

Continuous quantum phase transitions

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A quantum system can undergo a continuous phase transition at the absolute zero of temperature as some parameter entering its Hamiltonian is varied. These transitions are particularly interesting for, in contrast to their classical finite-temperature counterparts, their dynamic and static critical behaviors are intimately intertwined. Considerable insight is gained by considering the path-integral description of the quantum statistical mechanics of such systems, which takes the form of the *classical* statistical mechanics of a system in which time appears as an extra dimension. In particular, this allows the deduction of scaling forms for the finite-temperature behavior, which turns out to be described by the theory of finite-size scaling. It also leads naturally to the notion of a temperature-dependent dephasing length that governs the crossover between quantum and classical fluctuations. Using these ideas, a scaling analysis of experiments on Josephson-junction arrays and quantum-Hall-effect systems is presented. [S0034-6861(97)00501-1]

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I. INTRODUCTION

A century subsequent to Andrews' discovery of critical opalescence¹ in carbon dioxide, continuous phase

¹Opalescence is the strong reflection of light by a system (such as an opal) due to fluctuations in its index of refraction on length scales comparable to the wavelengths of visible light.

transitions continue to be a subject of great interest to physicists. The appeal of the subject is twofold. First, the list of systems that exhibit interesting phase transitions continues to expand; it now includes the Universe itself! Second, the formal theory of equilibrium phase transitions has found applications in problems as diverse as constructing field and string theories of elementary particles, the transition to chaos in dynamical systems, and the long-time behavior of systems out of equilibrium.

Our purpose in this Colloquium is to give a brief and qualitative account of some basic features of a species of phase transitions,² termed quantum phase transitions (QPTs), that have attracted much interest in recent years. These transitions take place at the absolute zero of temperature, where crossing the phase boundary means that the quantum ground state of the system changes in some fundamental way. This is accomplished by changing not the temperature, but some parameter in the Hamiltonian of the system. This parameter might be the charging energy in Josephson-junction arrays (which controls their superconductor-insulator transition), the magnetic field in a quantum-Hall sample (which controls the transition between quantized Hall plateaus), doping in the parent compound of a high- T_c superconductor (which destroys the antiferromagnetic spin order), or

A liquid-vapor system near its critical point has large density fluctuations on length scales that can reach microns. This causes the system, which is normally transparent, to have a cloudy appearance.

²Henceforth we shall use phase transitions as a shorthand for continuous phase transitions.

disorder in a conductor near its metal-insulator transition (which determines the conductivity at zero temperature). These and other QPTs raise new and fascinating issues for theory and experiment, most notably the inescapable necessity of taking quantum effects into account.

Exactly what quantum effects are at issue is a bit subtle. As a corollary of our definition, all finite-temperature³ transitions are to be considered “classical,” even in highly quantum-mechanical systems like superfluid helium or superconductors. It is not that quantum mechanics is unimportant in these cases, for in its absence there would not be an ordered state, i.e., the superfluid or superconductor. Nevertheless, sufficiently close to the critical point, quantum fluctuations are important at the microscopic scale but not at the longer length scales that control the critical behavior; in the jargon of statistical mechanics, quantum mechanics is needed for the existence of an order parameter,⁴ but it is classical thermal fluctuations that govern it at long wavelengths. For instance, near the superfluid lambda transition in ⁴He, the order parameter is a complex-valued field that is related to the underlying condensate wave function. However, its critical fluctuations can be captured exactly by doing classical statistical mechanics with an effective Hamiltonian for the order-parameter field [for instance, the phenomenological Landau-Ginsburg free-energy functional (Goldenfeld, 1992)].

The physics behind the classical nature of finite-temperature transitions is the following: phase transitions are quite generally accompanied by a divergent correlation length and correlation time, i.e., the order parameter (e.g., the magnetization in a ferromagnet) fluctuates coherently over increasing distances and ever more slowly. The latter implies that there is a frequency ω^* associated with the critical fluctuations that vanishes at the transition. A quantum system behaves classically if the temperature exceeds all frequencies of interest, and, since $\hbar\omega^* \ll k_B T_c$ close to the transition, the critical fluctuations will behave classically.

This argument also shows that QPTs, where $T_c=0$, are qualitatively different, and their critical fluctuations must be treated quantum mechanically. In the following we will describe the language and physical pictures that enable such a treatment and that have come into common usage among practitioners in the field in the last few years. Although much of this wisdom, which has its roots in work on quantum Ising models (Young, 1975;

Suzuki, 1976), dates back to the work of Hertz (1976),⁵ it remains unknown or poorly understood in the wider community, and extracting it from the literature remains a daunting task. It is our hope here to communicate this set of ideas to a wider audience and, in particular, to be helpful to newcomers to this field, experimentalists and theorists alike.

Our discussion is organized as follows. In Sec. II we introduce the statistical mechanics of quantum systems and the path-integral (Feynman, 1972) approach to it, which is an extremely useful source of intuition in these problems. A running theme throughout this discussion is the intertwining of dynamics and thermodynamics in quantum statistical mechanics. In Sec. III we describe the general features of a QPT at $T=0$ and how a non-zero temperature alters the physics. This leads naturally, in Sec. IV, to a discussion of what kind of scaling behavior in experiments is evidence of an underlying QPT. We illustrate this using particular examples from phase transitions in quantum-Hall systems. We end in Sec. V with a brief summary and pointers to work on QPTs in other interesting systems. Readers interested in a highly informative discussion at a higher technical level should consult the recent beautiful review article by Sachdev (1996).

II. QUANTUM STATISTICAL MECHANICS: GENERALITIES

Before we discuss what happens in the vicinity of a QPT, let us recall some very general features of the statistical mechanics of quantum systems. The quantities of interest are the partition function of the system, which is governed by a Hamiltonian H ,

$$Z(\beta) = \text{Tr} e^{-\beta H}, \quad (1)$$

and the expectation values of an arbitrary operator O ,

$$\langle O \rangle = \frac{1}{Z(\beta)} \text{Tr}(O e^{-\beta H}). \quad (2)$$

In writing these formal expressions we have assumed a finite temperature, $k_B T = 1/\beta$. To get at what happens exactly at $T=0$ we take the $T \rightarrow 0$ limit. Upon doing so, the free energy, $F = -(1/\beta) \ln Z(\beta)$, becomes the ground-state energy, and the various thermal averages become ground-state expectation values. From Z we can get all the thermodynamic quantities of interest. Expectation values of operators of the form $O \equiv A(\mathbf{r}t)A(\mathbf{r}'t')$, where A is a space-time-dependent operator, are related to the results of dynamical scattering and linear-response measurements. For example, A might be the local electron density (X-ray scattering) or current (electrical transport).

⁵We should note that the contemporaneous explosion of work on the one-dimensional electron gas (Emery, 1979) provided important, early illustrations of these ideas.

³In an almost standard abuse of language, we refer to non-zero temperatures as finite.

⁴An order parameter is a quantity which is zero in the disordered phase and nonzero in the ordered state. In systems that spontaneously break some symmetry in the ordered state, the nature (and value) of the order parameter reflects this broken symmetry. Thus, for example, in an Ising ferromagnet the magnetization is a positive or negative real number indicating the difference in populations of the up and down spins (see Goldenfeld, 1992).



FIG. 1. Schematic representation of a 1D Josephson-junction array. The crosses represent the tunnel junctions between superconducting segments, and θ_i are the phases of the superconducting order parameter in the latter.

A. Partition functions and path integrals

Let us focus for now on the expression for Z . Notice that the operator density matrix $e^{-\beta H}$ is the same as the time-evolution operator $e^{-iHT/\hbar}$, provided we assign the imaginary value $T = -i\hbar\beta$ to the time interval over which the system evolves. More precisely, when the trace is written in terms of a complete set of states,

$$Z(\beta) = \sum_n \langle n | e^{-\beta H} | n \rangle, \quad (3)$$

Z takes the form of a sum of imaginary-time transition amplitudes for the system to start in some state $|n\rangle$ and return to the same state after an imaginary time interval $-i\hbar\beta$. Thus we see that calculating the thermodynamics of a quantum system is the same as calculating transition amplitudes for its evolution in imaginary time, with the total time interval fixed by the temperature of interest. The fact that the time interval happens to be imaginary is not central. The key idea we hope to transmit to the reader is that Eq. (3) should evoke an image of quantum dynamics and temporal propagation.

This way of looking at things can be given a particularly beautiful and practical implementation in the language of Feynman's path-integral formulation of quantum mechanics (Feynman, 1972). Feynman's prescription is that the net transition amplitude between two states of the system can be calculated by summing amplitudes for all possible paths between them. The path taken by the system is defined by specifying the state of the system at a sequence of finely spaced intermediate time steps. Formally we write

$$e^{-\beta H} = [e^{-(1/\hbar)\delta\tau H}]^N, \quad (4)$$

where $\delta\tau$ is a time interval⁶ that is small on the time scales of interest ($\delta\tau = \hbar/\Gamma$, where Γ is some ultraviolet cutoff) and N is a large integer chosen so that $N\delta\tau = \hbar\beta$. We then insert a sequence of sums over complete sets of intermediate states into the expression for $Z(\beta)$:

$$\begin{aligned} Z(\beta) = & \sum_n \sum_{m_1, m_2, \dots, m_N} \langle n | e^{-(1/\hbar)\delta\tau H} | m_1 \rangle \\ & \times \langle m_1 | e^{-(1/\hbar)\delta\tau H} | m_2 \rangle \langle m_2 | \dots | m_N \rangle \\ & \times \langle m_N | e^{-(1/\hbar)\delta\tau H} | n \rangle. \end{aligned} \quad (5)$$

This rather messy expression actually has a rather simple physical interpretation. Formally inclined readers

⁶For convenience we have chosen $\delta\tau$ to be real, so that the small interval of imaginary time that it represents is $-i\delta\tau$.

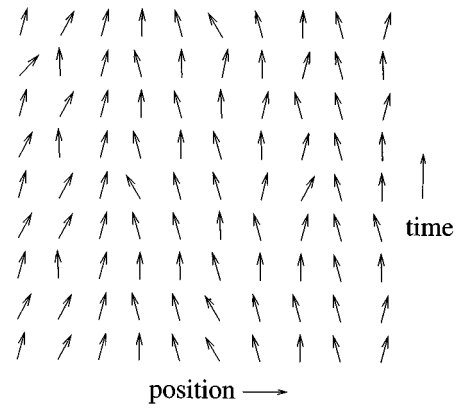


FIG. 2. Typical path or time history of a 1D Josephson-junction array. Note that this is equivalent to one of the configurations of a 1+1D classical XY model. The long-range correlations shown here are typical of the superconducting phase of the 1D array or, equivalently, of the ordered phase of the classical model.

will observe that the expression for the quantum partition function in Eq. (5) has the form of a classical partition function, i.e., a sum over configurations expressed in terms of a transfer matrix, if we think of imaginary time as an additional spatial dimension. In particular, if our quantum system lives in d dimensions, the expression for its partition function looks like a classical partition function for a system with $d+1$ dimensions, except that the extra dimension is finite in extent— $\hbar\beta$ in units of time. As $T \rightarrow 0$ the system size in this extra “time” direction diverges, and we get a truly $d+1$ -dimensional, effective classical system.

Since this equivalence between a d -dimensional quantum system and a $d+1$ -dimensional classical system is crucial to everything else we have to say and since Eq. (5) is probably not very illuminating for readers not used to a daily regimen of transfer matrices, it will be very useful to consider a specific example in order to be able to visualize what Eq. (5) means.

B. Example: One-dimensional Josephson-junction arrays

Consider a one-dimensional array comprising a large number L of identical Josephson junctions as illustrated in Fig. 1. Such arrays have recently been studied by Haviland and Delsing (1996). A Josephson junction is a tunnel junction connecting two superconducting metallic grains. Cooper pairs of electrons are able to tunnel back and forth between the grains and hence communicate information about the quantum state on each grain. If the Cooper pairs are able to move freely from grain to grain throughout the array, the system is a superconductor. If the grains are very small, however, it costs a large charging energy to move an excess Cooper pair onto a grain. If this energy is large enough, the Cooper pairs fail to propagate and become stuck on individual grains, which causes the system to be an insulator.

The essential degrees of freedom in this system are the phases of the complex superconducting order pa-

parameter on the metallic elements connected by the junctions⁷ and their conjugate variables, the charges (excess Cooper pairs, or equivalently the voltages) on each grain. The intermediate state $|m_j\rangle$, at time $\tau_j \equiv j \delta\tau$, that enters the quantum partition function of Eq. (5) can thus be defined by specifying the set of phase angles $\{\theta(\tau_j)\} \equiv [\theta_1(\tau_j), \theta_2(\tau_j), \dots, \theta_L(\tau_j)]$, where $\theta_i(\tau_j)$ is the phase angle of the i th grain at time τ_j . Two typical paths or time histories on the interval $[0, \hbar\beta]$ are illustrated in Figs. 2 and 3, where the orientation of the arrows (“spins”) indicates the local phase angle of the order parameter. The statistical weight of a given path, in the sum in Eq. (5), is given by the product of the matrix elements

$$\prod_j \langle \{\theta(\tau_{j+1})\} | e^{-(1/\hbar)\delta\tau H} | \{\theta(\tau_j)\} \rangle, \quad (6)$$

where

$$H = \frac{C}{2} \sum_j V_j^2 - E_J \cos(\hat{\theta}_j - \hat{\theta}_{j+1}) \quad (7)$$

is the quantum Hamiltonian of the Josephson-junction array. Here $\hat{\theta}_j$ is the operator representing the phase of the superconducting order parameter on the j th grain,⁸ $V_j \equiv -i(2e/C)(\partial/\partial\theta_j)$ is conjugate to the phase⁹ and is the voltage on the j th junction, and E_J is the Josephson coupling energy. The two terms in the Hamiltonian then represent the charging energy of each grain and the Josephson coupling of the phase across the junction between grains.

As indicated previously, we can map the quantum statistical mechanics of the array onto classical statistical mechanics by identifying the statistical weight of a space-time path in Eq. (6) with the Boltzmann weight of a two-dimensional spatial configuration of a classical system. In this case the classical system is therefore a two-dimensional X - Y model, i.e., its degrees of freedom are planar spins, specified by angles θ_i , that live on a two-dimensional square lattice. (Recall that at temperatures above zero the lattice has a finite width $\hbar\beta/\delta\tau$ in the temporal direction.) While the degrees of freedom are

⁷It is believed that neglecting fluctuations in the magnitude of the order parameter is a good approximation (see Bradley and Doniach, 1984; Wallin *et al.*, 1994).

⁸Our notation here is that $\{\theta(\tau)\}$ refers to the configuration of the entire set of angle variables at time slice τ . The $\hat{\theta}$'s appearing in the Hamiltonian in Eq. (7) are angular coordinate operators, and j is a site label. The state at time slice τ is an eigenfunction of these operators: $\cos(\hat{\theta}_j - \hat{\theta}_{j+1}) | \{\theta(\tau)\} \rangle = \cos[\theta_j(\tau) - \theta_{j+1}(\tau)] | \{\theta(\tau)\} \rangle$.

⁹It is useful to think of this as a quantum rotor model. The state with wave function $e^{im_j\theta_j}$ has m_j units of angular momentum representing m_j excess Cooper pairs on grain j . The Cooper-pair number operator in this representation is $n_j = -i(\partial/\partial\theta_j)$ (see Wallin *et al.*, 1994). The cosine term in Eq. (7) is a “torque” term which transfers units of conserved angular momentum (Cooper pairs) from site to site. Note that the potential energy of the bosons is represented, somewhat paradoxically, by the kinetic energy of the quantum rotors and vice versa.

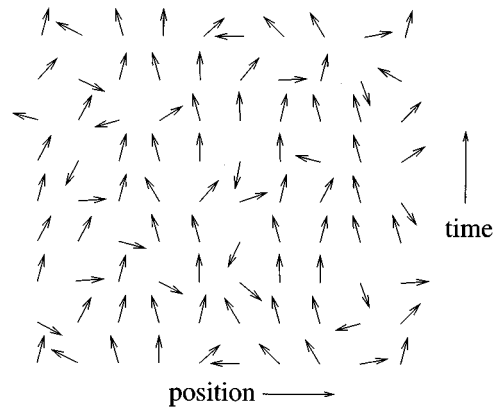


FIG. 3. Typical path or time history of a 1D Josephson-junction array in the insulating phase, where correlations fall off exponentially in both space and time. This corresponds to the disordered phase in the classical model.

easily identified, finding the classical Hamiltonian for this X - Y model is somewhat more work and requires an explicit evaluation of the matrix elements, which interested readers can find in the Appendix.

It is shown in the Appendix that, in an approximation that preserves the universality class of the problem,¹⁰ the product of matrix elements in Eq. (6) can be rewritten in the form $e^{-H_{XY}}$, where the Hamiltonian of the equivalent classical X - Y model is

$$H_{XY} = \frac{1}{K} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \quad (8)$$

and the sum runs over nearest-neighbor points in the two-dimensional (space-time) lattice.¹¹ The nearest-neighbor character of the couplings identifies the classical model as the 2D X - Y model, extensively studied in the context of superfluid and superconducting films (Goldenfeld, 1992; Chaikin and Lubensky, 1995). We emphasize that, while the straightforward identification of the degrees of freedom of the classical model in this example is robust, this simplicity of the resulting classical Hamiltonian is something of a minor miracle.

It is essential to note that the dimensionless coupling constant K in H_{XY} , which plays the role of the temperature in the classical model, depends on the ratio of the capacitive charging energy $E_C = (2e)^2/C$ to the Josephson coupling E_J in the array,

$$K \sim \sqrt{E_C/E_J}, \quad (9)$$

and has nothing to do with the physical temperature (see the appendix). The physics here is that a large Josephson coupling produces a small value of K , which favors

¹⁰That is, the approximation is such that the universal aspects of the critical behavior, such as the exponents and scaling functions, will be given without error. However, nonuniversal quantities, such as the critical coupling, will differ from an exact evaluation. Technically, the neglected terms are irrelevant at the fixed point underlying the transition.

¹¹Notice this crucial change in notation from Eq. (7), where j refers to a point in 1D space, not 1+1D space-time.

coherent ordering of the phases. That is, a small value of K makes it unlikely that θ_i and θ_j will differ significantly, even when sites i and j are far apart (in space and/or time). Conversely, a large charging energy leads to a large value of K , which favors zero-point fluctuations of the phases and disorders the system. That is, a large value of K means that the θ 's are nearly independent and all values are nearly equally likely.¹² Finally, we note that this equivalence generalizes to d -dimensional arrays and $d+1$ -dimensional classical XY models.

C. Quantum-classical analogies

This specific example of the equivalence between a quantum system and a classical system with an extra “temporal” dimension illustrates several general correspondences between quantum systems and their effective classical analogs.

Standard lore tells us that the classical XY model has an order-disorder phase transition as its temperature K is varied. It follows that the quantum array has a phase transition as the ratio of its charging and Josephson energies is varied. One can thus see why it is said that the superconductor-insulator quantum phase transition in a 1-dimensional Josephson-junction array is in the same universality class as the order-disorder phase transition of the 1+1-dimensional classical XY model. [One crucial caveat is that the XY -model universality class has strict particle-hole symmetry for the bosons (Cooper pairs) on each site. In reality, Josephson-junction arrays contain random “offset charges” that destroy this symmetry and change the universality class (Wallin *et al.*, 1994), a fact which is all too often overlooked.]

We emphasize again that K is the temperature only in the effective classical problem. In the quantum case the physical temperature is presumed to be nearly zero and only enters as the finite size of the system in the imaginary-time direction. The coupling constant K , the fake “temperature,” is a measure not of thermal fluctuations, but of the strength of quantum fluctuations, or zero-point motion of the phase variables.¹³ This notion is quite confusing, so the reader might be well advised to pause here and contemplate it further. It may be useful to examine Fig. 4, where we show a space-time lattice for the XY model corresponding to a Josephson-junction array at a certain temperature and at a temperature half as large. The size of the lattice constant in the time direction [$\delta\tau$ in the path integral in Eq. (5)] and K are the *same* in both cases even though the physical

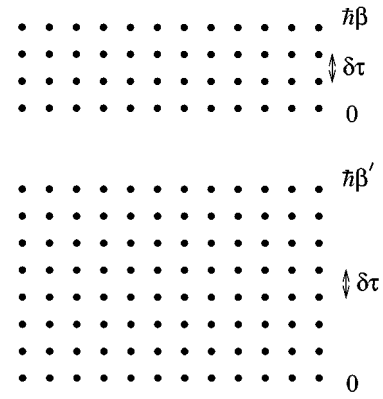


FIG. 4. Illustration of discrete space-time lattices with the same coupling K . The lattice constant in the time direction is the same $\delta\tau$ in both cases, but the corresponding physical temperature (determined from $\hbar\beta = L_c \delta\tau$) for the lattice in the lower panel is half that of the lattice in the upper panel.

temperature is not the same. The only difference is that one lattice is larger in the time direction than the other.

In developing intuition about this picture, it may be helpful to see how classical physics is recovered at very high temperatures. In that limit, the time interval $\hbar\beta$ is very short compared to the periods associated with the natural frequency scales in the system, and typical time histories will consist of a single static configuration that is the same at each time slice. The dynamics therefore drops out of the problem, and a Boltzmann weight $\exp(-\beta H_{\text{classical}})$ is recovered from the path integral.

The thermodynamic phases of the array can be identified from those of the XY model. A small value of K corresponds to low temperature in the classical system, so the quantum system will be in the ordered ferromagnetic phase of the XY model, as illustrated in Fig. 2. There will be long-range correlations in both space and time of the phase variables.¹⁴ This indicates that the Josephson coupling dominates over the charging energy, and the order parameter is not fluctuating wildly in space or time, so that the system is in the superconducting phase. For large K the system is disordered, and the order parameter fluctuates wildly. The correlations decay exponentially in space and time as illustrated in Fig. 3. This indicates that the system is in the *insulating* phase, where the charging energy dominates over the Josephson coupling energy.

Why can we assert that correlations that decay exponentially in imaginary time indicate an excitation gap characteristic of an insulator? This is readily seen by noting that the Heisenberg representation of an operator in imaginary time is

$$A(\tau) = e^{H\tau/\hbar} A e^{-H\tau/\hbar}, \quad (10)$$

and thus the (ground state) correlation function for any operator can be expressed in terms of a complete set of states as

¹²Because particle number is conjugate to the phase [$\hat{n}_j = -i\partial/\partial\theta_j$], a state of indefinite phase on a site has definite charge on that site, as would be expected for an insulator.

¹³Zero-point motion of the phase variables is caused by the fact that there is an uncertainty relation between the phase and the number of Cooper pairs on a superconducting grain. The more well-defined the phase is, the larger the uncertainty in the charge is. This costs capacitive energy.

¹⁴In this special 1+1D case, the correlations happen to decay algebraically rather than being truly of infinite range.

TABLE I. Analogies.

Quantum	Classical
d space, 1 time dimensions	$d+1$ space dimensions
Coupling constant K	Temperature T
Inverse temperature β	Finite size L_τ in “time” direction
Correlation length ξ	Correlation length ξ
Inverse characteristic energy $\hbar/\Delta, \hbar/k_B T_c$	Correlation length in the “time” direction ξ_τ

$$G(\tau) \equiv \langle 0|A(\tau)A(0)|0\rangle$$

$$= \sum_m e^{-(\epsilon_m - \epsilon_0)\tau/\hbar} |\langle 0|A|m\rangle|^2, \quad (11)$$

where ϵ_m is the energy of the m th excited state. The existence of a finite minimum-excitation gap $\Delta_{01} \equiv \epsilon_1 - \epsilon_0$ guarantees that for long (imaginary) times the correlation function will decay exponentially,¹⁵ i.e.,

$$G(\tau) \sim e^{-\Delta_{01}\tau/\hbar}. \quad (12)$$

To recapitulate, we have managed to map the finite-temperature 1D quantum problem into a 2D classical problem with one finite dimension that diverges as $T \rightarrow 0$. The parameter that controls the fluctuations in the effective classical problem does *not* involve T but instead is a measure of the quantum fluctuations. The classical model exhibits two phases, one ordered and one disordered. These correspond to the superconducting and insulating phases in the quantum problem. In the former the zero-point or quantum fluctuations of the order parameter are small. In the latter they are large. The set of analogies developed here between quantum and classical critical systems is summarized in Table I.

Aside from the beautiful formal properties of the analogy between the quantum path integral and $(d+1)$ -dimensional statistical mechanics, there are very practical advantages to this analogy. In many cases, particularly for systems without disorder, the universality class of the quantum transition is one that has already been studied extensively classically, and a great deal may already be known about it. For new universality classes, it is possible to do the quantum mechanics by classical Monte Carlo or molecular-dynamics simulations of the appropriate $(d+1)$ -dimensional model.

Finally, there is a special feature of our particular example that should be noted. In this case the quantum system, the 1D Josephson-junction array (which is also the 1D quantum X - Y model), has mapped onto a classical model in which space and time enter in the same fashion, i.e., the isotropic 2D classical X - Y model. Consequently, the dynamical exponent z (to be defined below) is unity. This is not true in general—depending upon the quantum kinetics, the coupling in the time direction can have a very different form, and the effective

classical system is then intrinsically anisotropic and not so simply related to the starting quantum system.

D. Dynamics and thermodynamics

We end this account of quantum statistical mechanics by commenting on the relationship between dynamics and thermodynamics. In classical statistical mechanics, dynamics and thermodynamics are separable, i.e., the momentum and position sums in the partition function are totally independent. For example, we do not need to know the mass of the particles to compute their positional correlations. In writing down simple nondynamical models, e.g., the Ising model, we typically take advantage of this simplicity.

This freedom is lost in the quantum problem because coordinates and momenta do not commute.¹⁶ It is for this reason that our path-integral expression for Z contains information on the imaginary time evolution of the system over the interval $[0, \hbar\beta]$, and, with a little bit of care, that information can be used to get the dynamics in real time by the analytic continuation,

$$G(\tau) \rightarrow G(+it) \quad (13)$$

in Eq. (11). Stating it in reverse, one cannot solve for the thermodynamics without also solving for the dynamics—a feature that makes quantum statistical mechanics more interesting but that much harder to do!

Heuristically, the existence of \hbar implies that energy scales that enter thermodynamics necessarily determine time scales, which then enter the dynamics and vice-versa. Consider the effect of a characteristic energy scale, such as a gap Δ , in the spectrum. By the uncertainty principle there will be virtual excitations across this gap on a time scale \hbar/Δ , which will appear as the characteristic time scale for the dynamics. Close to the critical point, where Δ vanishes, and at finite tempera-

¹⁵At $T \neq 0$ and for very long times (comparable to $\hbar\beta$), the finiteness in the time direction will modify this result. Also, we implicitly assume here that $\langle 0|A|0\rangle = 0$.

¹⁶Stated more formally, calculating Z classically only requires knowledge of the form of the Hamiltonian function and not of the equations of motion, while both enter the quantum calculation. Recall that H alone does not fix the equations of motion; one also needs the Poisson brackets/commutators among the phase-space variables. While these determine the classical oscillation frequencies, they do not enter the classical calculation of Z . In quantum mechanics \hbar multiplied by the classical oscillation frequencies yields the energy-level spacing. Hence the commutators are needed to find the eigenvalues of the quantum H needed to compute the quantum Z .

ture, this argument gets modified—the relevant uncertainty in the energy is now $k_B T$, and the characteristic time scale is $\hbar\beta$. In either case, the linkage between dynamics and thermodynamics is clear.

III. QUANTUM PHASE TRANSITIONS

We now turn our attention to the immediate neighborhood of a quantum critical point. In this region the mapping of the quantum system to a $(d+1)$ -dimensional classical model will allow us to make powerful general statements about the former using the extensive lore on critical behavior in the latter. Hence most of the following will consist of a reinterpretation of standard ideas in classical statistical mechanics in terms appropriate for $d+1$ dimensions, where the extra dimension is imaginary time.

A. $T=0$: Dynamic scaling

In the vicinity of a continuous quantum phase transition we will find several features of interest. First, we will find a correlation length that diverges as the transition is approached. That diverging correlation lengths are a generic feature of classical critical points immediately tells us that diverging lengths and diverging times are automatically a generic feature of quantum critical points, since one of the directions in the $(d+1)$ -dimensional space is time. This makes sense from the point of view of causality. It *should* take a longer and longer time to propagate information across the distance of the correlation length.

Actually, we have to be careful—as we remarked earlier, the time direction might easily involve a different set of interactions than the spatial directions, which lead to a distinct correlation “length” in the time direction. We will call the latter ξ_τ , reserving the symbol ξ for the spatial correlation length. Generically, at $T=0$ both $\xi(K)$ and $\xi_\tau(K)$ diverge as $\delta \equiv K - K_c \rightarrow 0$ in the manner,¹⁷

$$\begin{aligned}\xi &\sim |\delta|^{-\nu}, \\ \xi_\tau &\sim \xi^z.\end{aligned}\quad (14)$$

These asymptotic forms serve to define the correlation length exponent ν and the dynamical-scaling exponent z . The nomenclature is historical and refers to the extension of scaling ideas from the study of static classical critical phenomena to dynamics in the critical region associated with critical slowing down (Ferrell, 1968, Hohenberg and Halperin, 1977). In the classical problem the extension was a nontrivial step, which deserves a proper label. As remarked before, the quantum problem involves statics and dynamics on the same footing, and so nothing less is possible. For the case of the Josephson-junction array considered previously, we

found the simplest possible result, $z=1$. As noted, however, this is a special isotropic case, and, in general, $z \neq 1$.

As a consequence of the diverging ξ and ξ_τ , it turns out that various physical quantities in the critical region close to the transition have (dynamic) scaling forms, i.e., their dependence on the independent variables involves homogeneity relations of the form

$$\mathcal{O}(k, \omega, K) = \xi^{d_\mathcal{O}} \mathcal{O}(k\xi, \omega\xi_\tau), \quad (15)$$

where $d_\mathcal{O}$ is called the scaling dimension¹⁸ of the observable \mathcal{O} measured at wave vector k and frequency ω . The meaning of (and assumption behind) these scaling forms is simply that, close to the critical point, there is no characteristic length scale other than ξ itself¹⁹ and no characteristic time scale other than ξ_τ . Thus the specific value of the coupling K does not appear explicitly on the right-hand side of Eq. (15). It is present only implicitly through the K dependence of ξ and ξ_τ .

If we specialize to the scale-invariant critical point, the scaling form in Eq. (15) is no longer applicable, since the correlation length and time have diverged to infinity. In this case the only characteristic length left is the wavelength $2\pi/k$ at which the measurement is being made, whence the only characteristic frequency is $\bar{\omega} \sim k^z$. As a result we find the simpler scaling form

$$\mathcal{O}(k, \omega, K_c) = k^{-d_\mathcal{O}} \tilde{\mathcal{O}}(k^z/\omega), \quad (16)$$

which reflects the presence of quantum fluctuations on all length and time scales.²⁰

The utility and power of these scaling forms can be illustrated by the following example. In an ordinary classical system at a critical point in d dimensions where the correlation length has diverged, the correlations of many operators typically fall off as a power law

$$\tilde{G}(r) \equiv \langle \mathcal{O}(\mathbf{r}) \mathcal{O}(\mathbf{0}) \rangle \sim \frac{1}{r^{(d-2+\eta_d)}}, \quad (17)$$

so that the Fourier transform diverges at small wave vectors like

$$G(k) \sim k^{-2+\eta_d}. \quad (18)$$

Suppose that we are interested in a QPT for which the $(d+1)$ -dimensional classical system is effectively isotro-

¹⁸The scaling dimension describes how physical quantities change under a renormalization-group transformation in which short-wavelength degrees of freedom are integrated out. As this is partly a naive change of scale, the scaling dimension is often close to the naive (“engineering”) dimension of the observable, but (except at special, nongeneric fixed points) most operators develop “anomalous” dimensions (see Goldenfeld, 1992).

¹⁹For a more precise statement that includes the role of cutoff scales, see Goldenfeld (1992).

²⁰Equivalently, we could have argued that the scaling function on the right-hand side of Eq. (15) must, for large arguments x and y , have the form $\mathcal{O}(x, y) \sim x^{-d_\mathcal{O}} \tilde{\mathcal{O}}(x^z y^{-1})$ in order for the observable to have a sensible limit as the critical point is approached.

¹⁷Here and in the following we do not write the microscopic length and time scales that are needed to make dimensional sense of these equations (see Goldenfeld, 1992).

pic, and the dynamical exponent $z = 1$. Then the Fourier transform of the correlation function for the $(d+1)$ -dimensional problem is

$$G(k, \omega_n) \sim [\sqrt{k^2 + \omega_n^2}]^{-2 + \eta_{d+1}}, \quad (19)$$

where the $(d+1)$ st component of the “wave vector” is simply the Matsubara frequency used to Fourier transform in the time direction. Analytic continuation to real frequencies via the usual prescription (Mahan, 1990) $i\omega_n \rightarrow \omega + i\delta$ yields the retarded correlation function

$$G_R(k, \omega + i\delta) \sim [k^2 - (\omega + i\delta)^2]^{-(2 + \eta_{d+1})/2}. \quad (20)$$

Instead of a pole at the frequency of some coherently oscillating collective mode, we see instead that $G_R(k, \omega + i\delta)$ has a branch cut for frequencies above $\omega = k$ (we have implicitly set the characteristic velocity to unity). Thus we see that there is no characteristic frequency other than k itself [in general, k^z as in Eq. (16)], as we discussed above. This implies that collective modes have become overdamped, and the system is in an incoherent diffusive regime. The review by Sachdev contains some specific examples that nicely illustrate these points (Sachdev, 1996).

Finally, three comments are in order. First, as we saw in the example of the Josephson-junction array, a finite temporal correlation length means that there is a gap in the spectrum of the quantum problem. Conversely, critical systems are gapless. Second, the exponent z is a measure of how skewed time is, relative to space, in the critical region. This does not, *a priori*, have anything to do with what happens in either of the phases. For example, one should resist the temptation to deduce the value of z via $\omega \sim q^z$ from the dispersion of any Goldstone mode²¹ in the ordered phase. This is incorrect since the exponent z is a property of the critical point itself, not of the ordered phase. Third, we should restate the well-known wisdom that the diverging lengths and the associated scaling of physical quantities are particularly interesting because they represent *universal* behavior, i.e., behavior insensitive to microscopic details within certain global constraints such as symmetry and dimensionality (Goldenfeld, 1992).

B. $T \neq 0$: Finite-size scaling

So far we have described the framework, appropriate to the system at $T=0$, that would describe the underly-

²¹A Goldstone mode is a gapless excitation that is present as a result of a broken continuous symmetry in the ordered phase of a system. Broken continuous symmetry means that the energy is degenerate under a continuous family of uniform *global* symmetry transformations, for example, uniform rotation of the magnetization in an *XY* magnet. This implies that nonuniform but long-wavelength rotations must cost very little energy, and hence there exists a low-energy collective mode in which the order parameter fluctuates at long wavelengths (see Goldenfeld, 1992; Chaikin and Lubensky, 1995).

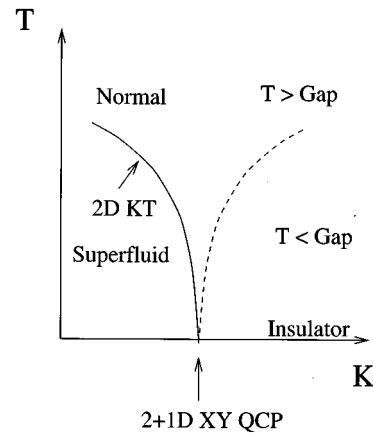


FIG. 5. Illustration of the phase diagram for a Josephson-junction array in two dimensions. K is the quantum-fluctuation parameter, and T is the physical temperature. The solid line represents the Kosterlitz-Thouless critical temperature for the phase transition from the normal state to the superfluid. The solid line ends at the quantum critical point (QCP), where the critical temperature goes to zero. For K greater than its critical value, the system is insulating at zero temperature. For any finite temperature it is not insulating. The dashed line represents the crossover from temperatures smaller than the ($T=0$ insulating) gap to temperatures greater than the gap. This is not a true phase transition; however, the conductivity can be expected to increase rapidly as the temperature goes above this line.

ing QPT in any system. As the experimentally accessible behavior of the system necessarily involves a nonzero temperature, we need to understand how to modify the scaling forms of the previous section for the $T \neq 0$ problem.

The crucial observation for this purpose is, as noted earlier and illustrated in Fig. 4, that the *only* effect of taking $T \neq 0$ in the partition function of Eq. (5) is to make the temporal dimension *finite*; in particular, there is no change in the coupling K with physical temperature. The effective classical system now resembles a hyperslab with d spatial dimensions (taken to be infinite in extent) and one temporal dimension of size $L_\tau \equiv \hbar\beta$. As phase transitions depend sensitively upon the dimensionality of the system, we expect the finiteness of L_τ to modify the critical behavior, since at the longest length scales the system is now d dimensional.

This modification can take two forms. First, it can destroy the transition entirely so that the only critical point is at $T=0$. This happens in the case of the 1D Josephson array. Its finite-temperature physics is that of an *XY* model on an infinite strip, which, being a one-dimensional classical system with short-range forces, is disordered at all finite values of K (finite temperatures in the classical language).

In the second form, the modification is such that the transition persists to $T \neq 0$ but crosses over to a different universality class. For example, the problem of a 2D Josephson-junction array maps onto a $3(=2+1)$ dimen-

sional classical XY model. Its phase diagram is illustrated in Fig. 5. At $T=0$ the QPT for the transition from superconductor to insulator is characterized by the exponents of the 3D XY model. That is, it looks just like the classical lambda transition in liquid helium, with $K-K_c$ playing the role of $T-T_c$ in the helium. However, at $T \neq 0$ the system is effectively two dimensional and undergoes a transition of the 2D Kosterlitz-Thouless²² XY variety at a new, smaller value of K , much like a helium film. The Kosterlitz-Thouless (KT) transition occurs on the solid line in Fig. 5. We see that it is necessary to reduce the quantum fluctuations (by making K smaller) in order to allow the system to order at finite temperatures. Above some critical temperature the system will become disordered (via the KT mechanism) owing to thermal fluctuations. Of course, if we make K larger the quantum fluctuations are then so large that the system does not order at any temperature, even zero. The region on the K axis (i.e., at $T=0$) to the right of the quantum critical point (QCP) in Fig. 5 represents the quantum disordered superconductor, that is, the insulator. At finite temperatures, no system with a finite gap is ever truly insulating. However, there is a crossover regime, illustrated by the dotted line in Fig. 5, separating the regimes where the temperature is smaller and larger than the gap.

At this point the reader might wonder how one learns anything at all about the QPT if the effects of temperature are so dramatic. The answer is that even though the finiteness of L_τ causes a crossover away from the $T=0$ behavior, sufficiently close to the $T=0$ critical point, it does so in a fashion controlled by the physics at that critical point. This is not an unfamiliar assertion. In the language of the renormalization group, critical points are always unstable fixed points and lead to scaling, not because they decide where the system ends up, but because they control how “long” it takes to get there. Here, instead of moving the system away from the critical fixed point by tuning a parameter, we do so by reducing its dimensionality.

Since the physics has to be continuous in temperature, the question arises of how large the temperature has to be before the system “knows” that its dimension has been reduced. The answer to this is illustrated in Fig. 6. When the coupling K is far away from the zero-temperature critical coupling K_c , the correlation length ξ is not large, and the corresponding correlation time $\xi_\tau \sim \xi^z$ is small. As long as the correlation time is smaller than the system “thickness” $\hbar\beta$, the system does not realize that the temperature is finite. That is, the characteristic fluctuation frequencies obey $\hbar\omega \gg k_B T$, and so are quantum mechanical in nature. However, as the critical coupling is approached, the correlation time grows and eventually exceeds $\hbar\beta$. (More precisely, the

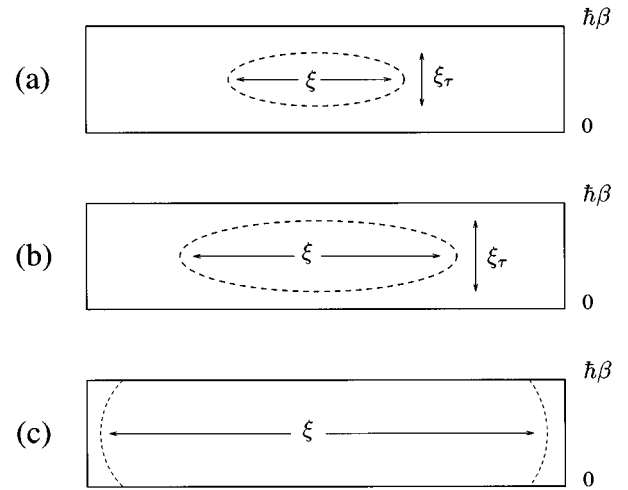


FIG. 6. Illustration of the growing correlation volume as the ($T=0$) critical coupling K_c is approached in a system with finite extent in the temporal direction. In (a) the correlation time is much shorter than $\hbar\beta$. In (b) it is comparable. In (c) the system is very close to the critical point, and the correlation time (that the system would have had at zero temperature) greatly exceeds $\hbar\beta$. Once $\hbar\beta < \xi_\tau \sim \xi^z$, the system realizes that it is effectively d dimensional and not $d+1$ -dimensional. The actual correlation time saturates at $\hbar\beta$, and the corresponding $T=0$ correlation length at which this occurs is the quantum-to-classical crossover length.

correlation time that the system would have had at zero temperature exceeds $\hbar\beta$; the actual fluctuation correlation time is thus limited by temperature.) At this point the system “knows” that the temperature is finite and realizes that it is now effectively a d -dimensional classical system rather than a $(d+1)$ -dimensional system.

The formal theory of the effect of reduced dimensionality near critical points, which quantifies the above considerations, is called *finite-size scaling* (Privman, 1990). For our problem it asserts that for $[K - K_c]/K_c \ll 1$ and $T \rightarrow 0$, physical quantities have the finite-size scaling form,

$$O(k, \omega, K, T) = L_\tau^{d_{\mathcal{O}}/z} O(kL_\tau^{1/z}, \omega L_\tau, L_\tau/\xi_\tau). \quad (21)$$

The interpretation of this form is the following. The quantity $L_\tau \equiv \hbar\beta$ defined above leads, as discussed in more detail below, to a characteristic length $\sim L_\tau^{1/z}$ associated with the temperature. Hence the prefactor $L_\tau^{d_{\mathcal{O}}/z}$ is the analog of the corresponding prefactor in Eq. (15). This same characteristic length is the only one against which to measure the wave vector k . The associated time L_τ is the time scale against which to measure the frequency in the second term. Finally, the distance to the zero-temperature critical coupling is measured via the ratio of L_τ to the zero-temperature correlation time ξ_τ . The message here is that what matters is the ratio of the finite size in the time direction to the $T=0$ correlation length in that direction. We will return to the uses of this form shortly.

²²The Kosterlitz-Thouless phase transition is a special transition exhibited by two-dimensional systems having a continuous XY symmetry. It involves the unbinding of topological vortex defects (see Goldenfeld, 1992; Chaikin and Lubensky, 1995).

Our considerations in this section also show us why the phase boundary (solid line) and the crossover line (dashed line) in Fig. 5 reach zero temperature in a singular way as the quantum critical point is approached. Simple dimensional analysis tells us that the characteristic energy scale (Δ for the insulator, T_{KT} for the superfluid) vanishes near the critical point like \hbar/ξ_τ , implying

$$\begin{aligned}\Delta &\sim |K - K_c|^{\nu z} \theta(K - K_c), \\ T_{KT} &\sim |K - K_c|^{\nu z} \theta(K_c - K).\end{aligned}\quad (22)$$

C. The quantum-classical crossover and the dephasing length

We now turn to a somewhat different understanding and interpretation of the effect of temperature that is conceptually of great importance. Recall that the $T=0$ critical points of interest to us are gapless and scale invariant; i.e., they have quantum fluctuations at all frequencies down to zero. Temperature introduces a new energy scale into the problem. Modes whose frequencies are larger than $k_B T/\hbar$ are largely unaffected, while those with frequencies lower than $k_B T/\hbar$ become occupied by many quanta, with the consequence that they behave classically. Put differently, the temperature cuts off coherent quantum fluctuations in the infrared.

What we want to show next is that this existence of a quantum to classical crossover frequency ($k_B T/\hbar$) leads to an associated length scale for the same crossover, as alluded to in the previous section. We shall refer to this length scale as the dephasing length, L_ϕ , associated with the QPT. The temperature dependence of L_ϕ is easy enough to calculate. From our imaginary-time formalism we recall that quantum fluctuations are fluctuations in the temporal direction. Evidently these cannot be longer ranged than the size of the system in the time direction, $L_\tau = \hbar\beta$. Since spatial and temporal correlations are linked via $\xi_\tau \sim \xi^z$, it follows that the spatial correlations linked with quantum fluctuations are not longer ranged than $L_\tau^{1/z}$. Since the spatial range of quantum fluctuations is the dephasing length, we find $L_\phi \sim 1/T^{1/z}$.

We use the term ‘‘dephasing’’ deliberately. Readers may know, from other contexts where quantum effects are observed, that such observation requires *phase coherence* for the relevant degrees of freedom. In other words, interference terms should not be wiped out by interactions with an ‘‘environment,’’ i.e., other degrees of freedom (Stern *et al.*, 1990). If dephasing takes place and the phase coherence is lost, the resulting behavior is classical. Thus our definition of L_ϕ is in line with that notion. However, readers familiar with the notion of a dephasing length ℓ_ϕ in mesoscopic condensed matter physics or the theory of Anderson localization might be concerned at our appropriation of this notion. The concern arises because, in the standard lore in those fields,

one is often dealing with models of noninteracting or even interacting electrons, whose quantum coherence is limited by degrees of freedom, e.g., phonons, that are not being considered explicitly. This has given rise to a tradition of thinking of ℓ_ϕ as being a length which is set externally. Unfortunately, this sort of separation of degrees of freedom should not be expected to work near a QPT, since there one needs to keep track of all relevant interactions. If a given interaction, e.g., the Coulomb interaction, is relevant then it already sits in the Hamiltonian we need to solve and enters the calculation of L_ϕ . In contrast, if an interaction, e.g., coupling to phonons, is irrelevant then we do not expect it to enter the low-energy physics as it should not in general affect the quantum-classical crossover either.

Another way of formulating this is in terms of dephasing rates. Since temperature is the only energy scale available, a generic quantum critical point will be characterized by a dephasing rate ξ_τ^{-1} that is linear in T , since we expect $\hbar/\xi_\tau \sim k_B T$. By definition, irrelevant interactions, e.g., phonons, have an effective coupling to the electrons that scales to zero as one examines the system on larger length and time scales. Hence such couplings will produce a dephasing rate that vanishes as T^p with $p > 1$ and will therefore become negligible compared to T at low temperatures.²³

Thus we conclude that L_ϕ is the unique dephasing length in the vicinity of a generic quantum critical point. Further discussion and explicit examples of the dissipative dynamics near a critical point can be found in the article by Sachdev (1996).

IV. EXPERIMENTS: QUANTUM PHASE TRANSITIONS IN QUANTUM-HALL SYSTEMS

We now turn from our somewhat abstract considerations to examples of how one actually looks for a QPT in experiments. The basic idea is relatively simple. We try to arrange that the system is close to the conjectured critical point in parameter space ($K \sim K_c$) and temperature ($T \sim 0$) and look for mutually consistent evidence of scaling with various relevant parameters. By these we mean either the deviation of the quantum coupling constant from its critical value $K - K_c$, the temperature, or the wave vector, frequency, and amplitude of a probe. We call these relevant parameters since, when they are

²³Equivalently, the associated length scale will diverge faster than L_ϕ as $T \rightarrow 0$ and hence will not control the quantum-classical crossover of the relevant degrees of freedom, since it is the shortest length that counts. There are times when this sort of reasoning can break down. These situations involve operators that are irrelevant, i.e., decrease in the infrared, but cannot be naively set to zero, since that produces extra singularities. Such operators are known in the trade as ‘‘dangerous irrelevant operators,’’ and we will meet an example in the next section in the context of current scaling.

nonzero, the response of the system has no critical singularities due to the quantum critical point—hence the analogy to relevant operators in renormalization-group theory.²⁴ Additionally, we can look for universal critical amplitudes or amplitude ratios that are implicit in scaling forms for various quantities.

To see how this works, we will consider as a specific example a set of recent experiments on phase transitions in quantum-Hall systems. We derive various scaling relations appropriate to the experiments, even though we do not actually know the correct theory describing the QPT in this disordered, interacting Fermi system. The very generality of the scaling arguments we will apply implies that one need not have a detailed microscopic understanding of the physics to extract useful information. Nevertheless, we will start with some introductory background for readers unfamiliar with the quantum-Hall effect (MacDonald, 1989; Prange and Girvin, 1990; Stone, 1992; Kivelson *et al.*, 1993; Chakraborty and Pietsiläinen, 1995; Das Sarma and Pinczuk, 1996).

The quantum-Hall effect (QHE) is a property of a two-dimensional electron system placed in a strong transverse magnetic field, $B \sim 10$ T. These systems are produced, using modern semiconductor fabrication techniques, at the interface of two semiconductors with different band gaps. The electronic motion perpendicular to the interface is confined to a potential well ~ 100 Å thick. Because the well is so thin, the minimum excitation energy perpendicular to the 2D plane (~ 200 K) is much larger than the temperature (~ 1 K), and so motion in this third dimension is frozen out, leaving the system dynamically two dimensional.

As the ratio of the density to the magnetic field is varied at $T=0$, the electrons condense into a remarkable sequence of distinct thermodynamic phases.²⁵ These phases are most strikingly characterized by their unique electrical transport properties, as illustrated in Fig. 7. Within each phase the current flow is dissipationless, in that the longitudinal resistivity ρ_L that gives the electric field along the direction of current flow ($E_L = \rho_L j$) vanishes. At the same time that the longitudinal resistivity vanishes, the Hall resistivity ρ_H that gives the electric field transverse to the direction of cur-

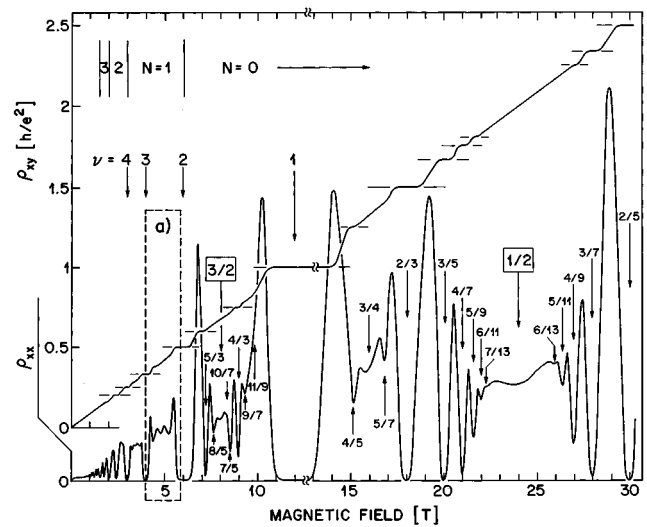


FIG. 7. Transport data in the quantum-Hall regime (Tsui, 1990). The diagonal trace is ρ_H , while the oscillating trace is ρ_L for a sample with density $3 \times 10^{11}/\text{cm}^2$ and mobility $1.3 \times 10^{11} \text{cm}^2/\text{Vs}$ at $T=85$ mK. Note that the latter tends to vanish when the former has a plateau. The arrows mark various filling factors, while the N values at the top mark the Landau level occupied by the electrons.

rent flow ($E_H = \rho_H j$) becomes quantized in rational multiples of the quantum of resistance

$$\rho_H = \frac{h}{\nu_B e^2}, \quad (23)$$

where ν_B is an integer or simple rational fraction that serves to distinguish between the different phases. This quantization has been verified to an accuracy of about one part in 10^7 and to an even higher precision.

The QHE arises from a commensuration between electron density and magnetic flux density, i.e., a sharp lowering of the energy when their ratio, the filling factor ν_B , takes on particular rational values. This commensuration is equivalent to the existence of an energy gap in the excitation spectrum at these “magic” filling factors and leads to dissipationless flow at $T=0$, since degrading a small current requires making arbitrarily small energy excitations that are unavailable. In the absence of disorder Eq. (23) follows straightforwardly, for example, by invoking Galilean invariance (Prange and Girvin, 1990). As the magnetic field (or density) is varied away from the magic filling factor, it is no longer possible for the system to maintain commensuration over its entire area, and it is forced to introduce a certain density of defects to take up the discommensuration. In the presence of disorder, these defects, which are the quasiparticles of the system, become localized and do not contribute to the resistivities, which remain at their magic values. In this fashion we get a QH phase or a plateau.

Transitions between QH phases occur when too many quasiparticles have been introduced into the original QH state, and it becomes energetically favorable to move to the “basin of attraction” of a different state and its associated defects. It might appear that these transi-

²⁴Consider for example a weak magnetic field applied to a system undergoing ferromagnetic ordering. The magnetic field is relevant and removes the sharp singularity in magnetization at the critical temperature, replacing it with a rapid but smooth increase in magnetization. Likewise, measurements at a non-zero frequency and wave vector do not exhibit singularities across a transition. For quantum systems, changes in the coupling and the temperature can produce more subtle effects; they will cut off the critical fluctuations coming from the proximity to the quantum critical point, but either Goldstone modes coming from a broken continuous symmetry or purely classical (thermal) fluctuations could lead to independent singularities in the thermodynamics and response. We saw an example of the latter in the persistence of a phase transition at finite temperatures for the 2D Josephson-junction array.

²⁵The exact membership of the sequence is sample specific but obeys certain selection rules (Kivelson *et al.*, 1992).

tions between neighboring phases are first order, since ρ_H jumps discontinuously by a discrete amount between them, but they are not. Qualitatively, they involve the quasiparticles of each phase, which are localized on a length scale, the localization length, that diverges as the transition is approached from either side. However, as these quasiparticles are always localized at the longest length scale away from criticality, they do not lead to dissipation ($\rho_L=0$) and do not renormalize the Hall resistivities of their respective phases. Exactly at the transition they are delocalized and lead to a nonzero ρ_L . The shift in ρ_H on moving through the transition can be understood in terms of either set of quasiparticles condensing into a fluid state, as there is an underlying duality in this description.

In our description of the QH phases and phase transitions we have employed a common language for all of them. We should note that this does not *ipso facto* imply that all quantum-Hall transitions are in the same universality class; however, experiments, as we discuss later, do seem to suggest that conclusion. The reason for this caution is that different QH states can arise from quite different physics at the microscopic level. States with integer ν_B arise, to first approximation, from single-particle physics. An electron in a plane can occupy a Landau level which comprises a set of degenerate states with energy $(n+1/2)\hbar\omega_c$. These reflect the quantization of the classical cyclotron motion having frequency $\omega_c=eB/m$ and the arbitrariness of the location of that motion in the plane. When an integer number of Landau levels are full (corresponding to an integer filling factor) excitations involve the promotion of an electron across the cyclotron gap, and we have the commensuration/gap nexus necessary for the observation of the (integer) QHE. In contrast, fractional filling factors imply fractional occupations of the Landau levels, with attendant macroscopic degeneracies, and they exhibit a gap only when the Coulomb interaction between the electrons is taken into account (the fractional QHE).²⁶ Readers interested in the details of this magic trick are encouraged to peruse the literature.

Before proceeding to the details of experiments, we need to discuss two important points about the units of the quantities measured in electrical transport where two spatial dimensions are rather special. Experiments measure resistances, which are ratios of total voltages to current, and these are related to local resistivities by ratios of cross-sectional “areas” to lengths. In two dimensions, a cross-sectional area is a length, and consequently no factor of length intervenes between the global and local quantities. In the QH phases this has the important implication that no geometrical factor affects the measurement of the Hall resistance, which is why the ratio of fundamental constants h/e^2 , and hence

the fine structure constant, can be measured to high accuracy on samples whose geometry is certainly not known to an accuracy of one part in 10^7 .

What we have said above is a statement about the engineering dimensions of resistance and resistivity. Remarkably, this also has an analog when it comes to their scaling dimensions at a quantum critical point, i.e., their scaling dimensions vanish.²⁷ Consequently, the resistivities vary as the zeroth power of the diverging correlation length on approaching the transition, i.e., will remain constant on either side. Precisely at criticality they will be independent of the length scale used to define them but can take values distinct from the neighboring phases.

In this fashion we have recovered from a purely scaling argument our earlier conclusion that even though the quantum-Hall transitions are continuous, the resistivities at $T=0$ differ from the quantized values *only* at critical points. Detecting the continuous transitions then requires measurements at a nonzero temperature, frequency, or current, all of which lead to a more gradual variation, which can then be examined for scaling behavior. Below are some examples of how that works.

A. Temperature and frequency scaling

Consider the caricature of a typical set of data shown in Fig. 8. Note that ρ_H interpolates between its quantized values over a transition region of nonzero width, while ρ_L is peaked in the same region but is extremely small outside the region. The change in the shape of these curves with temperature can be understood on the basis of the finite-size scaling form

$$\rho_{L/H}(B, T, \omega) = f_{L/H}(\hbar\omega/k_B T, \delta/T^{1/\nu_z}), \quad (24)$$

where $\delta \equiv (B - B_c)/B_c$ measures the distance to the zero-temperature critical point. This form is equivalent to the general finite-size scaling form in Eq. (21) except that we have assumed the limit $k=0$ and used the previously cited result that the scaling dimension of the resistivity vanishes in $d=2$ (Fisher *et al.*, 1990; Cha *et al.*, 1991). The first argument in the scaling function here is the same as the second in Eq. (21). The second argument in the scaling function here is simply a power of the third argument in Eq. (21). This change is inconsequential; it can be simply absorbed into a redefinition of the scaling function.

First, let us consider a DC or $\omega=0$ measurement. In this case our scaling form implies that the resistivities are not independent functions of δ (or B) and T but instead are functions of the single scaling variable $\delta/T^{1/\nu_z}$. Hence the effect of lowering T is to rescale the deviation of the field from its critical value by the factor T^{1/ν_z} . It follows that the transition appears sharper and sharper as the temperature is lowered; its width vanishes

²⁶This leads to the remarkable feature that, while the quasiparticles of the integer states are essentially electrons, those of the fractional states are fractionally charged and obey fractional statistics.

²⁷See (Fisher *et al.*, 1990; Cha *et al.*, 1991). This is analogous to the behavior of the superfluid density at the classical Kosterlitz-Thouless phase transition (Chaikin and Lubensky, 1995) and leads to a universal jump in it.

as a universal power of the temperature, $\Delta B \sim T^{1/\nu z}$. In Fig. 9 we show the pioneering data of Wei *et al.* (1988) that indeed shows such an algebraic dependence for several different transitions, all of which yield the value $1/\nu z \approx 0.42$. These transitions are between integer quantum-Hall states. Remarkably, this temperature-scaling behavior seems to be ubiquitous at quantum-Hall transitions and suggests that there is a single underlying fixed point for all of them. It was shown by Engel *et al.* (1990) that it holds at transitions between two fractional quantum-Hall states. Subsequently, Wong *et al.* (1995) found the same scaling for the transition between a Hall state and the insulator. In Fig. 10 we show some recent data of Shahar (1995) near another such transition, plotted both as a function of the magnetic field at several values of T and against the scaling variable $\delta/T^{1/\nu z}$. This illustrates the data collapse characteristic of the critical region.

Consider now the results of measurements at nonzero frequencies. In their full generality these require a two-variable scaling analysis (Engle *et al.*, 1995), but we focus instead on two distinct regimes. In the regime $\hbar\omega \ll k_B T$ we expect that the behavior of the scaling function will be governed by its $\omega=0$ limit analyzed previously, i.e., at small ω we expect the scaling to be dominated by T . In the second regime, $\hbar\omega \gg k_B T$, we expect the scaling to be dominated by ω and the scaling function to be independent of T . In order for the temperature to drop out, the scaling function in Eq. (24) must have the form

$$f(x, y) \sim \tilde{f}(yx^{-1/\nu z}) \quad (25)$$

for large x so that the scaling variables conspire to appear in the combination

$$\left(\frac{\hbar\omega}{k_B T}\right)^{-1/\nu z} \frac{\delta}{T^{1/\nu z}} \sim \frac{\delta}{\omega^{1/\nu z}}. \quad (26)$$

It follows that at high frequencies the resistivities are functions of the scaling variable $\delta/\omega^{1/\nu z}$ and that the width of the transition regions scales as $\omega^{1/\nu z}$. Figure 11 shows frequency-dependent conductivity²⁸ data of Engel *et al.* (1993), which exhibits this algebraic increase in the width of the transition region with frequency and yields a value of νz consistent with the temperature scaling.

We should note an important point here. As the ratio $\hbar\omega/k_B T$ is varied in a given experiment we expect to see a crossover between the T - and ω -dominated scaling regimes. The criterion for this crossover is $\hbar\omega \approx k_B T$. The observation by Engel *et al.* (1990), that this is indeed the correct crossover criterion (see Fig. 11), is important for two reasons. First, it involves \hbar and clearly implies that quantum effects are at issue. Second, it implies that T is the relevant infrared scale. If dephasing effects coming from coupling to some irrelevant degree of freedom were important, one would expect the crossover to take place when $\omega\tau \approx 1$, where $1/\tau$ is some microscopic scat-

tering or relaxation rate associated with this coupling. Since the coupling is irrelevant, it will, as noted earlier, give a scattering rate that vanishes as AT^p where p is greater than unity and A is nonuniversal (Sondhi and Kivelson, 1992) (e.g., it depends on the precise value of the electron-phonon coupling constant for the material). In contrast, what is observed is that the relaxation rate obeys $1/\tau = Ck_B T/\hbar$, where C is a *universal* (Sachdev, 1996) dimensionless constant of order unity.

It is important to note that frequency scaling does not give us any new information on exponents that we did not already have from the temperature scaling. The main import of frequency scaling is its ability to confirm the quantum critical nature of the transition by showing that the characteristic time scales have diverged, leaving the temperature itself as the only frequency scale.

B. Current scaling

A third relevant parameter that is experimentally useful is the magnitude of the measuring current or, equivalently, of the applied electric field. In talking about resistivities we have assumed that there is an ohmic regime at small currents, i.e., a regime in which the voltages are linear in the current. In general, there is no reason to believe that the nonlinear response can be related to equilibrium properties, i.e., there is no fluctuation-dissipation theorem beyond the linear regime. However, in the vicinity of a critical point we expect the dominant nonlinearities to come from critical fluctuations. At $T=0$ the electric-field scale for these can be estimated from an essentially dimensional argument. We start by defining a characteristic length ℓ_E associated with the electric field. Imagine that the system is at the critical point so that ℓ_E is the only length scale available. Then the only characteristic time for fluctuations of the system will scale like ℓ_E^{+z} . We can relate the length ℓ_E to the electric field that produces it by

$$eE\ell_E \sim \hbar\ell_E^{-z}. \quad (27)$$

This expression simply equates the energy gained from the electric field by an electron moving a distance ℓ_E to the characteristic energy of the equilibrium system at that same scale. Thus

$$\ell_E \sim E^{-1/(1+z)}. \quad (28)$$

If the system is not precisely at the critical point, then it is this length ℓ_E that we should compare to the correlation length

$$\frac{\ell_E}{\xi} \sim \delta^\nu E^{-1/(1+z)} \sim \left(\frac{\delta}{E^{1/\nu(z+1)}}\right)^\nu. \quad (29)$$

From this we find that the nonlinear DC resistivities for a 2D system obey the scaling forms

$$\rho_{L/H}(B, T, E) = g_{L/H}(\delta/T^{1/\nu z}, \delta/E^{1/\nu(z+1)}). \quad (30)$$

This is a very useful result because it tells us that electric-field scaling will give us new information not available from temperature scaling alone. From tem-

²⁸The conductivities scale in exactly the same fashion as the resistivities.

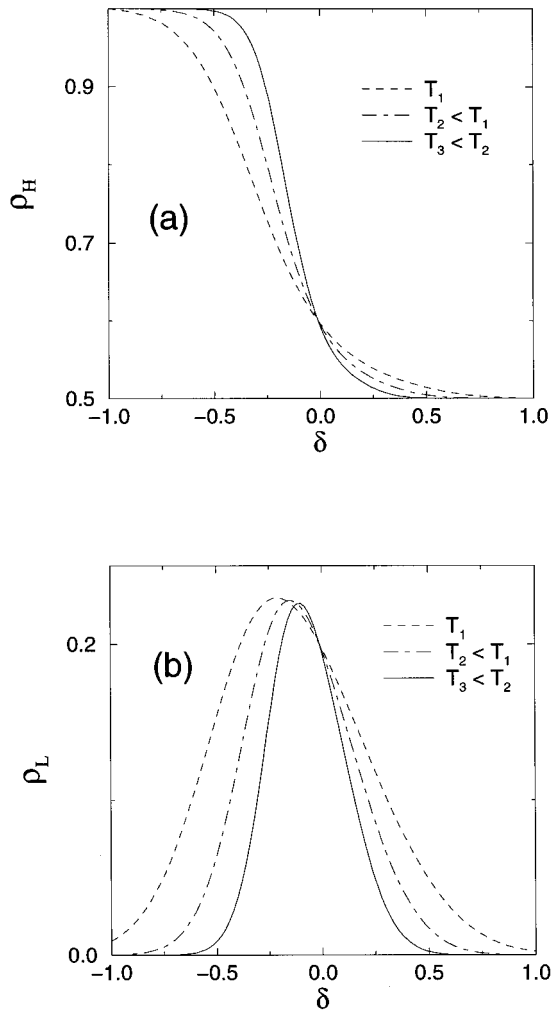


FIG. 8. Caricature of the (ideal) magnetic-field dependence of the Hall (a) and longitudinal (b) components of the resistivity tensor near the quantum-Hall critical point for the $\nu_B=1\rightarrow 2$ transition, i.e., from the $\rho_H=h/e^2$ plateau to the $\rho_H=h/2e^2$ plateau. The parameter $\delta\equiv(B-B_c)/B_c$ measures the distance to the zero-temperature critical point. The three curves in each panel correspond to three values of the temperature $T_1>T_2>T_3$. At finite temperature the variation of ρ_L and ρ_H is continuous, but sharpens up as the temperature is lowered.

perature scaling we can measure the combination of exponents νz . Because an electric field requires multiplication by one power of the correlation length to convert it to a temperature (energy), electric-field scaling measures the combination of exponents $\nu(z+1)$. Thus the two measurements can be combined to separately determine ν and z . The data of Wei *et al.* (1994), Fig. 12, confirm this and yield the value $\nu(z+1)\approx 4.6$. Together the T -, ω - and I -scaling experiments lead to the assignment $\nu\approx 2.3$ and $z\approx 1$.

Equation (30) tells us that there are two scaling regimes. At sufficiently high temperatures, $L_\phi\ll\ell_E$, and the scaling is controlled by the temperature. Below a crossover temperature scale

$$T_0(E)\sim\ell_E^{-z}\sim E^{z/(1+z)}, \quad (31)$$

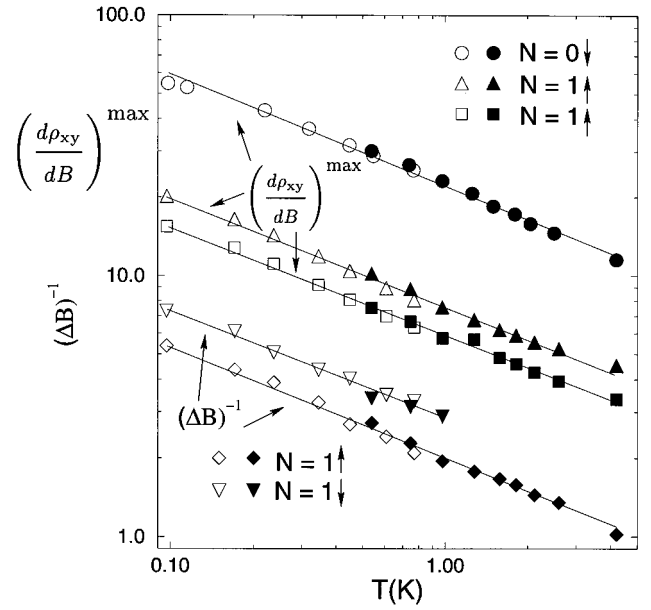


FIG. 9. Data of Wei *et al.* (1988) showing the power-law behavior of the width of the transition regions with temperature. This double-logarithmic plot shows two measures of the inverse width, the maximum value of $\partial\rho_H/\partial B$ and the inverse of the field separation, ΔB , between $\partial\rho_L/\partial B$ maxima as a function of temperature. Both are expected to scale as $T^{1/\nu z}$, and this data yields $1/\nu z=0.42\pm 0.04$. The labels for the symbols refer to different transitions: $N=0\downarrow$ to the $\nu_B=1\rightarrow 2$ transition, $N=1\uparrow$ to the $\nu_B=2\rightarrow 3$ transition, and $N=1\downarrow$ to the $\nu_B=3\rightarrow 4$ transition. The units of ΔB are Tesla (T), and the units of $\partial\rho_H/\partial B$ are $10^3\Omega/T$.

$L_\phi>\ell_E$, and the scaling is controlled by the electric field E (or equivalently, the applied current I). One might be tempted to identify $T_0(E)$ as the effective temperature of the electrons in the presence of the electric field, but this is not strictly appropriate since the system is assumed to have been driven out of equilibrium on length scales larger than ℓ_E .

This quantum critical-scaling picture explicitly assumes that the slow internal time scales of the system near its critical point control the response to the electric field and implicitly assumes that we can ignore the time scale that determines how fast the Joule heat can be removed by phonon radiation. Thus this picture is quite distinct from that of a simple heating scenario in which the electron gas itself equilibrates rapidly but undergoes a rise in temperature if there is a bottleneck for the carrying away of the deposited Joule heat by the phonons. This effect can give rise to an apparent nonlinear response that is, in fact, the linear response of the electron gas at the higher temperature. The power radiated into phonons at low electron temperatures scales as

$$P_{\text{ph}}=AT_e^\theta, \quad (32)$$

where $\theta=4-7$ depending on details (Chow *et al.*, 1996). Equating this to the Joule heating (assuming a scale-invariant conductivity) yields an electronic temperature

$$T_{\text{elec}}\sim E^{2/\theta}. \quad (33)$$

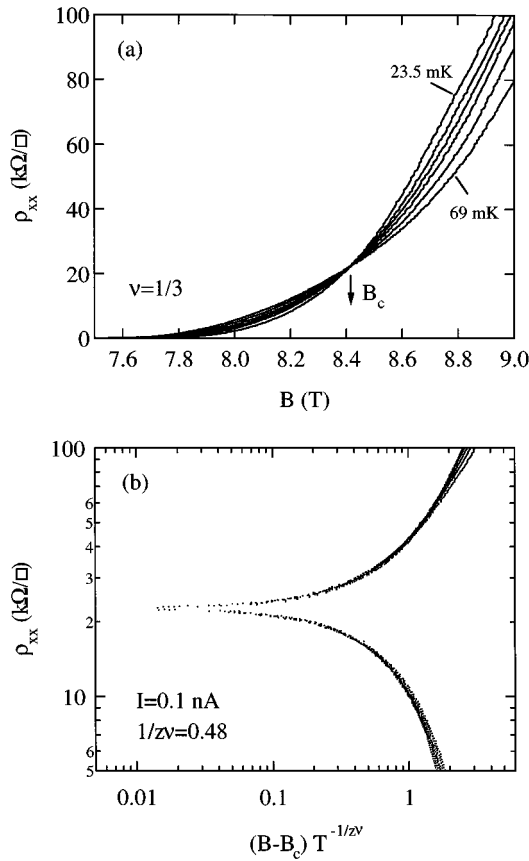


FIG. 10. ρ_L data (in units of $k\Omega$ per square) of Shahar *et al.* (1995) near the $\nu_B=1/3$ to insulator transition. (a) The raw data showing a critical magnetic field with a T -independent $\rho_L^c \approx h/e^2$. (b) Double-logarithmic scaling plot of the same data. Note the systematic improvement in the collapse close to the critical point. (The data points at the far right lie outside the scaling regime.) The apparent discrepancy between the value $1/\nu z=0.48$ obtained here and the earlier reported values is not significant. Reliable error estimates for critical exponents are notoriously hard to obtain.

We now have a paradox. The more irrelevant phonons are at low temperatures—i.e., the larger θ is, the smaller is the exponent $2/\theta$, and hence the more singular is the temperature rise produced by the Joule heat. Comparing Eqs. (31) and (33) we see that for

$$\frac{2}{\theta} < \frac{z}{z+1}, \quad (34)$$

we have

$$T_{\text{elec}} > T_0(E). \quad (35)$$

That is, we then have that the temperature rise needed to radiate away sufficient power is larger than the characteristic energy (“temperature”) scale predicted by the quantum critical-scaling picture. In this case the phonons are “dangerously irrelevant,” and the simple quantum critical-scaling prediction fails. It happens that for the case of GaAs, which is piezoelectric, $2/\theta=1/2$, which gives the same singularity exponent as the quantum critical model $z/(z+1)=1/2$ (since $z=1$). Hence

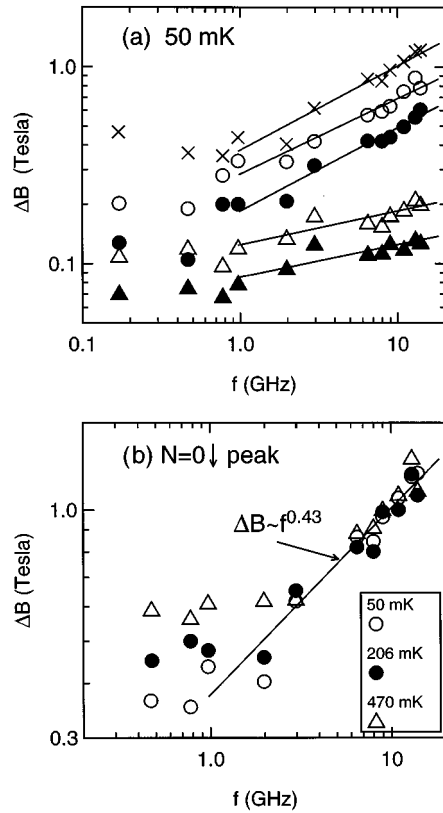


FIG. 11. Data of Engel *et al.* (1990) showing the power-law behavior of the width of the transition regions with the frequency of the measurement. (a) Double-logarithmic plot showing the width, ΔB , defined as the field separation between $\partial\sigma_L/\partial B$ maxima, at a fixed temperature of 50 mK, as a function of frequency at five different transitions: $\nu_B=1\rightarrow 2$ (crosses), $\nu_B=2\rightarrow 3$ (open circles), $\nu_B=3\rightarrow 4$ (filled circles), $\nu_B=4\rightarrow 6$ (open triangles), and $\nu_B=6\rightarrow 8$ (filled triangles). The last two are believed to be two transitions too closely spaced to be resolved at accessible temperatures and show anomalous scaling as a consequence. (b) Data for the $\nu_B=1\rightarrow 2$ transition at three different temperatures. Note that ΔB is independent of frequency below a crossover value roughly equal to the temperature (1 GHz is roughly 50 mK), and, above the crossover, scales as $f^{1/\nu z}$, with $1/\nu z=0.43$ consistent with the value obtained from the temperature scaling.

both quantum-critical and heating effects are important. (The phonon coupling is “marginally dangerous.”) This result is discussed in more detail elsewhere (Chow *et al.*, 1996; Girvin *et al.*, 1996).

C. Universal resistivities

The final signatures of critical behavior that we wish to discuss are universal amplitudes, and, more generally, amplitude ratios. These are readily illustrated in the quantum-Hall problem without considering their general setting, for which we direct the reader to the literature (Aharony and Hohenberg, 1976; Hohenberg *et al.*, 1976; Kim and Weichman, 1991; Chubukov and Sachdev, 1993; Sachdev, 1996). Note that the scaling forms of

Eqs. (24) and (30) imply that the resistivities at $B = B_c$ in the critical region are independent of T , ω , and I . Under certain assumptions it is possible to argue that they are, in fact, universal (Fisher *et al.*, 1990; Kivelson *et al.*, 1992). The observation of such universality between microscopically different samples would then be strong evidence for an underlying QPT as well.

Recently Shahar *et al.* (1995) have carried out a study of the critical resistivities at the transition from the $\nu_B = 1$ and $1/3$ quantum-Hall states to the insulating state. An example of their data is shown in Fig. 10. Notice that there exists a critical value of B field at which the resistivity is temperature independent. For $B < B_c$ the resistivity scales upward with decreasing T , while for $B > B_c$ it scales downward with decreasing T . Since we can think of lowering T as increasing the characteristic length scale L_ϕ at which we examine the system, we see that the point where all the curves cross is the scale-invariant point of the system and hence must be the critical point.

Shahar *et al.* (1995) find that, at these critical points, ρ_L is independent of the sample studied and in fact appears to be h/e^2 within experimental error for both transitions. Preliminary studies (Shahar, 1995) also seem to find sample-independent values of ρ_H at the critical points, with values of h/e^2 and $3h/e^2$ for the two transitions.

D. Unresolved issues

As we have tried to indicate, the success of experimental work in making a case for universal critical behavior at transitions in the quantum-Hall regime is impressive. However, not everything is settled on this score. Apart from the delicate issues surrounding the interpretation of the current scaling data mentioned earlier, there is one significant puzzle. This concerns the failure of ρ_L at the transition between two generic QH states to exhibit a T -independent value at a critical field even as the width of the curve exhibits algebraic behavior.²⁹ This is generally believed to stem from macroscopic inhomogeneities in the density, and some recent theoretical work offers support for this notion (Ruzin *et al.*, 1996). Nevertheless, this is an issue that will need further work. The transitions to the insulator studied more recently are believed to be much less sensitive to this problem, and hence the consistency of the data on those is encouraging. However, in these cases the temperature range over which there is evidence for quantum critical scaling is quite small, as in the data in Fig. 10 which leads us to a general caveat.

Evidence for power laws and scaling should properly consist of overlapping data that cover several decades in the parameters. The various power-law dependences that we have exhibited span at best two decades, most of them fewer, and the evidence for data collapse within the error bars of the data exists only over a small range

²⁹Hence our unwillingness to plot the actual traces in Fig. 8.

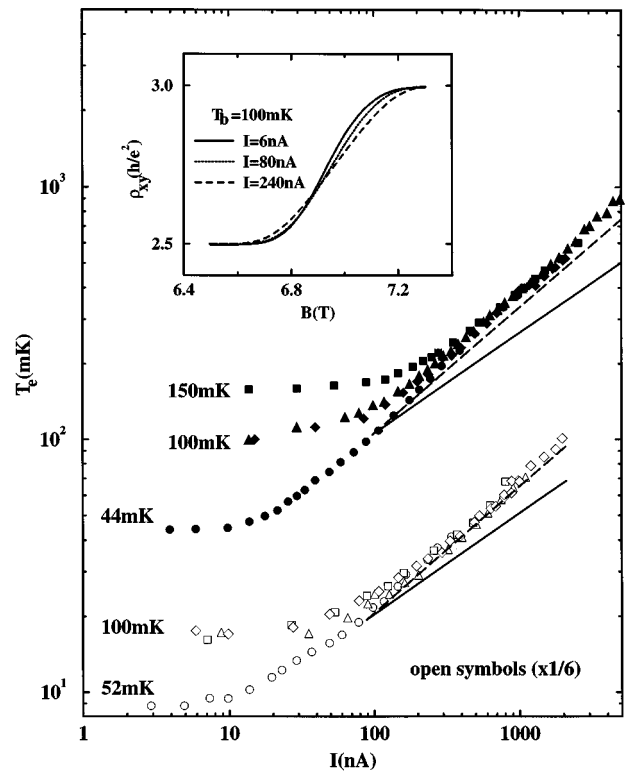


FIG. 12. Data of Wei *et al.* (1994) showing power-law behavior of the effective electron “temperature” T_e with current I at two different transitions: $\nu_B = 5 \rightarrow 6$ (closed symbols) and $\nu_B = 1/3 \rightarrow 2/5$ (open symbols, reduced by a factor of 6). T_e is experimentally defined as the equilibrium temperature at which the maximum of the slope $\partial\rho_h/\partial B$ in linear response equals the measured maximum slope of the same quantity in nonlinear response at low bath temperatures. In the quantum critical model, T_e is not interpreted as a true temperature, but rather as the disequilibrium energy scale $T_0(E)$ in Eq. (31). In the heating model, T_e is interpreted as the quasiequilibrium electronic temperature T_{elec} in Eq. (33). The temperature labels in units of mK at the left-hand side represent the bath temperature at which the current-dependent transport is measured. The dashed reference line under each data set has a slope of 0.5, and the solid line 0.4. If we assume $T_e \sim T_0$ (see text), where $T_0 \sim I^{z/(1+z)}$, the data suggest the assignment $z = 1$. Analysis in terms of the heating model yields the exponent $\theta = 4$ (see text). Inset: ρ_H vs B at $T = 100$ mK for three different I 's in the fractional quantum-Hall effect for $1/3 < \nu < 2/5$. This illustrates how the transition between Hall plateaus sharpens up as the current is reduced, much as it does when the temperature is reduced.

of the scaled variables. Consequently, though the overall picture of the different types of data is highly suggestive, it cannot really be said that it does more than indicate consistency with the scaling expected near a quantum critical point. Regrettably, there is at present no example of a quantum critical phase transition as clean as the remarkable case of the classical lambda transition in superfluid helium, for which superb scaling can be demonstrated (Ahler, 1980).

On the theoretical front the news is mixed. Remarkably, the experimental value of the correlation length

exponent $\nu \approx 2.3$ is consistent with numerical calculations of the behavior of noninteracting electrons in a strong magnetic field (Huckenstein, 1995). Also, the critical resistivities at the transition from the $\nu_B=1$ state to the insulator are also consistent with these calculations (Huckenstein, 1995). This agreement is still a puzzle at this time, especially as the value of the dynamic scaling exponent $z \approx 1$ strongly suggests that Coulomb interactions are playing an important role. The evidence for a super-universality of the transitions, however, does have some theoretical support in the form of a set of physically appealing “correspondence rules” (Kivelson *et al.*, 1992). Unfortunately, their *a priori* validity in the critical regions is still unclear (Lee and Wang, 1996). In summary, theorists have their work cut out for them!

V. CONCLUDING REMARKS

Let us briefly recapitulate our main themes. Zero-temperature phase transitions in quantum systems are fundamentally different from finite-temperature transitions in classical systems in that their thermodynamics and dynamics are inextricably mixed. Nevertheless, by means of the path-integral formulation of quantum mechanics, one can view the statistical mechanics of a d -dimensional $T=0$ quantum system as the statistical mechanics of a $(d+1)$ -dimensional classical system with a fake temperature that is some measure of zero-point fluctuations in the quantum system. This allows one to apply ideas and intuition developed for classical critical phenomena to quantum critical phenomena. In particular this leads to an understanding of the $T \neq 0$ behavior of the quantum system in terms of finite-size scaling and to the identification of a T -dependent length scale L_ϕ that governs the crossover between quantum and classical fluctuations. The identification of QPTs in experiments relies upon finding scaling behavior with relevant parameters. These are the temperature itself and the frequency, wavelength, and amplitude of various probes. Additional signatures are universal values of certain dimensionless critical amplitudes, such as the special case of resistivities at critical points in conducting systems in $d=2$, and, more generally, amplitude ratios.

In this Colloquium we have illustrated these ideas in the context of a single system, the two-dimensional electron gas in the quantum-Hall regime. The ideas themselves are much more widely applicable. Interested readers may wish to delve, for example, into work on the one-dimensional electron gas (Luther and Peschel, 1975; Emery, 1979), metal-insulator transitions in zero magnetic field (“Anderson-Mott transitions”) (Belitz and Kirkpatrick, 1994), superconductor-insulator transitions (Chakravarty *et al.*, 1986, 1989; Weichman, 1988; Weichman and Kim, 1989; Batrouni *et al.*, 1990; Wen and Zee, 1990; Cha *et al.*, 1991; Krauth and Trivedi, 1991; Krauth *et al.*, 1991; Scalettar *et al.*, 1991; Runge, 1992; Sørensen *et al.*, 1992; Batrouni *et al.*, 1993; Fazio and Zappala, 1996; Kampf and Zimanyi, 1993; Makivić *et al.*, 1993; Wallin *et al.*, 1994), two-dimensional antiferromagnets associated with high-temperature supercon-

ductivity (Chakravarty *et al.*, 1989; Sachdev and Ye, 1992; Chubukov and Sachdev, 1993; Chubukov *et al.*, 1994; Sachdev, 1994; Sachdev *et al.*, 1994; Sachdev, 1996), and magnetic transitions in metals (Altshuler *et al.*, 1995; Hertz, 1976; Millis, 1993). This list is by no means exhaustive, and we are confident that it will continue to expand for some time to come!

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APPENDIX

In this Appendix we briefly outline the derivation (Wallin *et al.*, 1994) of the expression for the matrix elements

$$M \equiv \langle \{ \theta(\tau_{j+1}) \} | e^{-(\delta\tau/\hbar)H} | \{ \theta(\tau_j) \} \rangle \quad (\text{A1})$$

appearing in Eq. (6). The Hamiltonian contains a “kinetic energy”

$$T = \frac{C}{2} \sum_j V_j^2 = \frac{E_C}{2} \sum_j \left(-i \frac{\partial}{\partial \theta_j} \right)^2, \quad (\text{A2})$$

where $E_C \equiv (2e)^2/C$, and a “potential energy”

$$V \equiv -E_J \cos(\hat{\theta}_j - \hat{\theta}_{j+1}). \quad (\text{A3})$$

For sufficiently small $\delta\tau$ we can make the approximation

$$e^{-(\delta\tau/\hbar)H} \approx e^{-(\delta\tau/\hbar)T} e^{-(\delta\tau/\hbar)V}. \quad (\text{A4})$$

Inserting a complete set of angular-momentum eigenstates $| \{ m_k \} \rangle$ (defined for a single site by $\langle \theta_k | m_k \rangle = e^{im_k \theta_k}$) yields

$$M = \sum_{\{m\}} \langle \{ \theta(\tau_{j+1}) \} | e^{-(\delta\tau/\hbar)T} | \{ m_k \} \rangle \times \langle \{ m_k \} | e^{-(\delta\tau/\hbar)V} | \{ \theta(\tau_j) \} \rangle. \quad (\text{A5})$$

We can now take advantage of the fact that V is diagonal in the angle basis and T is diagonal in the angular-momentum basis to obtain

$$M = \sum_{\{m\}} e^{-(\delta\tau/2\hbar)E_C} \sum_k m_k^2 e^{im_k [\theta_k(\tau_{j+1}) - \theta_k(\tau_j)]} \times e^{+(\delta\tau/\hbar)E_J} \sum_k \cos[\theta_k(\tau_{j+1}) - \theta_k(\tau_j)]. \quad (\text{A6})$$

Because $\delta\tau$ is small, the sum over the $\{m\}$ is slowly convergent. We may remedy this by using the Poisson summation formula (Wallin *et al.*, 1994)

$$\sum_m e^{-(\delta\pi 2\hbar)E_C m^2} e^{im\theta} = \sqrt{(\pi\hbar/\delta\tau E_C)} \times \sum_n e^{-(\hbar/2E_C\delta\tau)(\theta+2\pi n)^2}. \quad (\text{A7})$$

This periodic sequence of very narrow Gaussians is (up to an irrelevant constant prefactor) the Villain approximation to

$$e^{+(\hbar/E_C\delta\tau)\cos(\theta)}. \quad (\text{A8})$$

Strictly speaking, we should keep $\delta\tau$ infinitesimal. However, we may set it equal to the natural ultraviolet cutoff, the inverse of the Josephson plasma frequency $\delta\tau = \hbar/\sqrt{E_C E_J}$, without changing the universality class. Substituting this result into Eq. (A6) yields Eq. (8) with the same coupling constant $K = \sqrt{E_C/E_J}$ in both the space and time directions.

REFERENCES

- Aharony, A., and P.C. Hohenberg, 1976, *Phys. Rev. B* **13**, 3081.
- Ahlers, G., 1980, *Rev. Mod. Phys.* **52**, 489.
- Altshuler, B.L., L.B. Ioffe, and A.J. Millis, 1995, *Phys. Rev. B* **52**, 5563.
- Batrouni, G.G., B. Larson, R.T. Scalettar, J. Tobochnik, and J. Wang, 1993, *Phys. Rev. B* **48**, 9628.
- Batrouni, G.G., R.T. Scalettar, and G.T. Zimanyi, 1990, *Phys. Rev. Lett.* **65**, 1765.
- Belitz, D., and T.R. Kirkpatrick, 1994, *Rev. Mod. Phys.* **66**, 261.
- Bradley, R. M., and S. Doniach, 1984, *Phys. Rev. B* **30**, 1138.
- Cha, Min-Chul, M.P.A. Fisher, S.M. Girvin, M. Wallin, and A.P. Young, 1991, *Phys. Rev. B* **44**, 6883.
- Chaikin, P.M., and T.C. Lubensky, 1995, *Principles of Condensed Matter Physics* (Cambridge University, New York).
- Chakraborty, T., and P. Pietiläinen, 1995, *The Quantum Hall Effects: Integral and Fractional*, 2nd Ed., Springer Series in Solid State Sciences **85** (Springer, New York).
- Chakravarty, S., G.-L. Ingold, S. Kivelson, and A. Luther, 1986, *Phys. Rev. Lett.* **56**, 2303.
- Chakravarty, S., B.I. Halperin, and D.R. Nelson, 1989, *Phys. Rev. B* **39**, 2344.
- Chakravarty, S., S. Kivelson, G.T. Zimanyi, and B.I. Halperin, 1987, *Phys. Rev. B* **35**, 7256.
- Chow, E., H.P. Wei, S.M. Girvin, and M. Shayegan, 1996, *Phys. Rev. Lett.* **77**, 1143.
- Chubukov, A.V., and S. Sachdev, 1993, *Phys. Rev. Lett.* **71**, 169.
- Chubukov, A.V., S. Sachdev, and J. Ye, 1994, *Phys. Rev. B* **49**, 11 919.
- Das Sarma, S., and A. Pinczuk, 1996, Eds., *Perspectives in Quantum Hall Effects: Novel Quantum Liquids in Low-Dimensional Semiconductor Structures* (Wiley, New York).
- Emery, V. J., 1979, in *Highly Conducting One-Dimensional Solids*, edited by J. T. DeVreese, R. P. Evrard, and V. E. Van Doren (Plenum, New York), pp. 247–303.
- Engel, L.W., D. Shahar, C. Kurdak, and D.C. Tsui, 1993, *Phys. Rev. Lett.* **71**, 2638.
- Engel, L.W., D. Shahar, C. Kurdak, and D.C. Tsui, 1995, in *Proceedings of the 11th International Conference on High Magnetic Fields in the Physics of Semiconductors*, edited by D. Heiman (World Scientific, Singapore), p. 236.
- Engel, L.W., H.P. Wei, D.C. Tsui, and M. Shayegan, 1990, *Surf. Sci.* **229**, 13.
- Fazio, R., and D. Zappala, 1996, *Phys. Rev. B* **53**, R8883.
- Ferrell, R.A., N. Menyhard, H. Schmidt, F. S. Schwabl, and P. Szepfalusy, 1968, *Ann. Phys. (N.Y.)* **47**, 565.
- Feynman, R.P., 1972, *Statistical Mechanics* (Benjamin, New York).
- Fisher, M.P.A., G. Grinstein, and S.M. Girvin, 1990, *Phys. Rev. Lett.* **64**, 587.
- Girvin, S.M., S.L. Sondhi, and M.P.A. Fisher, 1996, unpublished.
- Goldenfeld, N., 1992, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA).
- Haviland, D.B., and P. Delsing, 1996, unpublished.
- Hertz, J.A., 1976, *Phys. Rev. B* **14**, 1165.
- Hohenberg, P.C., A. Aharony, B.I. Halperin, and E.D. Siggia, 1976, *Phys. Rev. B* **13**, 2986.
- Hohenberg, P.C., and B.I. Halperin, 1977, *Rev. Mod. Phys.* **49**, 435.
- Huckestein, B., 1995, *Rev. Mod. Phys.* **67**, 357.
- Kampf, A.P., and G.T. Zimanyi, 1993, *Phys. Rev. B* **47**, 279.
- Karlhede, A., S.A. Kivelson, and S.L. Sondhi, 1993, in *Correlated Electron Systems*, edited by V.J. Emery (World Scientific, Singapore), pp. 242–347.
- Kim, K., and P.B. Weichman, 1991, *Phys. Rev. B* **43**, 13 583.
- Kivelson, S.A., D.H. Lee, and S.C. Zhang, 1992, *Phys. Rev. B* **46**, 2223.
- Krauth, W., and N. Trivedi, 1991, *Europhys. Lett.* **14**, 627.
- Krauth, W., N. Trivedi, and D. Ceperley, 1991, *Phys. Rev. Lett.* **67**, 2307.
- Lee, Dung-Hai, and Ziqiang Wang, 1996, *Phys. Rev. Lett.* **76**, 4014.
- Luther, A., and I. Peschel, 1975, *Phys. Rev. B* **12**, 3908.
- MacDonald, A.H., 1989, Ed., *Quantum Hall Effect: A Perspective* (Kluwer, Boston).
- Mahan, G.D., 1990, *Many-Particle Physics*, 2nd Ed. (Plenum, New York).
- Makivić, M., N. Trivedi, and S. Ullah, 1993, *Phys. Rev. Lett.* **71**, 2307.
- Millis, A.J., 1993, *Phys. Rev. B* **48**, 7183.
- Prange, R.E., and S.M. Girvin, 1990, Eds., *The Quantum Hall Effect*, 2nd Ed. (Springer, New York).
- Privman, V., 1990, in *Finite Size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore), p. 1.
- Runge, K., 1992, *Phys. Rev. B* **45**, 13 136.
- Ruzin, I.M., N.R. Cooper, and B.I. Halperin, 1996, *Phys. Rev. B* **53**, 1558.
- Sachdev, S., 1994, *Z. Phys. B* **94**, 469.
- Sachdev, S., 1996, in *Proceedings of the 19th IUPAP International Conference on Statistical Physics*, Xiamen, China, 1995, edited by Hao Bailin (World Scientific, Singapore) p. 289 [preprint cond-mat/9508080].
- Sachdev, S., A. V. Chubukov, and A. Sokol, 1995, *Phys. Rev. B* **51**, 14 874.
- Sachdev, S., T. Senthil, and R. Shankar, 1994, *Phys. Rev. B* **50**, 258.
- Sachdev, S., and J. Ye, 1992, *Phys. Rev. Lett.* **69**, 2411.

- Scalettar, R.T., G.G. Batrouni, and G.T. Zimanyi, 1991, *Phys. Rev. Lett.* **66**, 3144.
- Shahar, D., 1996, unpublished.
- Shahar, D., 1995, Ph.D. thesis (Princeton University).
- Shahar, D., D.C. Tsui, M. Shayegan, R.N. Bhatt, and J.E. Cunningham, 1995, *Phys. Rev. Lett.* **74**, 4511.
- Sondhi, S.L., and S.A. Kivelson, 1992, unpublished.
- Sørensen, E.S., M. Wallin, S.M. Girvin, and A.P. Young, 1992, *Phys. Rev. Lett.* **69**, 828.
- Stern, A., Y. Aharonov, and Y. Imry, 1990, *Phys. Rev. A* **41**, 3436.
- Stone, Michael, 1992, Ed., *The Quantum Hall Effect* (World Scientific, Singapore).
- Suzuki, M., 1976, *Prog. Theor. Phys.* **56**, 1454.
- Tsui, D.C., 1990, *Physica B* **164**, 59.
- Wallin, M., E. Sørensen, S.M. Girvin, and A.P. Young, 1994, *Phys. Rev. B* **49**, 12 115.
- Wei, H.P., L.W. Engel, and D.C. Tsui, 1994, *Phys. Rev. B* **50**, 14 609.
- Wei, H.P., D.C. Tsui, M.A. Paalanen, and A.M.M. Pruisken, 1988, *Phys. Rev. Lett.* **61**, 1294.
- Weichman, P.B., 1988, *Phys. Rev. B* **38**, 8739.
- Weichman, P.B., and K. Kim, 1989, *Phys. Rev. B* **40**, 813.
- Wen, X.G., and Z. Zee, 1990, *Int. J. Mod. Phys. B* **4**, 437.
- Wong, L.W., H.W. Jiang, N. Trivedi, and E. Palm, 1995, *Phys. Rev. B* **51**, 18 033.
- Young, A. P., 1975, *J. Phys. C* **8**, L309.