

**PHYSICS 211B : CONDENSED MATTER PHYSICS**  
**HW ASSIGNMENT #4**

**(1)** Consider an anisotropic superconductor with free energy

$$F = \int d^d x \left\{ a |\Psi|^2 + \frac{1}{2} b |\Psi|^4 + K_{\mu\nu} \left( \partial_\mu \Psi^* - \frac{ie^*}{\hbar c} A_\mu \Psi^* \right) \left( \partial_\nu \Psi + \frac{ie^*}{\hbar c} A_\nu \Psi \right) + \frac{1}{8\pi} (\nabla \times \mathbf{A})^2 \right\} ,$$

where  $K_{\mu\nu}$  is a real symmetric matrix. We may write  $K_{\mu\nu} \equiv K N_{\mu\nu}$  where  $\det(N) = 1$  and  $K$  sets the overall scale of  $K_{\mu\nu}$ . We continue to define the quantity  $\lambda_L$  via Eq. 1.46 in the notes.

We showed in the Lecture Notes that

$$-\lambda_L^2 \nabla \times (\nabla \times \mathbf{B}) = \mathbf{B} + \phi_L \delta(\rho) \hat{\mathbf{z}} \quad .$$

This result holds for isotropic systems, *i.e.* when  $N$  is the identity matrix. In a layered anisotropic material, along symmetry axes we may write  $N = \text{diag}(N_\perp, N_\perp, N_\parallel)$ , with  $N_\perp^2 N_\parallel = 1$ .

(a) Show that if  $\mathbf{B}$  remains along  $\hat{\mathbf{z}}$  that the penetration depth in the above equation is replaced by the expression  $\lambda_{L\perp} = \lambda_L / N_\perp^{1/2}$ .

(b) If  $\mathbf{B}$  is along  $\hat{\mathbf{y}}$ , the above equation is modified. Find the modified form, and solve for the magnetic field profile of a vortex line. (*Hint: try rescaling one of your coordinates.*) Defining  $\lambda_{L\parallel} = \lambda_L / N_\parallel^{1/2}$ , derive an expression for  $H_{c1}$ .

(c) For the field orientation in part (b), derive an expression for  $H_{c2}$ .

**(2)** The nonlinear Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -K \nabla^2 \Psi + b(|\Psi|^2 - n_0) \Psi \quad ,$$

is often used to describe the dynamics of superfluids.

(a) Show that the NLSE follows from a variational principle based on the following Lagrangian density:

$$\mathcal{L} = i\hbar \Psi^* \frac{\partial \Psi}{\partial t} - K |\nabla \Psi|^2 - \frac{1}{2} b (|\Psi|^2 - n_0)^2 \quad .$$

(b) Show that by identifying a velocity scale  $c$  and a length scale  $\xi$ , that one can write

$$\Psi(\mathbf{x}, t) = \sqrt{n_0} \psi(\mathbf{x}/\xi, ct/\xi) \quad .$$

Find the dimensionless NLSE for  $\psi(\mathbf{r}, s)$ , where  $\mathbf{r} = \mathbf{x}/\xi$  and  $s = ct/\xi$ .

(c) Show that the NLSE respects Galilean invariance, in that if  $\psi(\mathbf{r}, s)$  is a solution, so is

$$\tilde{\psi}(\mathbf{r}, s) = e^{i\mathbf{v}\cdot\mathbf{r}} e^{-i\mathbf{v}^2 s/2} \psi(\mathbf{r} - \mathbf{v}s, s) \quad .$$

(d) The Madelung transformation converts the complex NLSE into two real equations for amplitude and phase variables. Writing  $\psi = \sqrt{n} e^{i\phi}$ , show that

$$\frac{\partial n}{\partial s} + \nabla \cdot (n \nabla \phi) = 0$$

$$\frac{\partial \phi}{\partial s} + n - 1 + \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2\sqrt{n}} \nabla^2 \sqrt{n} = 0 \quad .$$

Obtain the linearized equations for long wavelength oscillations and show that the  $c$  you found in part (b) is the speed of wave propagation. (The speed in terms of dimensionless variables should be unity.) Show that your dispersion interpolates between acoustic and ballistic as one goes from large to small wavelength.

(e) Write  $\psi(\mathbf{r}) = A(r) e^{i\varphi}$ , where  $\varphi = \tan^{-1}(y/x)$  is the planar azimuthal angle. Find the equation for  $A(r)$ , and solve it in the limits  $r \ll 1$  and  $r \gg 1$ . Make a sketch of  $A(r)$  based on what you think is a reasonable interpolation between these limits.

**(3)** Analyze the problem of a bound state in the vicinity of the bottom of a band where the dispersion behaves as  $\varepsilon(\mathbf{k}) \propto k^4$ . Under what conditions do you expect a bound state to form for arbitrarily weak attractive interactions?