PHYS 203A - Winter 2020

Hw 5:

1. Let $\phi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2}$ be a complex field (here $\phi_{1,2}$ are real). The Lagrangian density

$$\mathcal{L} = \eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - \mu^2|\phi|^2 - \lambda|\phi|^4$$

describes the dynamics of this field in the absence of other fields.

   (a) Find the equations of motion for ϕ and ϕ^*.

   (b) Determine the energy-momentum tensor, $T^{\mu\nu}$. Is it conserved?

   (c) Determine the conserved Noether current associated with invariance of $\mathcal{L}$ under $\phi(x) \to e^{i\alpha}\phi(x)$.

   (d) Determine the Hamiltonian density.

   (e) Replace $\partial_\mu + ieA_\mu(x)$ for ∂_μ in the lagrangian. Show that the resulting Lagrangian is gauge invariant, that is, invariant under $A_\mu \to A_\mu + \partial_\mu\alpha$ where $\alpha = \alpha(x)$ is an arbitrary function of space-time, provided one simultaneously transforms $\phi(x) \to e^{i\kappa\alpha(x)}\phi(x)$, for some choice of the constant κ. What is the electric current density $j_\mu(x)$ (in terms of the field ϕ and how does it differ from the conserved Noether current you computed above?

2. Let

$$\vec{E} = \begin{cases} E(0, \sin\theta, \cos\theta) & \text{for } z > 0 \\ 0 & \text{for } z < 0 \end{cases}$$

Find the force on a unit area of a surface on the xy plane.

3. Determine the force between two charges of equal magnitude and opposte sign a distance d apart, by integrating the stress tensor over the plane halfway between the charges. Why should this work? The case of two equal and same sign charges is trivial; why?

4. In class we showed that the total energy and momentum of a system consisting of the electromagnetic field and a collection of point charges is conserved, *e.g.*, for energy

$$\frac{d}{dt}\left[E_{\text{kin}} + \int_V d^3x\, u\right] = -\oint_{\partial V} d^2s\, \hat{n}\cdot\vec{S}$$

where E_{kin} is the kinetic energy of the point particles, and u and $\vec{S}$ are the energy density and Poynting vector of the electromagnetic field. Derive an analogous equation expressing the conservation of total angular momentum of this combined system.

5. The Lagrangian density for a real scalar field $\phi(x)$ is

$$\mathcal{L} = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

(a) Show that there exist spherical wave solutions to the equation of motion:

$$\phi(x) = \phi_0 \frac{\cos(k(r - x^0))}{r}$$

where $r^2 = \delta_{ij}x^i x^j$, and k and ϕ_0 are constants.

(b) Compute the angular momentum density and the total angular momentum for this solution.