# MHD Turbulence II

→ Anisotropic Cascades and Critical Balance → A Closer Look.

→ Extending the 4/5 Law.

→ Selective Decay and Relaxation.

→ 2D MHD – A Study in Turbulent Relaxation.

## i.) Anisotropic Cascades and Critical Balance – A Closer Look.

Recall: I-K Phenomenology:

$$\epsilon \cong \frac{Z(\ell)^2}{\tau_{Tr}(\ell)} \qquad (\underline{B_0} \text{ weak, but } B_{rms} \text{ generated})$$

$$1/\tau_{Tr}(\ell) \cong \frac{Z(\ell)^2}{\ell^2}\tau_A \qquad \tau_A \sim \frac{\ell}{b_0}$$

($b_0 \to$ rms)

$$\Rightarrow Z_\ell \sim (b_0 \epsilon)^{1/4} \ell^{1/4}$$

$$E(k) \sim \sqrt{\epsilon b_0}\, k^{-3/2}$$

and with strong $\underline{B_0}$:

$$\epsilon \sim \frac{Z(\ell_\perp)^2 Z(\ell_\perp^2)}{\ell_\perp^2 / |k_\parallel V_A|}$$

so W.T.T. Alfvenic cascade:

$$E(k_\perp, k_\parallel) \sim (\epsilon V_A)^{1/2} / k_\parallel^{1/2} k_\perp^2$$

~ "hard" in $k_\perp$ !

However, note:

$$z(\ell_\perp) \sim \delta B(\ell_\perp) \sim (\epsilon |k_\parallel V_A|)^{1/4} \ell_\perp^{1/2}$$

so

$$\delta B \cdot \nabla_\perp \sim \frac{1}{\ell_\perp^{1/2}} (\epsilon |k_\parallel V_A|)^{1/4}$$

But recall:

- Alfven wave:

$$\omega \simeq k_\parallel V_A$$

derived from:

$$\partial_t A = B_0 \nabla_\parallel \phi + \ldots$$

$$\partial_t \nabla^2 \phi = B_0 \nabla_\parallel J_\parallel + \ldots$$

$$\nabla_\parallel = \underset{\text{Linear}}{\partial_z} + \underset{\text{Nonlinear}}{\frac{\delta B_\perp}{B_0} \cdot \nabla_\perp}$$

Ratio $\frac{\text{Nonlinear}}{\text{Linear}} = Ku = \frac{\frac{\delta B_\perp}{B_0} \cdot \nabla_\perp}{\partial z}$

$\downarrow$

Kubo #

$\Rightarrow$

$$Ku \sim \frac{\delta B \, l_{ac}}{B_0 \, \Delta_\perp}$$

$l_{ac} \rightarrow$ parallel autocorrelation length

$\Delta_\perp \rightarrow \perp$ correlation length.

see stochastic fields discussion of Phys 235 2016.

Point:
$$B_0 \partial_z C + \delta B_\perp \cdot \nabla_\perp C = 0$$

$$\partial_z C + \frac{\delta B_\perp}{B_0} \cdot \nabla_\perp C = 0$$

$Ku < 1 \rightarrow$ C evolves by many kicks in $\Delta_\perp$

$\rightarrow$ <u>diffusion</u>

$\rightarrow$ in WTT wave interactions are diffusive in character.

$Ku > 1 \rightarrow$ C scattered $> \Delta_\perp$ in one step

$\rightarrow$ for transport in random media $\rightarrow$ percolation

Analogy $\partial_t C + \underline{v} \cdot \nabla C = 0$

$$Ku = \frac{\tilde{v}\, \tau_{ac}}{\Delta_\perp}$$

So we have a concern:

→ physics of [redacted] MHD turbulence understood in terms of Alfven wave interactions.

→ but scalings of WTT spectrum suggest that wave character lost as cascade progresses

i.e.

$$Ku \sim \frac{k_\parallel^{-1}}{\ell_\perp^{1/2}} \left[ \epsilon \, |k_\parallel V_A| \right]^{1/4}$$

$\uparrow$ as $\ell_\perp \downarrow$

i.e. How high can Ku # go and still be consistent with physics of Alfven Wave Cascade

⇒ $\boxed{\text{Critical Balance Conjecture}}$

(GS 95, K+P '78)

$\Rightarrow$ $\boxed{\text{MHD inertial range in strong field will set at } ku \sim 1.}$

i.e. $\rightarrow \delta B_\perp \cdot \nabla_\perp \sim \dfrac{Z(\ell_\perp)}{\ell_\perp} \sim B_0 \nabla_\parallel$

so $\dfrac{Z(\ell_\perp)}{\ell_\perp} \simeq k_\parallel V_A$

$\rightarrow \dfrac{\tau_A}{\tau_{eddy}} \rightarrow 1 \qquad \tau_{Tr} \rightarrow \tau_{eddy} \sim \tau_A.$

i.e. all timescales equalize

$\Rightarrow ku \sim 1$ is maximum $ku$ and still retain Alfvenic character.

$\rightarrow$ why?

Recall:

- WTT $\quad \tau_{Tc} \sim \tau_{ac\,wave}$ $\quad$ {Triad coherence set by wave dispersion

$$\rightarrow \pi\,\delta(\omega_{\underline{k}} - \omega_{\underline{k}''} - \omega_{\underline{k}'})$$

- STT - Renormalized Theory

$\tau_{Tc} \sim \tau_c$ $\quad$ {Triad coherence set by nonlinear scattering, itself.

$$\rightarrow 1/\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''}$$

$$= \Theta_{\underline{k},\underline{k}',\underline{k}''}$$

So, renormalized wave interaction theory $\Rightarrow$

$$\Theta_{\underline{k},\underline{k}',\underline{k}''} = \frac{\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''}}{(\omega_{\underline{k}} - \omega_{\underline{k}''} - \omega_{\underline{k}'})^2 + (\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''})^2}$$

$\rightarrow$ recovers both limits ✓

Now, $\Theta_{\underline{k},\underline{k}',\underline{k}''}$ clearly sets $\tau_{Tr}$.

So, can re-write phenomenological transfer balance as:

$$\epsilon \sim \frac{1}{\ell_1^2}\,\frac{Z(\ell_1)^2\, Z(\ell_1)^2}{\widetilde{\tau}_c(\ell_1)}$$

$$1/\tau_c(\ell_1) = 1\Big/\left[\overset{①}{(k_{\parallel} V_A)^2} + \overset{②}{\left(\frac{Z(\ell_1)}{\ell_1}\right)^2}\right]^{1/2}$$

comparable at $k_{\parallel} \sim 1$

by analogy with $\Theta_{\underline{k},\underline{k}',\underline{k}''}$.

① > ② $\rightarrow$ W.T.T.

① $\leq$ ② $\rightarrow$ S.T.T.

$$\epsilon \sim \frac{1}{\ell_\perp} \frac{z(\ell_\perp)^2 z(\ell_\perp)^2}{z(\ell_\perp)/\ell_\perp}$$

$$\sim z(\ell_\perp)^3/\ell_\perp$$

and $\boxed{z(\ell_\perp) \sim (\epsilon \ell_\perp)^{1/3}}$ — Back to $k$ 4).!

Point: $\langle z(k)^2\rangle \sim \epsilon^{2/3} k_\perp^{-5/3}$ — G-S spectrum.

but different physics!

— softer than W.T.T.

— Great Power Law on sky"

— $\frac{z(\ell_\perp)}{\ell_\perp}$ vs. $k_\parallel V_A$

$$\frac{(\epsilon \ell_\perp)^{1/3}}{\ell_\perp} \sim \frac{\epsilon^{1/3}}{\ell_\perp^{2/3}}$$

$\rightarrow$ rate increases as $\ell_\perp \downarrow$

$\rightarrow$ contrast constant $k_\parallel V_A$!

— $\frac{z(\ell_\perp)}{\ell_\perp} \sim \frac{\epsilon^{1/3}}{\ell_\perp^{2/3}} \rightarrow \frac{\delta B}{B_0} \cdot \nabla_\perp$

then $k_\parallel \sim 1$ ⇒

$$\frac{\epsilon^{1/3}}{\ell_\perp^{2/3}} \sim k_\parallel$$

⇒ $\boxed{k_\parallel \sim \epsilon^{1/3} k_\perp^{2/3}}$ — GS cone.

→ - Critical Balance is a hypothesis

- Plausible answer to question of "how maintain Alfvenic cascade in state of strong (i.e. non-weak) turbulence?"
- anisotropy of spectrum supported by simulations (cf. Galtier).

BUT

- hypothesis, only

→ Computational support semi-quantitative.

→ 5/3 vs 3/2 etc. still ongoing.

→ A word about triads.

In wave turbulence cascade, must satisfy:

$$\underline{k} = \underline{p} + \underline{q}$$

$$\omega_{\underline{k}} = \omega_{\underline{p}} \pm \omega_{\underline{q}} \qquad \text{(WTT)}$$

→ resonance criterion

Conditions satisfied by:

$$q_{\parallel} = 0 \qquad \text{(i.e. } \underline{q} \text{ is a cell, driven by beats)}$$

so $k_{\parallel} = p_{\parallel}$

$$\underline{k}_{\perp} = \underline{p}_{\perp} + \underline{q}_{\perp}$$

and $\omega_{k} = \omega_{p} \pm \cancel{0}$

⇒ – deformation of Alfvenic wave packet directly related to its interaction with 2D part of wave packet travelling in opposite direction.

– interaction passive w/r $k_{\parallel}$
→ $\perp$ transfer in long time limit.

ii.) 4/5 Law – See Lecture I.

iii.) Cascades and Relaxation

$\Rightarrow$ Selective Decay

Recall: Taylor Relaxation { 3D; 2D – "Taylor in Flatland"

Argued: $\int d^3x \, B^2/8\pi$ minimized subject to constraint of $\int d^3x \, A \cdot B$ conserved.

$\Rightarrow \quad J_{\parallel} = J \cdot B / B^2 \rightarrow$ const.

(2D $J/A \rightarrow$ const.).

Arguements heuristic. { Power counting ($k_\perp$); stoch. fields; ⋮ }

Now, – dissipation at small scale $\eta, \nu$

– expect energy transfer to small scale!

⋮

- Inverse {cascade / transfer} of magnetic helicity would set up "selective decay" scenario

i.e. magnetic energy scattered to small scale and dissipated
$\Rightarrow$ relaxation

magnetic helicity inverse cascades
$\Rightarrow$ avoids dissipation. Constraint; as survives.

c.f. {Frisch (75), Pouquet, et. al. (76) (posted)
see also: Montgomery

- Why, where from?

$\Rightarrow$ Primarily: Statistical Mechanics

$\Rightarrow$ c.f.: Frisch '75, though not transparent.

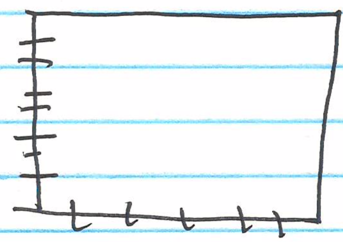
easier $\rightarrow$ "Taylor in Flatland" Problem.

Recall: Relaxation $\begin{cases} \text{minimizes } \langle B^2 \rangle \\ \text{conserving } \langle A^2 \rangle \end{cases}$.

Does this follow from Selective Decay?

$\Rightarrow$ Explore Absolute Equilibrium

i.e.



finite box

kmin kmax

- remove forcing, dissipation etc.
- input excitation.

For 2D MHD (ignoring cross helicity):

have $A \Rightarrow X_i$

$\hookrightarrow$ modal amplitude

so

$$E_M = \sum_{i=1}^{N} k_i^2 X_i^2$$

$$H = \sum_{i=1}^{N} X_i^2 \qquad - \langle A^2 \rangle$$

$$\phi \to Y_i$$

$$E_k = \sum_{i=1}^{N} k_i^2 Y_i^2$$

Now,

$$\left.\begin{array}{l} H \to \alpha \\ E = E_\alpha + E_k \to \beta \end{array}\right\} \text{conserved}$$

conserved, so pdf of this closed system is given by micro-canonical ensemble/distribution:

$$P(x,y) = \underset{\substack{\downarrow \\ \text{norm}}}{c} \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2 + \beta k_i^2 Y_i^2\right]$$

and can integrate out $y_i$ (KE) part, so:

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2\right]$$

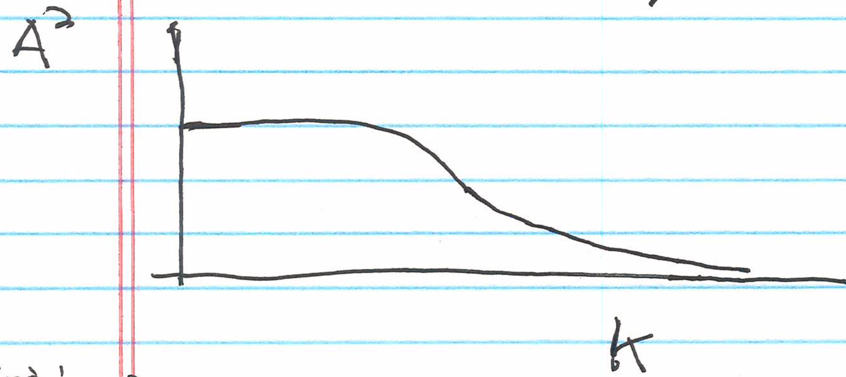
then:

$$\langle A^2(k) \rangle = \int dX_i \, X_i^2 \, P(X_i)$$

$$= 1/(\alpha + \beta k^2)$$

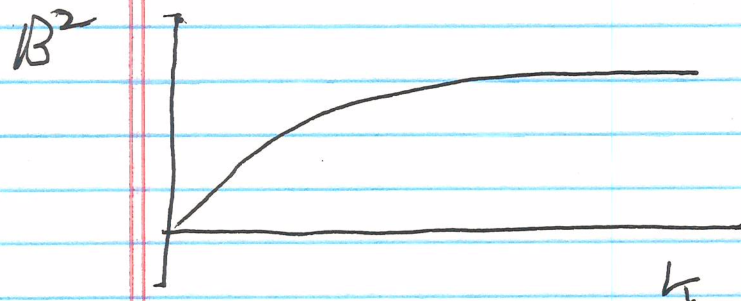
$$\langle B^2(k) \rangle = \left[k^2/(\alpha + \beta k^2)\right]$$

observe immediately:




"$A^2$ wants remain at large scale"

$k_{min} \to 0$




"$B^2$ approaches equipartition"

$\Rightarrow$ $A^2$ distribution most populated at larger scales. Decays at small

$\Rightarrow$ $B^2$ distribution most populated at smaller. Approaches equipartition at small scale.

$\therefore$ - suggests $A^2$ populates large scales, $B^2$ approaches equipartition.

- suggestive of inverse cascade of $A^2$, along with forward cascade of energy.

- supports Selective Decay Hypothesis as foundation for "Taylor in Flatland".

- similar story for Magnetic Helicity, though more laborious.

N.B. For 2D Fluid:

$$E = \int d^2x \, (\nabla\phi)^2 \qquad \text{- energy}$$

$$\Omega = \int d^2y \, (\nabla^2\phi)^2 \qquad \text{- enstrophy}$$

$$\Omega_i = k_i^2 E_i$$

$$\underline{V} \rightarrow \underline{X}_i$$

$\therefore$

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2\right]$$

so

$$E(k) = \langle V^2(k) \rangle = 1/(\alpha + \beta k^2)$$

$$\Omega(k) = k^2/(\alpha + \beta k^2)$$

similar suggestion of dual cascade and minimum enstrophy state.

→ Is this story true?

⇒ What does dynamics tell us?
∴ Consider interactions in 2D MHD.

Observe:

- Reduced MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = B_0 \partial_z \phi + \eta \nabla_\perp^2 \psi$$

$\Updownarrow$

- 2D MHD

$$\frac{\partial \psi}{\partial t} + \underline{\nabla}_\perp \phi \times \hat{z} \cdot \underline{\nabla}_\perp \psi = \eta \nabla_\perp^2 \psi$$

so; with strong $\underline{B_0}$:

$$\langle A \cdot B \rangle \rightarrow \langle \psi \rangle \underline{B_0}$$

so mean $\langle \psi \rangle$ in 2D ~ captures magnetic helicity dynamics in strongly magnetized system.

For $\langle A^2 \rangle_k$ transfer, consider closure of $\partial_t \langle A^2 \rangle$ equation, much akin to wave kinetics, though closure required.

See: Diamond, Hughes, Kim (posted).

Can write (see DHK):

$$\frac{1}{2}\left[\partial_t \langle A^2\rangle_k + T(k)\right] = -\Gamma_A(k)\frac{\partial \langle A\rangle}{\partial x} - \eta\langle B^2\rangle_k$$

triplet $\langle \nabla\cdot\langle \underline{V} A^2\rangle \rangle_k$

flux $\Gamma_A = \left[\tilde{\tau}_c^{\phi}(k)\langle V^2\rangle_k - \tilde{\tau}_c^{A}(k)\langle B^2\rangle_k\right]$

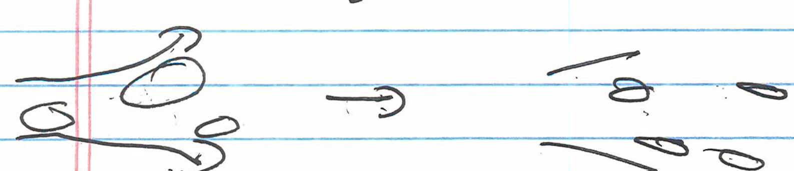
$$T_k = \sum_{\underline{k}'} \left(\underline{k}\cdot\underline{k}'\times\hat{z}\right)^2 \overset{(1)}{\Theta}_{k,k',k''}\left\{\langle\phi^2\rangle_{\underline{k}'} \overset{(2)}{-} \frac{(k'^2-k^2)}{|\underline{k}+\underline{k}'|^2}\langle A^2\rangle_{\underline{k}'}\right\}\{A^2_{\underline{k}}\}$$

$$\overset{(3)}{-\sum_{\substack{\underline{k}=\underline{p}+\underline{q}\\ \underline{p},\underline{q}}}} \left(\underline{p}\cdot\underline{q}\times\hat{z}\right)^2 \Theta_{\underline{k},\underline{p},\underline{q}}\langle A^2\rangle_{\underline{p}}\langle\phi^2\rangle_{\underline{q}}$$

(1), (3) → coherent damping, incoherent emission

→ akin to scattering of passive scalar, → small scale / chop-up.

→ conserve $\langle\psi^2\rangle$ upon $\sum_k$ together.



(2) → coherent damping/growth - from back reaction ($J \times B$) onto Ohm's Law:

→ reshuffle $\langle A^2 \rangle$ to larger scale. Sign $k'^2$ vs $k^2$ !

→ $\sum_k$ conserves $\langle A^2 \rangle$ independently.

→ correspondence to condensation of wires (currents) attracting.

(1) + (2) → net effective resistivity sign.

→ see $\Gamma_A$, too.
- 'negative resistivity'
- Alfvenized state

⇒ $E_K > E_M$ ⇒ $\langle A^2 \rangle_k$ shuffled to smaller scale.

$E_M < E_K$ ⇒ $\langle A^2 \rangle_k$ transferred to larger scale.

and

transfer need not be local!

⇒ In dynamics [redacted] $\langle A^2\rangle$ ; $\langle A\cdot B\rangle$ evolution is complex.

⇒ N.B. Recall flux expulsion:

$\frac{V_{A0}^2}{\tilde{V}^2} R_m < 1 \rightarrow$ A passive, B expelled

$> 1 \rightarrow J\times B$ disrupts vortex, expulsion stops

$$\Rightarrow \quad B_0^2 \lessgtr \rho_0 \langle \tilde{V}^2\rangle / R_m$$

but $\langle \tilde{B}^2\rangle \gg B_0^2$, upon stretching, weak $B_0$ is sufficient!

Zeldovich:

$$\frac{\partial A}{\partial t} + \underline{v}\cdot\underline{\nabla} A = -v_r \frac{\partial \langle A\rangle}{\partial x} + \eta \nabla^2 A$$

$\times A$ and avg. ⇒

$$\eta \langle \tilde{B}^2\rangle = \langle \tilde{v}_r \tilde{A}\rangle \frac{\partial \langle A\rangle}{\partial x}$$

$$\langle \tilde{B}^2\rangle = \frac{\eta_T}{\eta} B_0^2$$

$$= \frac{\tilde{v}\,\tilde{v}\,\tau_c}{\eta} B_0^2 \cong R_m B_0^2 \quad \checkmark.$$

so, crudely:

$$\langle \tilde{B}^2 \rangle / \cancel{R_m} \lessgtr \rho \langle \tilde{V}^2 \rangle / \cancel{R_m} \quad \checkmark$$

$\Rightarrow$ Questions still open!

$\therefore$ Taylor conjecture remains a conjecture!