\Rightarrow$

Since ω^2 real, ω^2 must pass thru $\omega^2 = 0$ as the system evolves from stable to unstable.

- this evolution is called "exchange of stabilities"

- $\Rightarrow$ marginal displacement solves $\underline{F}(\underline{\xi}) = 0$

N.B. $\Rightarrow$ Solution of $\underline{F}(\underline{\xi}) = 0$ gives linear stability boundary, in parameter space

ii.) orthogonality

consider two solutions to $\quad -\rho_0 \omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$,

$$-\rho_0 \omega_1^2 \underline{\xi}_1 = \underline{F}(\underline{\xi}_1) \qquad \underline{\times}\ \underline{\xi}_2$$

$$-\rho_0 \omega_2^2 \underline{\xi}_2 = \underline{F}(\underline{\xi}_2) \qquad \underline{\times}\ \underline{\xi}_1$$

$$-(\omega_1^2 - \omega_2^2) \int d^3x \rho_0 \underline{\xi}_1 \cdot \underline{\xi}_2 = \int d^3x \left[\underline{\xi}_2 \cdot \underline{F}(\underline{\xi}_1) - \underline{\xi}_1 \cdot \underline{F}(\underline{\xi}_2) \right]$$

$$= 0 \quad , \quad \text{by self-adjointness}$$

$$\omega_1^2 \neq \omega_2^2 \Rightarrow \int d^3x \, \rho_0 \, \underline{\xi}_1 \cdot \underline{\xi}_2 = 0$$

$\Rightarrow$ orthonormality, with weighting function ρ_0.

The point of all this is that now we can set up a variational quadratic form, ala' beloved Sturm-Liouville theory

$$-\rho\omega^2 \underline{\xi} = \underline{F}(\underline{\xi})$$

and $\otimes \ \underline{\xi}/2 \cdot \Rightarrow$

$$\boxed{\omega^2 = \frac{-\int d^3x \ \underline{\xi}\cdot\underline{F}(\underline{\xi})/2}{\int \rho_0 \underline{\xi}^2/2}}$$

$$= W_2(\underline{\xi}) \Big/ \int \rho \, \underline{\xi}^2/2 \, .$$

$\Rightarrow$ with $K(\underline{\xi}) \equiv \int d^3x \, \rho \, \underline{\xi}^2/2$, have

$$\boxed{\omega^2 = W_2(\underline{\xi}) / K(\underline{\xi})} \rightarrow \left\{ \begin{array}{l} \text{variational, quadratic} \\ \text{form} \end{array} \right.$$

and we know that, since all requirements satisfied, that

$\rightarrow$ any trial $\underline{\xi}$ plugged into $W_2(\underline{\xi})/K(\underline{\xi})$ — yields $\omega^2(\underline{\xi}) > \omega_T^2$
$\hookrightarrow$ the true eigenvalue.

i.e. variational result is always upper bound.

→ so, we know that

- if can find a trial $\underline{\xi}$ such that

$$W_2(\underline{\xi}) < 0$$

- then, configuration is surely unstable

∴ this yields the desired necessary and sufficient condition for instability namely that it be possible to find a $\underline{\xi}$ such that

$$W_2(\underline{\xi}) < 0.$$

hereafter, we write $W_2(\underline{\xi}) = \delta W(\underline{\xi})$,

so the MHD Energy Principle is just:

$$\boxed{\text{instability iff } \exists \text{ well behaved } \underline{\xi} \text{ s/t } \delta W(\underline{\xi}) < 0}$$

N.B.

- in physical terms, E.P. $\Rightarrow$ instability if can find a displacement which lowers the energy. Note linear instability $\Longleftrightarrow$ δW to $O(\xi^2)$ considered

- know $\delta W(\underline{\xi}) = -\frac{1}{2}\int d^3x\, \underline{\xi}\cdot \underline{F}(\underline{\xi})$

so, now must manipulate δW into physically useful form, i.e. recall

$$F(\underline{\xi}) = \frac{1}{4\pi}\Big\{\overbrace{\nabla\times[\nabla\times(\underline{\xi}\times\underline{B}_0)]\Big\}\times\underline{B}_0}^{\delta\underline{J}\times\underline{B}_0\ -①}$$

$$+\ \overbrace{\underline{J}_0\times[\nabla\times(\underline{\xi}\times\underline{B}_0)]}^{\underline{J}_0\times\delta\underline{B}\ -②}$$

$$+\ \overbrace{\underline{\nabla}[P_0\underline{\nabla}\cdot\underline{\xi}+\underline{\xi}\cdot\underline{\nabla}P_0)]}^{\nabla\delta p\ -③}\ +\ \overbrace{\nabla\cdot(\rho_0\underline{\xi})\underline{\nabla}\phi}^{\delta\rho\nabla\phi\ -④}$$

$$= F_① + F_② + F_③ + F_④$$

remember here, all $\underline{B}_0$, $\underline{P}_0$, $\underline{\rho}_0$ etc. inhomogeneous, and $\underline{\xi}\cdot\hat{n}$ and $\underline{B}\cdot\hat{n}$ on boundary.

- remains to manipulate $-\int \left[\underline{\xi}\cdot F(\underline{\xi})/2\right] d^3x$ into "illuminating" form

- key is sign of δW, so seek to extract quadratic terms, as unambiguous.

$\Rightarrow$ let the crank begin!

① $$\delta W_{①} = -\frac{1}{2}\int \underline{\xi}\cdot \underline{F}_{①}(\underline{\xi})\, d^3x$$

$$= -\frac{1}{2}\int d^3x\, \frac{\underline{\xi}}{4\pi}\cdot \left\{\left(\underline{\nabla}\times\left[\nabla\times(\underline{\xi}\times\underline{B}_0)\right]\right)\times\underline{B}_0\right\}$$

$$= \frac{1}{8\pi}\int d^3x\, \left(\underline{\nabla}\times\left[\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right]\right)\cdot\underline{\xi}\times\underline{B}_0$$

$$= \frac{1}{8\pi}\int d^3x\, \nabla\cdot\left[\nabla\times(\underline{\xi}\times B_0)\times(\underline{\xi}\times\underline{B}_0)\right]$$
$$+ \frac{1}{8\pi}\int d^3x\, \left(\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right)\cdot\left(\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0)\right)$$

if $\underline{Q}\equiv \nabla\times(\underline{\xi}\times B_0) = \delta\underline{B}$, from induction

$$\delta W_{①} = \int d^3x\, \frac{\underline{Q}^2}{8\pi} + \frac{1}{8\pi}\int d\underline{S}\cdot(\nabla\times(\underline{\xi}\times B_0))\times(\underline{\xi}\times\underline{B}_0)$$

$$\Rightarrow \quad \delta W_{①} \Big|_{\text{surface}} = -\frac{1}{8\pi}\int ds \left[\cancel{\hat{n}} \cdot \underline{B}_0 \; \underline{\xi}\cdot\underline{Q} - (\cancel{\hat{n}}\cdot\underline{\xi})\, \underline{B}_0\cdot\underline{Q} \right]$$

$$\delta W_{①} = \int d^3x \; \frac{Q^2}{8\pi}$$

$$\delta W_{②} = -\frac{1}{2}\int d^3x \; \underline{\xi}\cdot \underline{J}_0 \times \left[\nabla\times(\underline{\xi}\times \underline{B}_0)\right]$$

$$= -\frac{1}{2}\int d^3x \; \underline{\xi}\cdot (\underline{J}_0 \times \underline{Q})$$

$$= +\frac{1}{2}\int d^3x \; \underline{J}_0 \cdot (\underline{\xi}\times \underline{Q})$$

$$\delta W_{③} = -\frac{1}{2}\int d^3x \; \underline{\xi}\cdot \underline{\nabla}\left[p_0\, \underline{\nabla}\cdot\underline{\xi} + \underline{\xi}\cdot\underline{\nabla} p_0 \right]$$

ibp $\underline{\xi}\cdot\hat{n} = 0$ on boundary

$$\Rightarrow \delta W_{③} = \int \frac{d^3x}{2} \left[p_0 (\underline{\nabla}\cdot\underline{\xi})^2 + (\underline{\nabla}\cdot\underline{\xi})\, \underline{\xi}\cdot\underline{\nabla} p_0 \right]$$

& last but not least...

$$\delta W_{④} = -\int \frac{d^3x}{2}\ \underline{\xi} \cdot \nabla \cdot (\rho_0 \underline{\xi})\, \underline{\nabla}\phi$$

$$= -\frac{1}{2}\int d^3x\ (\underline{\xi}\cdot\underline{\nabla}\phi)\ \nabla\cdot(\rho_0\underline{\xi})$$

so, putting the whole mess together

$$\boxed{\begin{aligned}\delta W = \frac{1}{2}\int d^3x \Big\{ &\underbrace{\frac{\underline{Q}^2}{4\pi}}_{①} + \underbrace{\underline{J}_0(\underline{x})\cdot(\underline{\xi}\times\underline{Q})}_{②} \\ &+ \underbrace{\gamma P_0(\underline{x})(\underline{\nabla}\cdot\underline{\xi})^2}_{③} + \underbrace{(\underline{\xi}\cdot\underline{\nabla} P_0(\underline{x}))\,\underline{\nabla}\cdot\underline{\xi}}_{④} - \underbrace{(\underline{\xi}\cdot\underline{\nabla}\phi)\,\nabla\cdot(\rho_0\underline{\xi})}_{⑤}\Big\} \\ \underline{Q} = \nabla\times(\underline{\xi}\times\underline{B}_0)\end{aligned}}$$

note: general characteristics

- ① $\Rightarrow\ >0\ \Rightarrow$ field line bending
  ③ $\Rightarrow\ >0\ \Rightarrow$ compression
  $\Big\}\Rightarrow$ always stabilizing $\delta W>0$

free energy sources:
$\nabla\rho_0(\underline{x})$ in ⑤ $\rightsquigarrow$ density gradient
$\underline{J}_0(\underline{x})$ in ② $\rightsquigarrow$ current profile

$\underline{\nabla} P_0(\underline{x})$ in ④ $\rightsquigarrow$ pressure gradient
gravity and ρ_0 in ⑤

$\Rightarrow$ can make $\delta W<0$, for certain profiles and $\underline{\xi}$ $\Rightarrow$ free energy sources for instability.

Note:

→ δW is imprecise

→ δW does not reveal much about growth rates

but

→ very useful for simple quick assessment of stability

→ can elucidate
- complex problem
- problem in which info re: equilibrium not precise.

∴ further developments in theory remain, but better to consider some examples

⇒

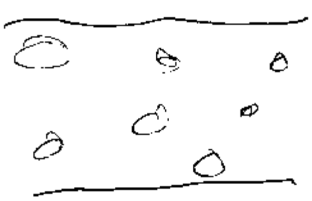
(iii) Convection and Interchange Instabilities
→ A Simple Application of the Energy Principle

consider 4 related examples:

a.) Convection and the Schwarzschild Criterion
b.) Rayleigh-Taylor Instability
c.) Interchange Instability
d.) Interchange Without Gravity

(i) Schwarzchild Criterion and Convection

i.e. stellar atmosphere



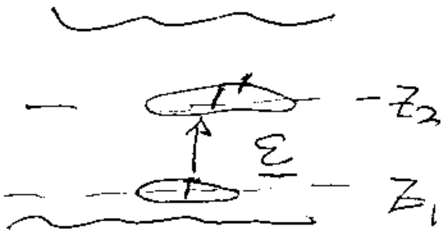
$$\left(\rho g = \frac{dp}{dz} \right)$$

$z \uparrow$ $\quad \frac{d\rho}{dz} < 0, \ \frac{dp}{dz} < 0$

and $\quad p \rho^{-\gamma} = \text{const.}$ $\quad$ (basic means of heat transport)

in basic idea of convection, consider a virtual displacement of a slug/blob of gas upward

⇒ physical argument



if $\rho_1' > \rho_2 \Rightarrow$ blob sinks
system stable

$\rho_1' < \rho_2 \Rightarrow$ blob rises,
system unstable

for infinitesimal displacement, $\varepsilon \sim \Delta z \Rightarrow$

$$\rho_2 = \rho_1 + \frac{d\rho_1}{dz} \Delta z$$

For ρ_1'; $\Rightarrow$ system is isentropic $\Rightarrow$
$P\rho^{-\gamma} = \text{const.}$ applies

$\Rightarrow$ displaced blob (i.e. ①$'$) comes to rapid pressure equilibration with surroundings

i.e. $\boxed{\frac{\Delta z}{c_s} \ll T_{rise}}$ $\Leftrightarrow$ $\gamma < k c_s$

$\sim$ 'nearly incompressible'

$$\therefore \quad P_1' = P_1 + \Delta z \frac{dP_1}{dz} = P_2$$

—

$$P_1 \rho_1^{-\gamma} = P_1' \rho_1'^{-\gamma}$$

$$\Rightarrow \quad p_1 \rho_1^{-\gamma} = \left(p_1 + \Delta z \frac{dp_1}{dz}\right) \rho_1'^{-\gamma}$$

$$\Rightarrow \quad \left(\frac{\rho_1'}{\rho_1}\right)^{\gamma} = 1 + \frac{\Delta z}{p_1}\frac{dp_1}{dz}$$

$$\frac{\rho_1'}{\rho_1} = \left(1 + \frac{\Delta z}{p_1}\frac{dp_1}{dz}\right)^{1/\gamma} \simeq 1 + \frac{\Delta z}{\gamma\, p_1}\frac{dp_1}{dz}$$

$$\frac{\rho_2}{\rho_1} = 1 + \frac{1}{\rho_1}\Delta z \frac{d\rho_1}{dz}$$

$\Rightarrow$ buoyant blob if:

$$\frac{\rho_1'}{\rho_1} < \frac{\rho_2}{\rho_1} \quad \Rightarrow \quad \frac{\Delta z}{\gamma}\frac{1}{p_1}\frac{dp_1}{dz} < \frac{\Delta z}{\rho_1}\frac{d\rho_1}{dz}$$

$$\Rightarrow \quad \boxed{\frac{1}{\gamma}\frac{1}{p_1}\frac{dp_1}{dz} < \frac{1}{\rho_1}\frac{d\rho_1}{dz}}$$

or, as both gradients negative

$$\boxed{\frac{1}{\gamma}\left|\frac{1}{p_1}\frac{dp_1}{dz}\right| > \frac{1}{\rho_1}\left|\frac{d\rho_1}{dz}\right|}$$

Schwarzchild criterion for Convective Instability

and as $S \equiv \ln\left(p\rho^{-\gamma}\right)$

$$\frac{dS}{dz} = \frac{1}{p}\frac{dp}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$\Rightarrow$ blob buoyant if $\frac{dS}{dz} < 0$ $\rightarrow$ "superadiabatically stratified"

sinks/restored if $\frac{dS}{dz} > 0 \rightarrow$ "subadiabatically stratified"

Marginal $dS/dz = 0$ $\rightarrow$ adiabatically stratified

Note: $\rightarrow$ Schwarzschild instability criterion $\leftrightarrow$ answers "is free energy available, locally" $\leftrightarrow$ ideal

$\rightarrow$ Rayleigh # criterion $\Rightarrow$ $Ra > Ra)_{crit}$
$\Rightarrow$ does free energy overcome dissipation?

Now, what does δW say?

Recall: $$\delta W = \frac{1}{2}\int d^3x \left[\frac{\underline{Q}^2}{4\pi} + \gamma P (\underline{\nabla}\cdot\underline{\xi})^2 + \underline{j}_0 \cdot (\underline{\xi} \times \underline{Q}) + (\underline{\xi}\cdot\underline{\nabla}P_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)\nabla\cdot(\rho_0 \underline{\xi}) \right.$$

pure hydro $\rightarrow$ $\underline{Q} = 0$, $\underline{j}_0 = 0$

$\frac{dp}{dz} = \rho g$ $\rightarrow$ hydrostatic equilibrium
$\underline{\nabla}p = \cancel{\underline{J}\times\underline{B}} + \rho \underline{g}$

$\underline{g} = \nabla \phi$, $\underline{g}$ downward

$\therefore$

$$2\delta W = \int d^3x \Big[\gamma p \, (\nabla\cdot\underline{\xi})^2 + (\underline{\xi}\cdot\nabla p)(\nabla\cdot\underline{\xi}) + (\underline{\xi}\cdot\underline{g})\left(\underline{\xi}\cdot\nabla\rho_0 + \rho_0 \nabla\cdot\underline{\xi}\right) \Big]$$

$$= \int d^3x \Big[\gamma p \, (\nabla\cdot\underline{\xi})^2 + (\nabla\cdot\underline{\xi})\left(\underline{\xi}\cdot(\nabla p + \underline{g}\,\rho_0)\right) + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

but $\nabla p = \rho \, \underline{g}$ (eqbrm condition) $\Rightarrow$

$$2\delta W = \int d^3x \Big[\gamma p \left((\nabla\cdot\underline{\xi})^2 + 2\frac{(\nabla\cdot\underline{\xi})}{\gamma p}(\underline{\xi}\cdot\nabla p) + \left(\frac{\underline{\xi}\cdot\nabla p}{\gamma p}\right)^2 \right) - \gamma p \left(\frac{\underline{\xi}\cdot\nabla p}{\gamma p}\right)^2 + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

$$= \int d^3x \Big[\gamma p \left(\nabla\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\nabla p}{\gamma p} \right)^2 - \frac{(\underline{\xi}\cdot\nabla p)^2}{\gamma p} + (\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\nabla\rho_0) \Big]$$

$$= \int d^3x \Big[\gamma p \left(\nabla\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\nabla p}{\gamma p} \right)^2 - \underline{\xi}\cdot\nabla p \left(\frac{\underline{\xi}\cdot\nabla p}{\gamma p} - \frac{\underline{\xi}\cdot\nabla\rho_0}{\rho_0} \right) \Big]$$

where used equilibrium condition again, so
$\Rightarrow$

$$2\delta W = \int d^3x \left[\gamma p \left(\underline{\nabla}\cdot\underline{\xi} + \frac{\underline{\xi}\cdot\underline{\nabla}p}{\gamma p} \right)^2 - \frac{\underline{\xi}\cdot\underline{\nabla}p}{\gamma}\ \underline{\xi}\cdot\underline{\nabla}\ \ln\left(p\rho^{-\gamma}\right) \right]$$

Now, object is to
- → explore possible displacements to see if $\delta W < 0$ possible
- → uncover any general condition

Now, expect $\underline{\xi}$ to have form:

$$\underline{\xi} = \mathrm{re}\left[\hat{\underline{\xi}}(z)\, e^{ikx} \right] \qquad \text{(must be real!)}$$

so can choose $\underline{\nabla}\cdot\underline{\xi} = -\dfrac{\underline{\xi}\cdot\underline{\nabla}p}{\gamma p}$

→ equivalent to setting a relation between ξ_x, ξ_z.

$$\rightarrow \quad \underline{\nabla}\cdot\underline{\xi} \sim \frac{\xi}{\gamma p}\frac{dp}{dz} \sim \frac{\xi}{\gamma L_p}$$

↳ pressure scale height

so $\dfrac{|\nabla\cdot\xi|}{|\xi|} \sim 1/L_p$ → "weakly compressible", in accord with physical argument contrast $\frac{|\underline{\nabla}\cdot\underline{\xi}|}{|\xi|} \sim |k| \rightarrow$ "strongly compressible" limit

so $2\delta W = -\int d^3x \left[\frac{\underline{\xi}\cdot\underline{\nabla}P \; \underline{\xi}\cdot\underline{\nabla} \ln (P\rho^{-\gamma})}{\gamma} \right]$

$\frac{dP}{dz}\hat{z} = \underline{\nabla}P$ and $\frac{dP}{dz} < 0 \Rightarrow$

if have any range of z over which

$$\frac{d}{dz} \ln (P\rho^{-\gamma}) < 0$$

$\Rightarrow$ have $\underline{\xi} \neq 0$ there, and $\delta W < 0$

$\Rightarrow$ instability, with criterion/condition that

$$\boxed{\frac{d \ln (P\rho^{-\gamma})}{dz} < 0}$$

$\rightarrow$ Schwarzschild Condition recovered

Now, can go further, and ask what is effect of magnetic field?

i.e. — consider $\underline{B} = B_0 \hat{x}$

then
$$\delta W = \underbrace{\delta W_0}_{\text{what we have}} + \int d^3x \, \frac{Q^2}{8\pi}$$

$$\underline{Q} = \underline{\nabla} \times (\underline{\xi} \times \underline{B_0})$$

$$\underline{Q} = \underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} - \underbrace{\underline{\xi} \cdot \underline{\nabla}\, \underline{B_0}}_{0 \text{ (homogeneous)}} - \underline{B_0}\, \underline{\nabla} \cdot \underline{\xi}$$

Now, to minimize δW, $\underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} = 0$

→ flute displacement

$k_\parallel = 0$

→ no bending energy expended

i.e. $Q = -B_0\, \underline{\nabla} \cdot \underline{\xi}$

$$\delta W = \delta W_0 + \int d^3x \, \frac{B_0^2}{8\pi} (\underline{\nabla} \cdot \underline{\xi})^2$$

but from before have, $\underline{\nabla} \cdot \underline{\xi} = -\dfrac{\underline{\xi} \cdot \underline{\nabla} p}{\gamma p}$

$$\delta W = \int d^3x \left[\frac{B_0^2}{8\pi} \left(\frac{\underline{\xi} \cdot \nabla p}{\gamma p} \right)^2 - \left(\frac{\underline{\xi} \cdot \underline{\nabla} p}{\gamma} \right) \frac{\underline{\xi}}{2} \cdot \nabla \ln (p \rho^{-\gamma}) \right]$$

$$\delta W \sim \int d^3x \left[P_{mag} \frac{\mathcal{E}^2}{\gamma^2 L_p^2} - \frac{P_{Th}}{\gamma L_p} \mathcal{E}^2 \left| \frac{dS}{dz} \right| \right]$$

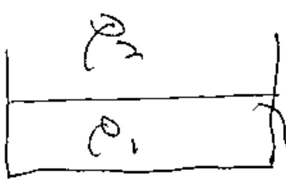
$$\delta W < 0 \quad \text{if} \quad \left| \frac{dS}{dz} \right| > \frac{P_{mag}}{P_{Th}\, \gamma L_p}$$

$$\Rightarrow \quad \frac{dS}{dz} < \frac{1}{\gamma \beta} \left(\frac{dP}{P\, dz} \right)$$

∴ indicates → magnetic field stabilizing
→ need critical entropy gradient $\sim \frac{1}{\beta L_p}$ for instability.

Moral of the story:
- → energy principle recovers essential physical criterion (Schwarzschild)
- → enables simple, quick, albeit imprecise insights into more complicated stability problems.

b.) Rayleigh-Taylor Instability → critical to implosions (ICF)



$\downarrow g$ $\quad \rho_2 > \rho_1$ $\quad p = 0$ (cold) (as will $\underline{\nabla} \cdot \underline{V} = 0$)

→ while nominally at equilibrium, configuration is unstable (heavy "falls" into light)

→ 

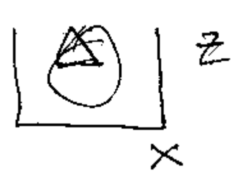
→ ripples, "spike and bubble"

$$\gamma^2 = |kg| \frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}$$

→ here $\underline{\nabla} \cdot \underline{V} = 0$

→ if continuous profile $\quad \downarrow g \quad \nabla\rho \uparrow$

$$\frac{\partial \tilde{V}}{\partial t} = -\frac{\underline{\nabla} \tilde{p}}{\rho_0} - g \frac{\tilde{\rho}}{\rho_0} \hat{z} \qquad g > 0$$

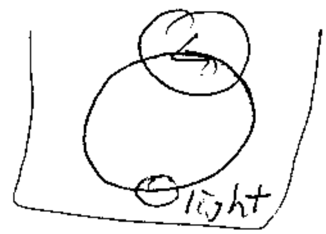


$$\frac{\partial}{\partial t} (\underline{\nabla} \times \underline{\tilde{V}}) \cdot \hat{y} = 0 - g \nabla \times \left(\frac{\tilde{\rho}}{\rho_0} \hat{z} \right)$$

$$\underline{V} = -\partial_z \phi \, \hat{x} + \partial_x \phi \, \hat{z}$$

$$\underline{\nabla} \cdot \underline{V} = 0$$

heavy



$$-\frac{\partial}{\partial t}\nabla^2\phi = g\,\partial_x\left(\frac{\tilde{\rho}}{\rho_0}\right)$$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\partial_x\tilde{\phi}\,\frac{d\rho_0}{dz} \quad \Rightarrow \quad \omega^2 = -\frac{k_x^2}{k^2}\frac{g}{L_\rho}$$

$$\boxed{\gamma^2 = \frac{k_x^2}{k^2}\,g/L_\rho} \qquad \begin{matrix} g>0 \\ 1/L_\rho > 0 \end{matrix}$$

interchange structure

Now, what would δW say?

$$\delta W = \frac{1}{2}\int d^3x\left[\frac{Q^2}{4\pi} + \gamma P(\underline{\nabla}\cdot\underline{\xi})^2 + \underline{J}_0\cdot(\underline{\xi}\times\underline{Q})\right.$$

$$\left. + (\underline{\xi}\cdot\underline{\nabla}P_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)(\nabla\cdot\rho_0\underline{\xi})\right]$$

$$\underline{Q}=0,\ \underline{j}=0,\ \underline{P}=0,\ \underline{\nabla}\cdot\underline{\xi}=0$$

$$\delta W = \int d^3x\left[-(\underline{\xi}\cdot\underline{\nabla}\phi)(\rho_0\underline{\nabla}\cdot\underline{\xi} + \underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$$= \int d^3x\left[+(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$$\delta W = \int\frac{d^3x}{2}\left[(\underline{\xi}\cdot\underline{g})(\underline{\xi}\cdot\underline{\nabla}\rho_0)\right]$$

$$\delta W = \int \frac{d^3x}{2} \left[(\underline{\xi} \cdot \underline{g}) (\underline{\xi} \cdot \underline{\nabla} \rho_0) \right]$$

$g < 0$ so if $\underline{\nabla} \rho_0 > 0$ $(d\rho_0/dz > 0)$ anywhere

$\Rightarrow \delta W < 0 \rightarrow$ instability

Now, if equilibrium hydrostatic:

$$\underline{\nabla} p = \rho \underline{g}, \quad \Rightarrow$$

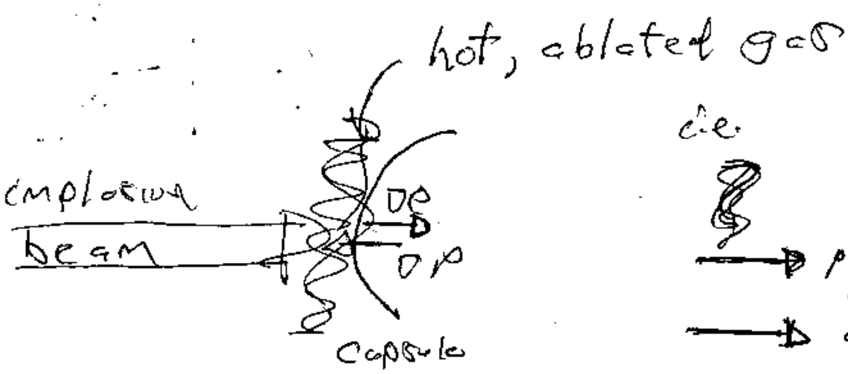
$$\boxed{\delta W = \int \frac{d^3x}{2} \left[(\underline{\xi} \cdot \underline{\nabla} p) \left(\frac{\underline{\xi} \cdot \underline{\nabla} \rho_0}{\rho_0} \right) \right]}$$

$\Rightarrow$ Rayleigh Taylor instability will result whenever $(\nabla P)(\nabla \rho) < 0$

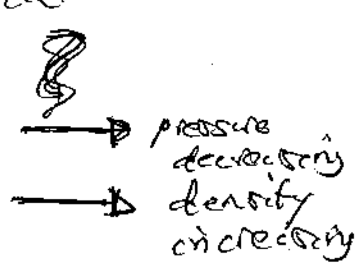
$\rightarrow$ pressure density gradients opposite

i.e. "heavy supported by light" (i.e. pressure highest at bottom)

i.e.

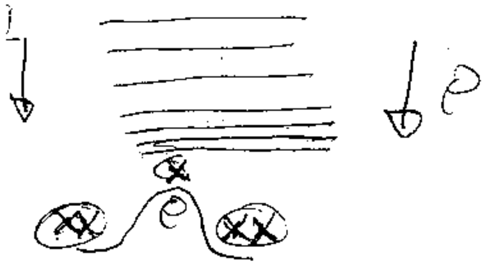


i.e.



iii.) Interchange Instability (basic confinement consideration)

→ consider plasma confined by magnetic pressure gradient



$$\underline{\nabla} p = \underline{J} \times \underline{B} + \rho \underline{g}$$

str. lines

$$\frac{dp}{dz} = -\nabla\left(\frac{B^2}{8\pi}\right) + \cancel{\frac{\underline{B}\cdot\underline{\nabla}\underline{B}}{4\pi}} + \rho \underline{g}$$

$\beta \ll 1$

$$\boxed{\underline{\nabla}\left(\frac{B^2}{8\pi}\right) = \rho \underline{g}}$$

$\underline{g} = -g\,\hat{z}$

$p \to 0$

equilibrium

$$\to \quad \delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + (\underline{\nabla}\cdot\underline{\xi})^2 \gamma p_0 + \underline{j}_0 \cdot (\underline{\xi} \times \underline{Q}) + (\underline{\xi}\cdot\underline{\nabla} p_0)(\underline{\nabla}\cdot\underline{\xi}) - (\underline{\xi}\cdot\underline{\nabla}\phi)\,\underline{\nabla}\cdot(\rho_0 \underline{\xi}) \right]$$

$j_0 = 0$

$p_0 = 0$

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + (\underline{\xi}\cdot\underline{g})\left(\underline{\xi}\cdot\underline{\nabla}\rho_0 + \rho_0 \underline{\nabla}\cdot\underline{\xi}\right) \right]$$

Here, must address $\underline{Q}$,

$$\underline{Q} = \underline{B_0}\cdot\underline{\nabla}\,\underline{\xi} - \underline{\xi}\cdot\underline{\nabla}\,B - B_0\,\underline{\nabla}\cdot\underline{\xi}$$

Now, can have $\underline{Q} = 0$ if:

$\rightarrow$ $\underline{B_0}\cdot\underline{\nabla}\,\xi = 0$ i.e. $\underline{\xi}$ constant along $\underline{B_0}$

$\Rightarrow k_{\parallel} = 0$

and

$$\rightarrow \quad \underline{\nabla}\cdot\underline{\xi} = -\frac{\underline{\xi}\cdot\underline{\nabla}B_0}{B_0}$$

$\therefore$

$$\delta W = \int d^3x \left[(\underline{\xi}\cdot\underline{g})\,\rho_0 \left(\frac{\underline{\xi}\cdot\underline{\nabla}\rho_0}{\rho_0} - \frac{\underline{\xi}\cdot\underline{\nabla}B_0}{B_0} \right) \right.$$

$$= \int d^3x \left[(\underline{\xi}\cdot\underline{g}\rho_0)\ \underline{\xi}\cdot\underline{\nabla}\ln(\rho/B) \right]$$

$g < 0 \Rightarrow$ if $\nabla \ln(\rho/B) > 0$ anywhere

$\therefore$ instability there !

Now:

→ obvious parallel to Rayleigh-Taylor is

$$\nabla \rho > 0 \quad \leftrightarrow \quad \nabla \ln(P/B) > 0$$

→ as $k_{\parallel} = 0$, field lines not bent

so can think of instability motion as interchange of flux tubes

Key question: Does interchange lower/raise potential energy?

interchange conserves magnetic flux

$$\Phi_2 = \int B_2 \, da = B_2 A_2$$

$$\Phi_1 = \int B_1 \, da = B_1 A_1$$

$$M_2 = \left(\frac{\rho}{B}\right)_2 \Phi_2$$

$M \Leftrightarrow m/\text{length}$

$$M_1 = \left(\frac{\rho}{B}\right)_1 \Phi_1$$

but $\phi_1 = \phi_2 \Rightarrow$

$$M_{2,1} = (P/B)_{2,1}\, \Phi$$

so $\nabla M \gtrless 0 \Rightarrow \nabla (P/B) \gtrless 0$

$\Rightarrow$ if P/B increases interchange will liberate gravitational potential energy, i.e. instability, a la' R-T.

$\rightarrow$ why care?:

- (interchange instability) severely degrades plasma confinement

- curing interchange stability is key element in device design $\rightarrow$ "minimum-B"

iv.) Interchange without Gravity

- in the context of magnetic confinement, "g" is a crutch, to represent curved field lines

– i.e. 

$$\underline{g} = \frac{V^2}{R_c} \rightarrow \underline{g}_{eff}$$

[centrifugal acceleration on particle]

– natural to investigate interchanges without "$\underline{g}$" $\Rightarrow$ pressure gradient drive (expansion free energy)

– now

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + \gamma p (\underline{\nabla}\cdot\underline{\xi})^2 + \underline{\xi}\cdot\underline{\nabla} p\, (\underline{\nabla}\cdot\underline{\xi}) + \underline{j}\cdot\underline{\xi}\times\underline{Q} \right]$$

Now, $\underline{Q} = 0$ $\rightarrow$ avoid bending, etc.

$$\underline{\nabla}\times(\underline{\xi}\times\underline{B}_0) = 0$$

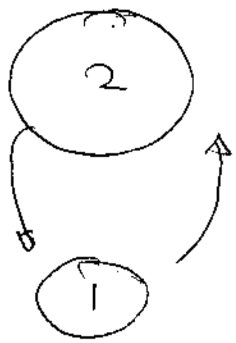
$$\Rightarrow \quad \underline{\xi}\times\underline{B}_0 = \nabla\phi$$

$\hookrightarrow$ some scalar potential

and $\underline{B}\cdot\underline{\nabla}\phi = 0$. $\Rightarrow$ ϕ constant along lines of force ...

and can formulate dW in terms ϕ, or ....

$\Rightarrow$ consider interchange, with flux conservation



$$\Phi_1 = \Phi_2$$

Does interchange raise or lower energy?

$$\Delta E = [\text{final energy of } 1] - [\text{initial energy of } 1] + [\text{final energy of } 2] - [\text{initial energy of } 2]$$

where interchange

a) "puts" 1 into 2 slot
"puts" 2 into 1 slot

b) keeps $p\rho^{-\gamma} = p V^{\gamma} = \text{const}$

$V \equiv$ volume of flux tube

$\Rightarrow$ final energy of 1 $\rightarrow \underbrace{(\text{new } p)}_{p_1'} V_2 / (\gamma - 1)$

final energy of 2 $\rightarrow \underbrace{(\text{new } p)}_{p_2'} V_1 / \gamma - 1$

so ...

$$\Delta E = \Delta W = \frac{1}{(\gamma-1)}\left[(P_1' V_2 - P_1 V_1) + (P_2' V_1 - P_2 V_2)\right]$$

and
$$\left.\begin{aligned} P_1' V_2^{\gamma} &= P_1 V_1^{\gamma} \\ P_2' V_1^{\gamma} &= P_2 V_2^{\gamma} \end{aligned}\right\}$$

from eqn. state
$P' \equiv$ pressures of displaced flux tubes
(argument akin to Schwarzschild)

$\Rightarrow$

$$(\gamma-1)\Delta W = \left\{ P_1\left[\left(\frac{V_1}{V_2}\right)^{\gamma} V_2 - V_1\right] + P_2\left[\left(\frac{V_2}{V_1}\right)^{\gamma} V_1 - V_2\right]\right\}$$

$$V_2 = V_1 + \delta V$$
$$P_2 = P_1 + \delta P$$

$$(\Delta W)(\gamma-1) = \left\{ P_1\left[\left(\frac{V_1}{V_1+\delta V}\right)^{\gamma}(V_1+\delta V) - V_1\right] + (P_1+\delta P)\left[\left(\frac{V_1+\delta V}{V_1}\right)^{\gamma} V_1 - (V_1+\delta V)\right]\right\}$$

$$(\gamma-1)\,\Delta W = \Big\{ P_1 V_1 \Big[\Big(1+\frac{\delta V}{V}\Big)^{-(\gamma-1)} - 1 \Big]$$

$$+ \; P_1 V_1 \Big(1+\frac{\delta P}{P}\Big)\Big[\Big(1+\frac{\delta V}{V}\Big)^{\gamma} - \Big(1+\frac{\delta V}{V}\Big) \Big] \Big\}$$

$$= P_1 V_1 \Big\{ \Big[\cancel{1} - (\gamma-1)\frac{\delta V}{V} + \frac{(\gamma-1)(\gamma)}{2}\Big(\frac{\delta V}{V}\Big)^2 - \cancel{1} \Big]$$

$$+ \Big(1+\frac{\delta P}{P}\Big)\Big[\cancel{1} + \gamma\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\Big(\frac{\delta V}{V}\Big)^2 - \cancel{1} - \frac{\delta V}{V} \Big] \Big\}$$

$$= P_1 V_1 \Big\{ \cancel{-(\gamma-1)\frac{\delta V}{V}} + \frac{(\gamma-1)(\gamma)}{2}\Big(\frac{\delta V}{V}\Big)^2$$

$$+ \cancel{\gamma\frac{\delta V}{V}} - \cancel{\frac{\delta V}{V}} + \frac{\delta P}{P}(\gamma-1)\frac{\delta V}{V} + \gamma\frac{(\gamma-1)}{2}\Big(\frac{\delta V}{V}\Big)^2 \Big\}$$

$$\boxed{\frac{\Delta W}{P_1 V_1} = \underbrace{\gamma\Big(\frac{\delta V}{V}\Big)^2}_{>0} + \underbrace{\frac{\delta P}{P}\frac{\delta V}{V}}_{>0 \text{ or } <0}}$$

$\Rightarrow$ generic expression for interchange δW

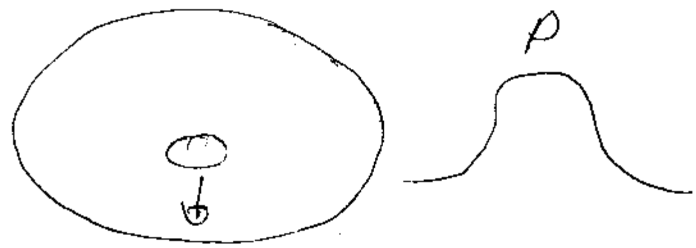
→ Clearly,

$$\frac{\delta V}{V} \sim (\underline{\nabla} \cdot \underline{\xi}) \quad , \quad \frac{\delta p}{p} \sim \quad \underline{\xi} \cdot \underline{\nabla} p$$

and

→ expansion free energy relaxation ⇒

$\underline{\delta p < 0}$ → i.e.



pressure higher in center so, $\delta p < 0$ ⇒ relaxation occurs.

∴ Key is sign $\frac{\delta V}{V}$ → > 0 → instability; < 0 → stability

→ Now, for flute perturbation $(k_{\parallel} = 0)$

$$V = \int S \, d\ell$$

$S \equiv$ cross-sectional area of tube.

but $\Phi = B(\ell)\, S(\ell) = \text{const}$

$\Rightarrow$

$$V = \Phi \int \frac{dl}{B} \quad \Rightarrow \quad \frac{\delta V}{V} < 0$$

$\Rightarrow$ $$\boxed{\delta \int \frac{dl}{B} < 0}$$ $\rightarrow$ condition for interchange stability

$$\underset{<0}{\delta p}\ \underset{<0}{\delta V} > 0$$

→ content of criterion is that configuration should have a minimum in B in the core, to confine pressure

i.e.

then stable if:

$$\delta \int \frac{dl}{B} < 0 \, .$$

$\Rightarrow$ "minimum B" criterion for stability.

→ if define ψ → label of surface enclosing const flux Φ

$\therefore$ $V(\psi) \equiv$ volume enclosed by flux surface

$p(\psi) \equiv$ pressure enclosed

$dp/d\psi < 0 \Rightarrow$ need $\dfrac{d^2V}{d\psi^2} > 0$

$\Longleftrightarrow$ minimum-B

→ Can re-write instability criterion

$$\delta W = p_1 \, \delta V \left(\gamma \frac{\delta V}{V_1} + \frac{\delta p}{p_1} \right)$$

$$= p_1 \, \delta V \left[\delta \ln (p V^{\gamma}) \right]$$

so $\delta (p V^{\gamma}) < 0$ → inst. (cf. Schwarzschild)

Also, if tube ① has flux ψ, then

$$V = u\,\psi$$

$\Rightarrow$

$$\frac{\delta w}{\psi} = p\,\delta u\,\frac{\delta(p u^{\gamma})}{p u^{\gamma}} < 0$$

→ What does it Mean ?

$$\underset{\text{volume}}{V} = \int dl\, A = \Phi \int \frac{dl}{B}$$

now $\underline{\nabla} p$ → "expansion free energy"

$\delta V > 0 \Rightarrow \delta \int \frac{dl}{B} > 0$ → fluid element expands
⇒ tends reduce W_p

$\delta V < 0 \Rightarrow \delta \int \frac{dl}{B} < 0$ → fluid element compresses
⇒ tends increase W_p

$\delta V > 0$ → 'maximum B'
$\delta V < 0$ → 'minimum B'

Can then define:

$$\underset{\text{potential energy of tube}}{E_p} = -p\, U \quad , \quad U = -\oint \frac{dl}{B}$$

(i.e. for sign convention)

∴ → can argue tube tends to move in direction of lower U.
→ equilibrium for $p = p(U)$

then, not surprisingly, can develop parallel between convection and interchange

i.e.

Convection	Interchange
gravitational potential energy	E_p → expansion energy
blob	flux tube
displace blob	displace tube
$\rho' < \rho_{ambient}$ ⟹ buoyant rise	$\frac{\delta V}{V} > 0$ ⟹ expansion continues (with $dp/d\psi < 0$) (squeezed out)
adiabatic profile $\frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}$	adiabatic displacement $-\gamma p \frac{\delta \psi}{\psi} = \frac{dp}{d\psi} \delta\psi$
Schwarzschild Criterion $\frac{\nabla p}{p} < \gamma \frac{\nabla \rho}{\rho}$ for instability	Interchange Criterion $\frac{dp}{d\psi} \delta\psi < -\gamma p \frac{\delta\psi}{\psi}$ ⟹ $\left\lvert \frac{dp}{d\psi} \right\rvert > -\frac{\gamma p}{\psi}$ ∇p sufficiently steep → overcome compr.

∴ for instability: $\left|\frac{dP}{dU}\right| > \frac{-\gamma P}{U}$

$\left|\frac{dP}{dU}\right|$: change from relaxation

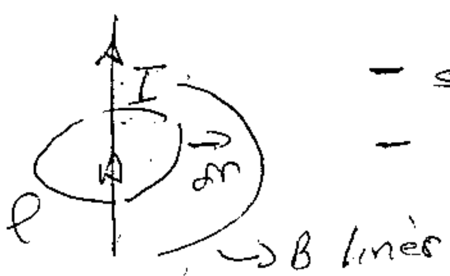
$\frac{-\gamma P}{U}$ → adiabatic pressure change

for stability, need:

$$\boxed{\left|\frac{dP}{dU}\right| < \frac{\gamma P}{|U|}}$$

→ Consider some configurations (magnetic)

a.) single wire



- stability to displacement δr
- $\nabla P/P$ limit ?

now $\delta \int \frac{d\ell}{B}$

$$d\ell = 2\pi r$$

$$B = 2I/r$$

$$d\ell/B \sim \frac{\pi}{I} r^2$$

→ wire is not "minimum-B"!

i.e. actually maximal

→ will have a ∇P_{crit}.

for ∇p limit: $\quad \dfrac{dp}{dU} < \dfrac{\gamma p}{|U|}$

$$U \equiv -\int \frac{dl}{B} \sim -r^2$$

$$\frac{dp}{dU} = \frac{dp}{dr}\frac{dr}{dU}$$

$\boxed{U \text{ scalar} \Rightarrow \text{I cancels}}$

$$= \left|\frac{dp}{dr}\right|\left(\frac{1}{2r}\right) \quad \Rightarrow \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{\gamma(2r)}{r^2}$$

$$\therefore \left|\frac{1}{p}\frac{dp}{dr}\right| < \frac{2\gamma}{r} \quad \Rightarrow \boxed{\left|\frac{d\ln p}{d\ln r}\right| < 2\gamma}$$

$\Rightarrow$ imposes limit on pressure gradient for interchange stability. $\Rightarrow$ "β limits"

b)

can approach point dipole similarly $\rightarrow$ i.e. earth.

i.e. $B \sim 1/r^3$

$dl \sim r$ $\quad \Rightarrow \quad U \sim -r^4$

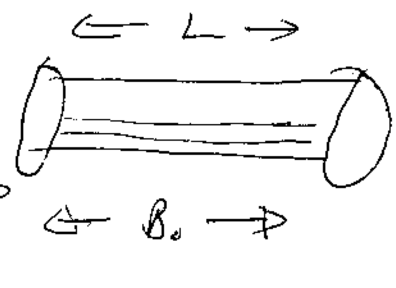
similar reasoning $\Rightarrow -\dfrac{d\ln p}{d\ln r} < 4\gamma$

→ Line Tying and Conducting End Plates

- Till now, have ignored boundary

⇒ consider plasma between two conducting end plates

i.e.

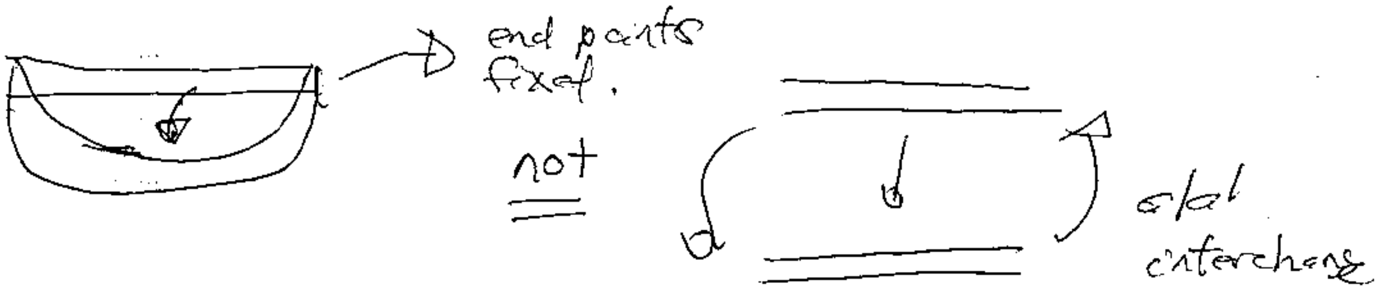


$\underline{E}_t = 0$ on plate

$\Rightarrow \underline{\xi}\Big|_{plate} = 0$

lines are fixed

i.e. displacement has form:



⇒ field lines bent!

now, $\underline{U}_0 = 0$ ⇒

$$\delta W = \int d^3x \left[\frac{\underline{Q}^2}{8\pi} + \gamma p (\underline{\nabla} \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \underline{\nabla} p_0)(\underline{\nabla} \cdot \underline{\xi}) \right]$$

$$\underline{Q} = \underline{\nabla} \times \underline{\xi} \times \underline{B_0}$$

$$= \underline{B_0} \cdot \underline{\nabla} \underline{\xi} - \underline{\xi} \cdot \underline{\nabla} \underline{B_0} - \underline{B_0}\, \underline{\nabla} \cdot \underline{\xi}$$

$\underline{\nabla} \cdot \underline{\xi} \mp 0$

new stabilizing effect

$$\delta W = \int d^3x \left[\frac{(\underline{B_0} \cdot \underline{\nabla} \underline{\xi} - \underline{B_0}\, \underline{\nabla} \cdot \underline{\xi})^2}{8\pi} + \gamma p (\underline{\nabla} \cdot \underline{\xi})^2 + (\underline{\xi} \cdot \underline{\nabla} p_0)\, \underline{\nabla} \cdot \underline{\xi} \right]$$

i.e. can't take $\underline{B_0} \cdot \underline{\nabla}\, \underline{\xi} = 0$ any more

so $\quad Q \sim B_0 \dfrac{\partial \xi_r}{\partial z}$ $\quad$ i.e. can make $(\underline{\nabla} \cdot \underline{\xi}) \underline{B_0}$ smaller .. i.e. $\underline{\nabla} \cdot \underline{V} \cong 0$.

$$\delta W \sim V \left[\underbrace{\frac{B_0^2}{8\pi} \left(\frac{\partial \xi_r}{\partial z} \right)^2}_{\text{new}} + \underbrace{\gamma p \left(\frac{\delta u}{u} \right)^2 + \delta p \frac{\delta u}{u}}_{\text{old}} \right]$$

i.e. schematic ....

$$\frac{\partial \xi_r}{\partial z} \sim \frac{\xi_r}{L} \qquad \frac{\delta u}{u} = \frac{\nabla u}{u} \xi_r$$

$$\delta p = \nabla p \, \xi_r$$

$\Rightarrow$

$$\delta W \sim V \left\{ \left(\frac{B_0^2}{8\pi L^2} + \gamma P \left(\frac{\nabla U}{U}\right)^2 + \nabla P \frac{\nabla U}{U} \right) \varepsilon^2 \right)$$

$\therefore$ $\delta W < 0 \to$ instability $\Rightarrow$

instability if $\quad -\nabla P \frac{\nabla U}{U} < \gamma P \left(\frac{\nabla U}{U}\right)^2 + \underbrace{\frac{B^2}{8\pi L^2}}_{\text{additional stabilizing effect}}$

$\Rightarrow$ line tying raises
critical pressure gradient

$\Rightarrow$ clearly stabilizing $\Rightarrow$ β limit!

Physics $\rightarrow$ fixing end points forces
bending of field lines

$\rightarrow$ lose interchange structure

$\Rightarrow$ energy expended coupling to
plucking magnetic field lines.