PHYS218B: Final Project
Hyper Resistivity in Presence of Stochastic Fields

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Abstract

In 1974, J. B. Taylor (Taylor, 1974) proposed a self-organized theory of turbulent magnetic relaxation. This theory describes that a topological configuration of magnetic field at the end-state will evolve to a configuration that has minimum magnetic energy. This turbulent relaxation process is subjected to the conservation of large-scale magnetic helicity $\mathcal{H}$, which is defines as $\mathcal{H}_0 \equiv \int_{\text{all space}} \mathbf{A}\cdot\mathbf{B} d^3x = const.$, where $\mathbf{A}$ is the magnetic potential field and $\mathbf{B}$ is the magnetic field. This theory has no rigorous justification and its evolution process remains unknown. At this stage, we know that the evolution of Taylor relaxation is related to hyper resistivity and that, with criteria of the helicity does not change the large-scale $\mathcal{H}_0$ and it dissipates magnetic energy, the hyper resistivity can be derived. We are interested in the hyper resistivity with a tangle magnetic field in presence of a weak mean field which is strong enough to break the symmetry but weak enough to kill all the flow dynamo and to be modified the current and hence defines the hyper resistivity. This weak mean field regime require a model beyond simple quasi-linear closure. Here we present a "double-averag" method and use the mean field theory to obtain the hyper resistivity analytically.

1 Taylor Relaxation and Helicity

Taylor (1974) proposed that small scale tube surfaces are destroyed by magnetic reconnections that caused by microturbulence, resistivity, and some other departure from perfect conductivity. In classes, we have prove that the magnetic helicity will inverse decay and accumulate at large scale by using Lagrange multiplier and minimizing the magnetic energy at the relaxation state. That is

$$\delta \int d^3x\left(\frac{b^2}{8\pi} + \lambda \mathbf{A}\cdot\mathbf{B}\right) = 0. \tag{1}$$

Taylor relaxation stating that any departure from the perfectly conductive leads to the invariant of magnetic helicity at large scales, indicating that the topological structure of magnetic fields is invariant at large scale. This invariant is unique, and depends only on the helicity at larger scale $\mathcal{H}_0 \equiv \int_{\text{all space}} \mathbf{A}\cdot\mathbf{B} d^3x = const$ and the toroidal flux ψ in a toroidal container (Taylor, 1974), or equivalently, on the pinch ratio $\theta = \mathcal{H}/\psi^2$. All tube surfaces are broken by the mechanism of reconnection and hence the small scale helicity is not conserved in every individual tube. Time asymptotically, only the large scale helicity will survive, leading to the

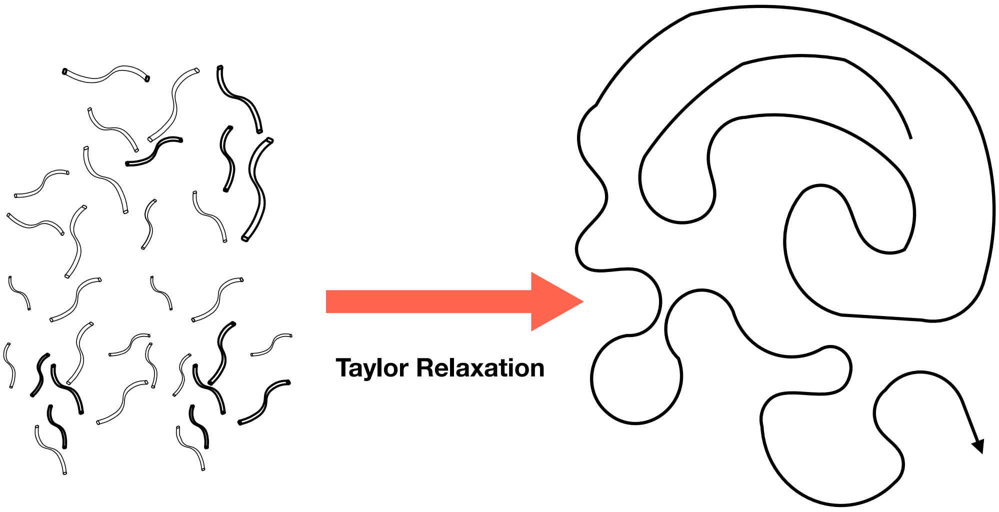

Figure 1: Taylor Relaxation. All the small scale tube surfaces are destroyed while the magnetic helicity decay to larger scales. At the relaxation state, large scale magnetic field can be pictured as a one distorted field line.

"ruggedness" of global helicity. This indicates that the magnetic field has one field line, which still demonstrates the stochasticity ("ruggedness") of the field at larger scales (see figure 2).

In 3D MHD, the energy is decay to small scales (forward cascade), while the magnetic energy is decay to larger scales, which have minimal coupling to the dissipation. At larger scale, magnetic energy decay, but the total energy (fluid energy plus the magnetic energy) remains invariant.

$$\int_{allspace} \langle \nabla\phi \rangle^2 + \langle B \rangle^2, \tag{2}$$

where ϕ is the stream function and the angle bracket $\langle\ \rangle$ is the ensemble average over a larger scale. The above discussion is also known as the **selective decay** in 3D MHD.

At the relaxation state, the current is "flattened", i.e. $\frac{\mathbf{J}\cdot\mathbf{B}}{B^2} \to const$ or $\langle J_{\|} \rangle_{\text{Taylor}} = const$. This indicates the free energy from current gradient in unavailable.

2 Hyper Resistivity Diffusion

One may ask *how does a system evolve to Taylor state*? Before the system evolve to the relaxation state, we'll have $J_{\|} \nrightarrow 0$. The diffusion of current in parallel direction contributes an additional term in Ohm's Law equation.

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta\mathbf{J} + \mathbf{S}, \tag{3}$$

where η is the classical Spitzer resistivity and $\mathbf{S}$ is "something" unsolved. The "something" $\mathbf{S}$ must satisfied two conditions:

- $\mathbf{S}$ must not change the magnetic helicity $\mathcal{H}_0$.
- $\mathbf{S}$ must dissipate magnetic energy at larger scale.

In classes, we use the first criterion ($\frac{\partial}{\partial t}\int d^3x\langle A\rangle\langle B\rangle = 0$) to derived that $\mathbf{S}$ must has a form so that it can be related to the flux of magnetic helicity (Γ_H):

$$\mathbf{S} = (\frac{\hat{\mathbf{b}}}{B})\nabla\cdot\Gamma_H, \tag{4}$$

where $\hat{\mathbf{b}}$ is the unit vector in the direction that is parallel to the magnetic field. The second criterion gives

$$\frac{\partial}{\partial t}\int d^3x\frac{\langle B^2\rangle}{8\pi} = -\int \eta J^2 d^3x + E_{\mathbf{S}}, \tag{5}$$

$$E_{\mathbf{S}} < 0, \tag{6}$$

where $E_{\mathbf{S}}$ is the power modification of magnetic energy caused by the "something". So we have

$$\Gamma_H = -\lambda\nabla\frac{J_{\|}}{B}, \tag{7}$$

where λ is an arbitrary positive function. Ans the parallel term in 3 can be rewritten with the diffusion of current:

$$\langle E_{\|}\rangle = \eta\langle J_{\|}\rangle - \nabla_{\perp}\lambda\nabla_{\perp}\langle J_{\|}\rangle. \tag{8}$$

Note that in above equation, the λ absorb the B and other terms. Equation 8 can be written as

$$\frac{\partial}{\partial t}\langle A_{\|}\rangle = \langle\hat{\mathbf{b}}\cdot\nabla\phi\rangle - \eta\langle J_{\|}\rangle + \nabla_{\perp}(-\lambda\nabla_{\perp}\nabla^2\langle A_{\|}\rangle) \tag{9}$$

$$= \langle\hat{\mathbf{b}}\cdot\nabla\phi\rangle - \eta\langle J_{\|}\rangle + \nabla_{\perp}(-l^2 D_J\nabla_{\perp}\nabla^2\langle A_{\|}\rangle), \tag{10}$$

where ϕ is velocity stream function and D_J is the current diffusivity with dimension $[\frac{m^2}{s}]$ and the positive function λ is the **hyper resistivity**. Notice that we define J and the vorticity w as

$$J = -\nabla^2 A, \tag{11}$$

$$w = -\nabla^2\psi. \tag{12}$$

So the positive parameter λ has a form

$$\lambda = l^2 D_J, \tag{13}$$

where l is a length need to be determined.

Now, we consider a prescribed, stochastic magnetic field in a slab. Note that we are interested in the $A_{\|}$, so we let the stochastic field depends on time, i.e. $\frac{\partial}{\partial t}A_{st} \neq 0$. Then, the parallel current can be written as

$$J \equiv J_0 + J_{st}, \tag{14}$$

where st denotes the stochastic field. We order the magnetic fields and currents by spatial scales as:

$$\begin{aligned}\text{potential field} \quad \mathbf{A} &= \mathbf{A_0} + \widetilde{\mathbf{A}} + \mathbf{A_{st}}\\ \text{magnetic field} \quad \mathbf{B} &= \mathbf{B_0} + \widetilde{\mathbf{B}} + \mathbf{B_{st}}\\ \text{magnetic current} \quad \mathbf{J} &= \mathbf{0} + \widetilde{\mathbf{J}} + \mathbf{J_{st}},\end{aligned} \tag{15}$$

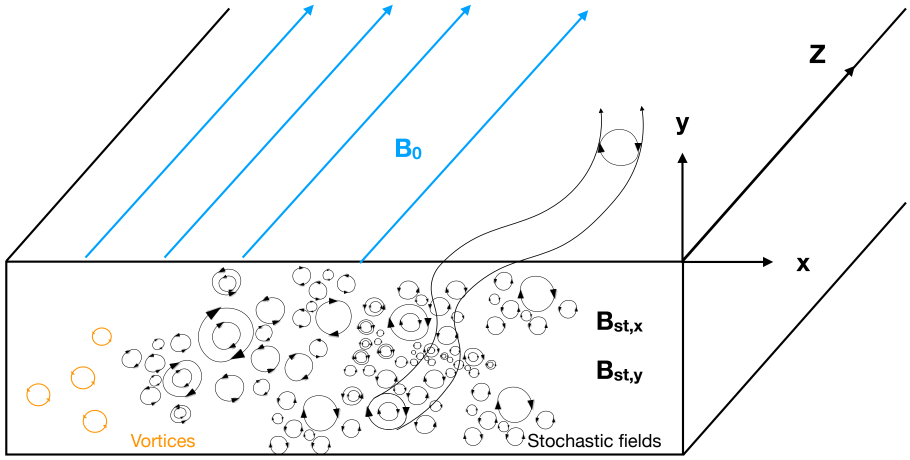

Figure 2: A slab with random magnetic field in x-y plane. The mean magnetic field is in z direction.

where $\mathbf{J_0} = 0$ for $\mathbf{B_0}$ is a constant. The waves are described hydrodynamically by:

$$\begin{aligned} \text{stream function} \quad \psi \quad &= \langle \psi \rangle + \widetilde{\psi} \\ \text{flow velocity} \quad \mathbf{u} \quad &= \langle \mathbf{u} \rangle + \widetilde{\mathbf{u}} \\ \text{vorticity} \quad \mathbf{w} \quad &= \langle \mathbf{w} \rangle + \widetilde{\mathbf{w}}, \end{aligned} \tag{16}$$

where the $\langle \quad \rangle$ is an average over a region that is in x-y plane and fast timescales. So the stochastic magnetic potentials are mainly on z direction, while magnetic potentials on x and y direction contribute to the mean magnetic field and hence they are static. Also note that the vortices are mainly on x-y plane, leading to $\mathbf{w} = (0, 0, w)$ (see figure 2). We introduce a “double-average” method which make this complex system approachable. A procedure to calculate the mean effect of the stochastic fields is to average over the random field, within a window of length scale $(1/|k_{avg}|)$:

$$\bar{F} = \int dR^2 \int dB_{st} \cdot P_{(B_{st,x}, B_{st,y})} F, \tag{17}$$

where $P_{(B_{st,\,i})}$ is the probability distribution function for the random field and F is the arbitrary function being averaged, and dR^2 refers to integration over a region containing random fields. And we assume that stochastic fields have the spatial symmetry in x and y direction $\overline{B_{st,i}} = 0$ (where $i = x, y$) and that there's no spatial correlation in x and y directions $\overline{B_{st,x} B_{st,y}} = 0$. Under this assumption, the autocorrelation length of stochastic fields be comes small $l_{ac} \to 0$ which insures the **magnetic Kubo number** small

$$Ku_{\text{mag}} < 1, \tag{18}$$

and hence validates the use of mean field theory. However, we are interested in the "hyper-resistive diffusion", so we let $\overline{B_{st,x}} \neq 0$. From above assumption, we have the Navier-Stoke equation and the induction equation

$$\frac{\partial}{\partial t}A_{st,z} \quad = (\mathbf{B_{st}} \cdot \nabla)\psi_z + \eta\nabla_\perp^2 A_{st,z} \tag{19}$$

$$\frac{\partial}{\partial t}\overline{w} \quad = -\frac{\overline{(\mathbf{B_{st}} \cdot \nabla)\nabla^2\mathbf{A}}}{\mu_0\rho} + \nu\nabla^2\overline{w}. \tag{20}$$

Then, the $\hat{z}$ component in equation 20 is

$$\frac{\partial}{\partial t}\overline{w} = \frac{1}{\eta\mu_0\rho}\frac{\partial}{\partial x}(\overline{B_{st,x}^2}\frac{\partial}{\partial x}\overline{\psi}) + \frac{1}{\eta\mu_0\rho}\frac{\partial}{\partial y}(\overline{B_{st,y}^2}\frac{\partial}{\partial y}\overline{\psi}) + \frac{B_0}{\mu_0\rho}\frac{\partial}{\partial z}(\nabla_\perp^2\overline{A}_z) + \nu\nabla_\perp^2\overline{w}. \tag{21}$$

The fourth term in equation 21 is the source for the hyper-resistivity diffusion. Now, we apply the mean field theory to equation 21 and have

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{w} = \quad & \frac{1}{\eta\mu_0\rho}\frac{\partial}{\partial x}(\langle\overline{B_{st,x}^2}\rangle\frac{\partial}{\partial x}\widetilde{\psi}) + \frac{1}{\eta\mu_0\rho}\frac{\partial}{\partial y}(\langle\overline{B_{st,y}^2}\rangle\frac{\partial}{\partial y}\widetilde{\psi}) + \frac{B_0}{\mu_0\rho}\frac{\partial}{\partial z}(\nabla_\perp^2\widetilde{A}_z) \\ & + \frac{\widetilde{B}_{st,x}}{\mu_0\rho}\frac{\partial}{\partial x}(\nabla_\perp^2\langle\overline{A}_z\rangle) + \nu\nabla_\perp^2\widetilde{w}.\end{aligned} \tag{22}$$

From equation 22, we have a linear response of vorticity to the stocahstic magnetic field:

$$\widetilde{w}_k = \left(\frac{i}{\omega + i\nu k_\perp^2 + \frac{i(\overline{B_{st,\,x}^2}k_x^2 + \overline{B_{st,\,y}^2}k_y^2)}{\mu_0\rho\eta k_\perp^2} + \frac{-B_0^2k_z^2}{\mu_0\rho(\omega + i\eta k^2)}}\right)\frac{\widetilde{B}_{st,\,x,k}}{\mu_0\rho}\frac{\partial}{\partial x}\nabla^2\overline{A}_z, \tag{23}$$

where $k_\perp^2 = k_x^2 + k_y^2$. And we also apply mean field theory on equation 19:

$$\frac{\partial}{\partial t}\langle\overline{A}_z\rangle = \frac{\partial}{\partial x}\langle\widetilde{B}_{st,x}\widetilde{\psi}\rangle + \eta\nabla_\perp^2\langle\overline{A}_z\rangle. \tag{24}$$

From equation 23, we have

$$\begin{aligned}\frac{\partial}{\partial x}\langle\widetilde{B}_{st,x}\widetilde{\psi}\rangle &= \langle\widetilde{B}_{st,x}(...)\frac{-1}{k_\perp^2}\frac{\widetilde{B}_{st,\,x,k}}{\mu_0\rho}\frac{\partial}{\partial x}\overline{J}_z\rangle \\ = \frac{\partial}{\partial x}\sum_k \mathbf{Re}&\left[\frac{i/k_\perp^2}{\omega + i\nu k_\perp^2 + \frac{i(\overline{B_{st,\,x}^2}k_x^2 + \overline{B_{st,\,y}^2}k_y^2)}{\mu_0\rho\eta k_\perp^2} + \frac{-B_0^2k_z^2}{\mu_0\rho(\omega + i\eta k^2)}}\right]\frac{|\widetilde{B}_{st,x}|^2}{\mu_0\rho}\frac{\partial}{\partial x}\nabla^2\langle\overline{A}_z\rangle\end{aligned} \tag{25}$$

Now, we compare the last term in equation 10 and the first term of RHS of equation 24:

$$\nabla_\perp(-\lambda\nabla_\perp\nabla^2\langle A_\parallel\rangle) = \frac{\partial}{\partial x}\langle\widetilde{B}_{st,x}\widetilde{\psi}\rangle \equiv -\frac{\partial}{\partial x}\langle\overline{\Gamma}\rangle_A, \tag{26}$$

where $\langle\overline{\Gamma}\rangle_A$ is the " double-average" of magnetic potential flux. Notice that here $\nabla_\perp \to \frac{\partial}{\partial x}$ since the "double-average" method allows us to drop $\frac{\partial}{\partial y}\langle\ \rangle \to 0$ terms for the system is periodic in y direction and these terms will be 0 after we take averages. So, we have hyper resistivity

$$\lambda = -\sum_k \mathbf{Re}\left[\frac{i}{\omega + i\nu k_\perp^2 + \frac{i(\overline{B_{st,\,x}^2}k_x^2 + \overline{B_{st,\,y}^2}k_y^2)}{\mu_0\rho\eta k_\perp^2} + \frac{-B_0^2k_z^2}{\mu_0\rho(\omega + i\eta k^2)}}\right]\frac{|\widetilde{B}_{st,x}|^2}{\mu_0\rho k_\perp^2} \tag{27}$$

We check the dimension and find that $\lambda = [\frac{m^4}{s}]$ which matches the equation 26. This the " double-average" of magnetic potential flux can be written as:

$$\langle\overline{\Gamma}\rangle_A = -\sum_k \frac{|\widetilde{B}_{st,\,x,\,k}|^2}{\mu_0\rho k_\perp^2}C_k\frac{\partial}{\partial x}\nabla_\perp^2\langle\overline{A}_z\rangle \equiv -D_A\frac{\partial}{\partial x}\langle\overline{A}_z\rangle, \tag{28}$$

where the resonance function (phase coherence) C_k, which defines the effective decorrelation time $\tau_{c,\,k}$, is:

$$C_k \equiv \frac{-(\nu k_\perp^2 + \frac{\omega_A^2 \eta k^2}{\omega^2 + \eta^2 k_\perp^4} + \frac{(\overline{B_{st,\,x}^2} k_x^2 + \overline{B_{st,\,y}^2} k_y^2)}{\mu_0 \rho \eta k_\perp^2})}{\omega^2 (1 - \frac{\omega_A^2}{\omega^2 + \eta^2 k_\perp^4})^2 + (\nu k^2 + \frac{\omega_A^2 \eta k_\perp^2}{\omega^2 + \eta^2 k_\perp^4} + \frac{(\overline{B_{st,\,x}^2} k_x^2 + \overline{B_{st,\,y}^2} k_y^2)}{\mu_0 \rho \eta k_\perp^2})^2} < 0, \tag{29}$$

where $\omega_A \equiv B_{0,z} k_z / \sqrt{\mu_0 \rho}$ is the Alfvén frequency of the mean field. So we have

$$\lambda = \boxed{\sum_k |C_k| \frac{|\widetilde{B}_{st,x}|^2}{\mu_0 \rho k_\perp^2}} > 0. \tag{30}$$

Thus, since $l_\perp^2 D_J = D_A$, we have current diffusivity

$$D_J = \sum_k \frac{|\widetilde{B}_{st,\,x,\,k}|^2}{\mu_0 \rho k_\perp^2} |C_k|. \tag{31}$$

As the mean-square stochastic fields ($\overline{B_{st,i}^2}$) becomes large, we'll have $|C_k| \to 0$ (see equation 30), we have

$$\lambda \to 0, \tag{32}$$

indicating that *the hyper diffusivity (or the current diffusivity) is suppressed by strong stochastic feilds*. In the form of magnetic potential evolution

$$\frac{\partial}{\partial t} \langle A_z \rangle = \eta \nabla^2 \langle A_z \rangle - \nabla_\perp (\lambda \nabla_\perp \nabla^2 \langle A_z \rangle). \tag{33}$$

The classical resistivity will fight with the hyper resistivity and control the evolution of magnetic potential (or the current diffusivity). The stochastic fields can modify the evolution of magnetic potential via hyper resistivity and hence affect the $\langle A_z \rangle$. But $\langle A_z \rangle$ will also affect the stochastic field itself via induction equation (see equation 19). Finally, notice that no matter how big or small stochastic fields are, the large scale magnetic helicity $\mathcal{H}_0$ remains the same so the topological structure of large-scale field remains unchanged, while the magnetic energy dissipates. Recall that the magnetic energy dissipation flux $\Gamma_H = -\lambda \nabla \frac{J_\parallel}{B}$, we know that when the stronger stochastic fields are ($\lambda \to 0$), the less magnetic energy dissipation we have. This can be visualized that stochastic fields act as an ensemble of tangle springs that *can store the magnetic energy and can slow down the rate of magnetic energy dissipation*.

References

Taylor, J. B. 1974, Phys. Rev. Lett., 33, 1139