Hw 2 (Revised 1/17/2020):

1. In Newtonian mechanics a body experiencing constant acceleration (for example, a rocket with constant thrust to mass ratio) exhibits rectilinear motion increasing its speed to eventually exceed the speed of light. This, of course, cannot happen in relativistic dynamics. Determine the motion of a body that experiences acceleration that is constant in its rest frame. *Hint:* Consider a "comoving" frame, that is, an inertial frame that is instantaneously at rest with the particle, and find its relation to a fixed frame in which you will describe the motion.

2. This problem extends the class discussion on infinitesimal Lorentz transformations, and the exponential map. You may find the following identity useful: $\epsilon^{ijk}\epsilon^{mnk} = \delta^{im}\delta^{jn} - \delta^{jm}\delta^{in}$.

   (a) An infinitesimal Lorentz transformation: Verify that to first order in ϵ $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$ satisfies
   $$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}. \tag{1}$$
   Here and elsewhere the indices are lowered by use of the metric $\eta_{\mu\nu}$ and raised with its inverse $\eta^{\mu\nu}$.

   (b) Exponentiation: A finite Lorentz transformation can be written as $\Lambda = \lim_{N\to\infty}(\mathbb{1} + \frac{1}{N}\epsilon)^N = \exp(\epsilon)$, where ϵ is no longer restricted to be infinitesimal but still satisfies (1). Show that one can write this 4×4 matrix as $\Lambda = \exp(\sum_{i=1}^{3}(\omega^i K^i + \theta^i J^i)$ where ω^i and θ^i are numbers that parametrize boosts and rotations, respectively, and correspondingly K^i and J^i are the matrices
   $$(K^i)^0{}_0 = 0, \quad (K^i)^j{}_0 = (K^i)^0{}_j = \delta_{ij}, \quad (K^i)^j{}_k = 0, \text{ (for } j,k = 1,2,3)$$
   and
   $$(J^i)^\mu{}_0 = (J^i)^0{}_\mu = 0, \quad (J^i)^j{}_k = -\epsilon_{ijk}, \text{ (for } j,k = 1,2,3)$$
   (Note: These expressions have unbalanced upper and lower indices. That's because the symbols δ_{ij} and ϵ_{ijk} on the right hand side of these expressions simply express the numerical values of entries in these fixed matrices).

   (c) Show that
   $$\begin{aligned} [J^i, J^j] &= \epsilon^{ijk} J^k \\ [J^i, K^j] &= \epsilon^{ijk} K^k \\ [K^i, K^j] &= -\epsilon^{ijk} J^k \end{aligned} \tag{2}$$

   (d) Using vector notation for the triplets of parameters, define unit "vectors" by $\hat\theta = \vec\theta/|\vec\theta|$ and $\hat\omega = \vec\omega/|\vec\omega|$. Show that
   $$\begin{aligned} (\hat\theta \cdot \vec J)^3 &= -\hat\theta \cdot \vec J \\ (\hat\omega \cdot \vec K)^3 &= \hat\omega \cdot \vec K \end{aligned}$$

(e) Show that

$$\exp(-\zeta\hat{\beta}\cdot\vec{K}) = \mathbb{1} - \hat{\beta}\cdot\vec{K}\,\sinh\zeta + (\hat{\beta}\cdot\vec{K}))^2(\cosh\zeta - 1).$$

What is the physical interpretation of this result?

(f) Similarly, compute $\exp(-\phi\,\hat{\theta}\cdot\vec{J})$ and give a physical interpretation of your result.

3. As we will see in class, the calculation of the angular velocity in Thomas precession arises from a mismatch in the orientation of boosts that relate a fixed inertial frame to frames comoving with an accelerated particle at two infinitesimally separated instants. Say the boost to the comoving frame at times t and $t+\delta t$ are Λ_1 and Λ_2, respectively. What is needed in the computation of Thomas precession is $\Lambda = \Lambda_2\Lambda_1^{-1}$.

(a) Writing $\Lambda_1 = e^K$ and $\Lambda_2 = e^{K+\delta K}$, and assuming δK is infinitesimal show that (to first order in δK)

$$\Lambda = \mathbb{1} + \delta K + \tfrac{1}{2!}[K,\delta K] + \tfrac{1}{3!}[K,[K,\delta K]] + \cdots \tag{3}$$

Hint: Consider $\Lambda(s) = e^{s(K+\delta K)}e^{-sK}$ as a function of the real parameter s, and Taylor expand, setting $s = 1$ at the end of the computation.

(b) With

$$K = -\hat{\beta}\cdot\vec{K}\tanh^{-1}\beta$$
$$K + \delta K = -\hat{\beta}'\cdot\vec{K}\tanh^{-1}\beta'$$

where $\vec{\beta}' = \vec{\beta} + \delta\vec{\beta}$, show that

$$\delta K = -\gamma^2\delta\vec{\beta}_{\|} - \frac{\delta\vec{\beta}_{\perp}\cdot\vec{K}\tanh^{-1}\beta}{\beta}$$

where $\delta\vec{\beta}_{\|}$ and $\delta\vec{\beta}_{\perp}$ are the components of $\delta\vec{\beta}$ parallel and transverse to $\vec{\beta}$, and $\gamma^2 = (1-\beta^2)^{-1}$.

(c) Use Eqs. 2 to compute explicitly the first few terms in the series (3), $[K,\delta K]$, $[K,[K,\delta K]]$, $[K,[K,[K,\delta K]]]$. One can use this to

$$\Lambda = \mathbb{1} - (\gamma^2\delta\vec{\beta}_{\|} + \gamma\delta\vec{\beta}_{\perp})\cdot\vec{K} - \frac{\gamma^2}{\gamma+1}(\vec{\beta}\times\delta\vec{\beta}_{\perp})\cdot\vec{J}$$

Although not a required part of this problem, if you see a pattern emerge from the computation of multiple commutators, you will see that you can easily sum (3).